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INTEGRAL INEQUALITIES RESEMBLING COPSON'S INEQUALITY

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Abstract

The present paper deals with two inequalities which resemble Copson's integral inequalities. From our theorems, we obtain two interesting corollaries.

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1. Introduction

Copson [4] has proved integral analogues of his inequalities for series which he originally proved with a view to generalizing Hardy's inequality for a series of nonnegative terms [5, Theorem 337]. A typical inequality of Copson is the following:

THEOREM A [4, Theorem 1]. *Let $\phi(x)$, $f(x)$ be non-negative for $x \geq 0$ and be continuous in $[0, \infty)$. Let*

$$\Phi(x) = \int_0^x \phi(t)\,dt, \qquad F(x) = \int_0^x f(t)\phi(t)\,dt. \tag{1.1}$$

Let $p \geq 1$, $c > 1$. If $0 < b \leq \infty$ and

$$\int_0^b F(x)^p \Phi(x)^{-c} \phi(x)\,dx \tag{1.2}$$

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converges at the lower limit of integration, then

$$\int_0^b F^p \Phi^{-c} \phi \, dx \le \left(\frac{p}{c-1}\right)^p \int_0^b f^p \Phi^{p-c} \phi \, dx. \tag{1.3}$$

The case $c = p$ and $\phi(x) = 1$ is Hardy's classical inequality [5, Theorem 327] which inspired many mathematicians including Copson. In fact, Theorem A mentioned above is one of the six inequalities established in [4]. Beesack [1] has proved six similar inequalities, two of which provide alternate proofs of [4, Theorems 5 and 6]; a further two are additional cases which complete Copson's list, and the remaining two inequalities deal with the case $p < 0$. Independent generalizations of Copson's inequalities referred to above have been established by Love [6, 7], Mohapatra and Russell [8]. Boas [2] has considered generalization of Hardy's inequality and in [3], Boas and Imoru have obtained interesting results on convolution inequalities.

2. Statement of results

Throughout the paper, $K(p)$ will denote a positive constant, depending only on p, which may be different at different occurrences. We shall define p' by $1/p + 1/p' = 1$, for $p > 1$. Also it is to be understood that $x \ge 0$ in all occurrences of x. We also assume that functions f and ϕ are real and continuous on $[0, \infty)$. Additional restrictions to be satisfied by ϕ will be mentioned whenever necessary.

We shall prove the following theorems.

THEOREM 1. *Let $\phi(x)$ be positive and locally absolutely continuous in $[0, \infty)$ and $f(x) \ge 0$ in $[0, \infty)$. Let $p > 1$,*

$$\Phi(x) = \int_0^x \phi(t) \, dt, \tag{2.1}$$

$$x|\phi'(x)| \le A\phi(x) \quad \text{and} \quad x\phi(x) \le B\Phi(x) \tag{2.2}$$

for all $x > 0$, where A and B are positive constants, and

$$\int_0^\infty \phi(x) \left\{ \Phi(x)^{-1} \int_0^\infty t^{-1} \phi(t)^{1/p'} \int_0^t \phi(u)^{1/p} f(u) \, du \, dt \right\}^p dx < \infty. \tag{2.3}$$

Then

$$\int_0^\infty \phi(x) \left\{ \frac{1}{\Phi(x)} \int_0^x \phi(t) f(t) \, dt \right\}^p dx \le (A+B)^p \int_0^\infty \left\{ \frac{1}{x} \int_0^x \phi(t)^{1/p} f(t) \, dt \right\}^p dx. \tag{2.4}$$

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REMARK. The outer integrals in (2.4) can be replaced by $\int_0^m$ for $0 < m \leq \infty$ by modifying the hypotheses appropriately.

By setting $\phi(u) = (u+1)^{-1}$ in Theorem 1 and using the inequality $\ln(x+1) > x/(x+1)$ $(x > 0)$, we get

COROLLARY 1. *Let $p \geq 1$ and $f(x)$ be non-negative on $x \geq 0$. Then*

$$\int_0^\infty \frac{1}{x+1}\left\{\frac{1}{\ln(x+1)}\int_0^x \frac{f(t)}{t+1}\,dt\right\}^p dx \leq 2^p \int_0^\infty \left\{\frac{1}{x}\int_0^x \frac{f(t)}{(t+1)^{1/p}}\,dt\right\}^p dx, \tag{2.5}$$

provided the integral in (2.3) converges when $\phi(u) = (u+1)^{-1}$.

THEOREM 2. *Let $p > 1$, $\phi(x)$ be positive, locally absolutely continuous in $[0,\infty)$, and $f(x)$ be non-negative on $[0,\infty)$. Further, let (2.1), (2.2) and the following hold:*

$$\phi(t)^{1/p} f(t)/t \in L(x,\infty), \quad \textit{for all } x > 0. \tag{2.6}$$

If the integral

$$\int_0^\infty \phi(x)\left(\int_x^\infty \frac{\phi(t)^{1/p'}}{\Phi(t)}\int_t^\infty u^{-1}\phi(u)^{1/p} f(u)\,du\,dt\right)^p dx \tag{2.7}$$

converges, then

$$\int_0^\infty \phi(x)\left(\int_x^\infty \frac{\phi(t) f(t)}{\Phi(t)}\,dt\right)^p dx \leq (B + Cp)^p \int_0^\infty \left(\int_x^\infty \frac{\phi(t)^{1/p} f(t)}{t}\,dt\right)^p dx \tag{2.8}$$

where $C = 1 + A + B$.

By setting $\phi(u) = (u+1)^{-1}$, we obtain

COROLLARY 2. *Let $p > 1$ and $f(x)$ be a non-negative function on $x \geq 0$. Further, let the integral in (2.7) converge with $\phi(u) = (u+1)^{-1}$. Then*

$$\int_0^\infty \frac{1}{x+1}\left(\int_x^\infty \frac{f(t)\,dt}{(t+1)\ln(1+t)}\right)^p dx \leq (1+3p)^p \int_0^\infty \left(\int_x^\infty \frac{f(t)}{t(t+1)^{1/p}}\,dt\right)^p dx. \tag{2.9}$$

Love [7] has developed an excellent method for obtaining integral inequalities when the kernel is homogeneous. Unfortunately that method cannot be employed here since we want to obtain corollaries in which the kernel contains a logarithmic function.

3. Statement of lemma

We shall need the following lemma for the proof of our theorems.

LEMMA (see [8, Lemma 1]). (a) *Let* $1 \leq p < \infty$ *and* $z(\cdot)$ *be non-negative and integrable over* $[0, x)$. *Then*

$$\left(\int_0^\infty z(t)\,dt\right)^p = p\int_0^x z(t)\left(\int_0^t z(u)\,du\right)^{p-1} dt. \tag{3.1}$$

(b) *Let* $1 \leq p < \infty$ *and* $z(\cdot)$ *be non-negative and integrable over* (x, ∞). *Then*

$$\left(\int_x^\infty z(t)\,dt\right)^p = p\int_0^\infty z(t)\left(\int_t^\infty z(u)\,du\right)^{p-1} dt. \tag{3.2}$$

4. Proof of theorems

PROOF OF THEOREM 1. Set

$$g(x) = \int_0^x \phi(t)^{1/p} f(t)\,dt. \tag{4.1}$$

Since $\phi(t)$ and $f(t)$ are continuous on $[0, \infty)$, we have

$$g'(x) = \phi(x)^{1/p} f(x). \tag{4.2}$$

Using (4.2), we have

$$\begin{aligned} &\int_0^\infty \phi(x)\left\{\frac{1}{\Phi(x)}\int_0^x \phi(t)f(t)\,dt\right\}^p dx \\ &\quad = \int_0^\infty \phi(x)\left\{\frac{1}{\Phi(x)}\int_0^x \phi(t)^{1/p'} g'(t)\,dt\right\}^p dx. \end{aligned} \tag{4.3}$$

On integrating by parts we obtain

$$\int_0^x \phi(t)^{1/p'} g'(t)\,dt = g(x)\phi(x)^{1/p'} - \frac{1}{p'}\int_0^x \phi(t)^{-1/p}\phi'(t)g(t)\,dt. \tag{4.4}$$

By (4.3), (4.4) and Minkowski's inequality,

$$(4.5)\quad \left[\int_0^\infty \phi(x)\left\{\frac{1}{\Phi(x)}\int_0^x \phi(t)f(t)\,dt\right\}^p dx\right]^{1/p}$$
$$\leq \left\{\int_0^\infty \phi(x)\left\{\frac{g(x)\phi(x)^{1/p'}}{\Phi(x)}\right\}dx\right\}^{1/p}$$
$$+\frac{1}{p'}\left[\int_0^\infty \phi(x)\left\{\frac{1}{\Phi(x)}\int_0^x \phi(t)^{-1/p}|\phi'(t)|g(t)\,dt\right\}^p dx\right]^{1/p}.$$
$$:= I_1 + I_2.$$

Now, observe that by (2.2) we can conclude that

$$(4.6)\qquad I_1^p = \int_0^\infty \phi(x)\left\{\frac{g(x)}{\Phi(x)}\phi(x)^{1/p'}\right\}^p dx \leq B^p\int_0^\infty\left(\frac{g(x)}{x}\right)^p dx.$$

Thus, it is enough to prove that

$$(4.7)\qquad I_2^p \leq A^p\int_0^\infty\left(\frac{g(x)}{x}\right)^p dx.$$

For the sake of brevity, let us write

$$(4.8)\qquad h(t) = g(t)\phi(t)^{-1/p}|\phi'(t)|$$

and

$$(4.9)\qquad H(t) = \int_0^t h(u)\,du.$$

Hence, $I_2^p = (p')^{-p}\int_0^\infty \phi(x)\{\frac{1}{\Phi(x)}\int_0^x h(t)\,dt\}^p\,dx$.

By the lemma,

$$\left(\int_0^x h(t)\,dt\right)^p = p\int_0^x h(t)\left(\int_0^t h(u)\,du\right)^{p-1}dt$$

and consequently,

$$(4.10)\qquad \int_0^\infty \phi(x)(\Phi(x))^{-p}H(x)^p\,dx$$
$$= p\int_0^\infty \phi(x)(\Phi(x))^{-p}\int_0^x h(t)H(t)^{p-1}\,dt\,dx$$
$$= p\int_0^\infty h(t)H(t)^{p-1}\int_t^\infty \frac{\phi(x)}{\Phi(x)^p}\,dx\,dt$$
$$\leq \frac{p}{p-1}\int_0^\infty h(t)\left\{\frac{H(t)}{\Phi(t)}\right\}dt.$$

As explanation of the last step, we note that $\Phi(x)$ is positive and increasing, so $\Phi(x)^{1-p}$ is positive and decreasing. Thus $\Phi(x)^{1-p}$ tends to a limit $I \geq 0$ as $x \to \infty$, and so

$$(4.11)\quad \int_t^\infty \Phi(x)^{-p}\phi(x)\,dx = (p-1)^{-1}\left\{\Phi(t)^{1-p} - I\right\} \leq (p-1)^{-1}\Phi(t)^{1-p}.$$

Next, writing $h(t) = \left(h(t)\phi(t)^{-1/p'}\right)\phi(t)^{1/p'}$, and using Hölder's inequality, we get

$$(4.12)\quad \int_0^\infty h(t)\left\{\frac{H(t)}{\Phi(t)}\right\}^{p-1} dt \\ \leq \left[\int_0^\infty \left\{\phi(t)^{-1/p'}h(t)\right\}^p dt\right]^{1/p}\left[\int_0^\infty \phi(t)\left\{\frac{H(t)}{\Phi(t)}\right\}^p dt\right]^{1/p'}$$

By (4.8) and (2.2),

$$\phi(t)^{-1/p'}h(t) = g(t)(\phi(t))^{-1}|\phi'(t)| \leq Ag(t)/t.$$

We can obtain from (4.10), (4.11) and (4.12)

$$(4.13)\quad \int_0^\infty \phi(x)\left\{\Phi(x)^{-1}H(x)\right\}^p dx \\ \leq A\left[\int_0^\infty \left(\frac{g(t)}{t}\right)^p dt\right]^{1/p}\left[\int_0^\infty \phi(t)\left\{(\Phi(t))^{-1}H(t)\right\}^p dt\right]^{1/p'}$$

It should be remarked that, by (2.2) and (2.3), the expression in the second square bracket is not infinite. However, if the expression is zero, $H(x)$ is zero for almost all $x > 0$, and hence so is $h(x)$ by (4.9). By (4.5), then $I_2 = 0$; so (4.7) is satisfied immediately and there is nothing further to be proved. If the expression referred to is nonzero and finite, we derived both sides of (4.13) by the expression in the second square bracket to get (4.7).

This completes the proof of Theorem 1.

PROOF OF THEOREM 2. Set

$$(4.14)\quad V(x) = \int_x^\infty t^{-1}\phi(t)^{1/p}f(t)\,dt.$$

Hence

$$\frac{dV}{dx} = -x^{-1}f(x)\phi(x)^{1/p}$$

and consequently

$$(4.15)\quad f(x) = -x\phi(x)^{-1/p}V'(x).$$

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For $m > x > 0$,

$$\begin{aligned}\int_x^m \frac{f(t)\phi(t)}{\Phi(t)}\,dt &= -\int_x^m \frac{t\phi(t)^{1/p'}}{\Phi(t)} V'(t)\,dt \\ &= -\frac{m\phi(m)^{1/p'}}{\Phi(m)} V(m) + \frac{x\phi(x)^{1/p'}}{\Phi(x)} V(x) + \int_x^m V(t) \frac{d}{dt}\left(\frac{t\phi(t)^{1/p}}{\Phi(t)}\right) dt \\ &\leq \frac{x\phi(x)^{1/p'}}{\Phi(x)} V(x) + \int_x^m V(t) \left|\frac{d}{dt}\left(\frac{t\phi(t)^{1/p'}}{\Phi(t)}\right)\right| dt.\end{aligned}$$

On making $m \to \infty$, we get

$$\int_x^\infty \frac{f(t)\phi(t)}{\Phi(t)}\,dt \leq \frac{x\phi(x)^{1/p'}}{\Phi(x)} V(x) + \int_x^\infty V(t) \left|\frac{d}{dt}\left(\frac{t\phi(t)^{1/p'}}{\Phi(t)}\right)\right| dt. \tag{4.16}$$

Now, by using (4.16) and Minkowski's inequality, we obtain

$$\left[\int_0^\infty \phi(x) \left\{\int_x^\infty \frac{\phi(t)f(t)}{\Phi(t)}\,dt\right\}^p dx\right]^{1/p} \leq E_1^{1/p} + E_2^{1/p}, \tag{4.17}$$

where

$$E_1 = \int_0^\infty \left\{x\phi(x)(\Phi(x))^{-1}V(x)\right\}^p dx$$

and

$$E_2 = \int_0^\infty \phi(x) \left\{\int_x^\infty \left|\frac{d}{dt}\left(\frac{t\phi(t)^{1/p'}}{\Phi(t)}\right)\right| V(t)\,dt\right\}^p dx.$$

Since (2.2) holds,

$$E_1 = \int_0^\infty \left(\frac{x\phi(x)}{\Phi(x)} V(x)\right)^p dx \leq B^p \int_0^\infty V(x)^p\,dx.$$

In order to handle E_2, observe that

$$\frac{d}{dt}\left(\frac{t\phi(t)^{1/p'}}{\Phi(t)}\right) = \frac{t\phi(t)^{1/p'}}{\Phi(t)} \left(\frac{1}{t} + \frac{\phi'(t)}{p'\phi(t)} - \frac{\phi(t)}{\Phi(t)}\right). \tag{4.18}$$

From (4.18) and (2.2),

$$\left|\frac{d}{dt}\left(\frac{t\phi(t)^{1/p'}}{\Phi(t)}\right)\right| \leq \frac{t\phi(t)^{1/p'}}{\Phi(t)} \left(\frac{1}{t} + \frac{A}{p't} + \frac{B}{t}\right) < C\frac{\phi(t)^{1/p'}}{\Phi(t)}, \tag{4.19}$$

where $C = 1 + A + B$ is actually independent of p. Thus

$$E_2 \leq C^p \int_0^\infty \phi(x) \left(\int_x^\infty V(s) \frac{\phi(x)^{1/p'}}{\Phi(s)}\,ds\right)^p dx := C^p I.$$

The hypothesis (2.7) shows that I is not infinite. If $I = 0$, then $E_2 = 0$ and consequently the inequality (2.8) follows from the inequality for E_1.

When $I \neq 0$, the proof of the theorem is completed by using Lemma (b), changing the order of integration and using Hölder's inequality, much as in the treatment of (4.10) of Theorem 1.

Finally, we shall get

$$E_2 \leq (Cp)^p \int_0^\infty V(s)^p \, ds. \tag{4.20}$$

On substituting the estimates for E_1 and E_2 in (4.17), we get

$$\left[\int_0^\infty \phi(x) \left\{\int_x^\infty \frac{\phi(t)f(t)}{\Phi(t)} \, dt\right\}^p dx\right]^{1/p} \leq (B + Cp) \left[\int_0^\infty V(x)^p \, dx\right]^{1/p}$$

where $C = 1 + A + B$.

This completes the proof of Theorem 2.

Remark. It is desirable to prove Theorem 1 and Theorem 2 without requiring the convergence of integrals in (2.3) and (2.7). It is more natural to assume the convergence of the integrals on the right side of (2.4) and (2.8). However, we have not succeeded in obtaining such proofs.

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References

[1] P. R. Beesack, 'On some integral inequalities of E. T. Copson', *General inequalities* II, pp. 151–159 (Birkhäuser, Basel, 1980).

[2] R. P. Boas, 'Some integral inequalities related to Hardy's inequality', *J. Analyse Math.* **23** (1970), 53–63.

[3] R. P. Boas and C. O. Imoru, 'Elementary convolution inequalities', *SIAM J. Math. Anal.* **6** (1973), 457–471.

[4] E. T. Copson, 'Some integral inequalities', *Proc. Roy. Soc. Edinburgh. Sect. A* **75** (1975/76), 157–164.

[5] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, (University Press, Cambridge, 1934).

[6] E. R. Love, 'Links between some generalizations of Hardy's integral inequality', *General inequalities* IV, pp. 47–57 (Proc 4th. Internat. Conf. on General Inequalities, Oberwolfach, Basel, 1984).

[7] E. R. Love, 'Generalization of Hardy's integral inequalities', *Proc. Roy. Soc. Edinburgh, Sect. A* **100** (1985), nos. 3–4, 237–262.
[8] R. N. Mohapatra and D. C. Russell, 'Integral inequalities related to Hardy's inequality', *Aequationes Math.* **28** (1985), 199–207.

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