

THE AVERAGE DISTANCE PROPERTY OF CLASSICAL BANACH SPACES II

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A Banach space X has the average distance property if there exists a unique real number r such that for each positive integer n and all x_1, \dots, x_n in the unit sphere of X there is some x in the unit sphere of X such that

$$\frac{1}{n} \sum_{k=1}^n \|x_k - x\| = r.$$

We show that l_p does not have the average distance property if $p > 2$. This completes the study of the average distance property for l_p spaces.

1. INTRODUCTION

The aim of this note is to finish the study of the average distance property of l_p and $L_p[0, 1]$ for $1 \leq p \leq \infty$ using and refining the method introduced in [1]. We start by giving a short review of that method. The reader is referred to [1] for further information and to the pointers to the literature therein.

A rendezvous number of a metric space (M, d) is a real number r with the property that for each positive integer n and $x_1, \dots, x_n \in M$ there exists $x \in M$ such that

$$\frac{1}{n} \sum_{k=1}^n d(x_k, x) = r.$$

We say that a (real or complex) Banach space X has the average distance property if its unit sphere has a unique rendezvous number. It is known that l_2 and $L_2[0, 1]$ have the average distance property ([4]) and that l_p and $L_p[0, 1]$ do not have the average distance property if $1 \leq p < 2$ and if $p \geq 3$, see [3] and [1], respectively. Here we prove the following result.

THEOREM 1. *For $p > 2$, l_p and $L_p[0, 1]$ do not have the average distance property.*

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In [1], using an improved Clarkson inequality, the study of the average distance property for l_p and L_p in the case $p > 2$ was reduced to the study of a scalar function as follows. For $n \in \mathbb{N}$, $p > 2$ and $x, y_1, \dots, y_n \in l_p$ or L_p such that $\|x\|^p = 1/n$ and $\sum_{i=1}^n \|y_i\|^p = 1$ define

$$(1) \quad \sigma_i := \frac{\|x - y_i\|^p}{\|x\|^p + \|y_i\|^p} \quad \text{and} \quad \alpha_i := \frac{\|x\|^p + \|y_i\|^p}{2}.$$

It follows that

$$\frac{1}{2n} \leq \alpha_i \leq \frac{n+1}{2n} \quad \text{and} \quad \sum_{i=1}^n \alpha_i = 1.$$

Let $u_i \in [-1, +1]$ be defined by the relation

$$(2) \quad \sigma_i = \frac{(1 - u_i)^p}{1 + |u_i|^p}.$$

and let

$$\varphi(x, y_1, \dots, y_n) := 2^{-n} \sum_{\varepsilon_1, \dots, \varepsilon_n = \pm 1} \left(\sum_{i=1}^n \alpha_i \frac{(1 + \varepsilon_i u_i)^p}{1 + |u_i|^p} \right)^{1/p}.$$

As pointed out in [1], in order to prove Theorem 1 for a fixed $p > 2$, it suffices to find n such that $\varphi > 1$ for $(u_1, \dots, u_n) \neq (0, \dots, 0)$.

Considering the case $u_i = 1$ for $i = 1, \dots, n$, $\alpha_i = 1/(2n)$ for $i = 1, \dots, n - 1$, and $\alpha_n = (n + 1)/(2n)$ yields that

$$\begin{aligned} & 2^{-n} \sum_{\varepsilon_1, \dots, \varepsilon_n = \pm 1} \left(\sum_{i=1}^n \alpha_i \frac{(1 + \varepsilon_i u_i)^p}{1 + |u_i|^p} \right)^{1/p} \\ &= 2^{1-2/p} 2^{-n} \sum_{\varepsilon_1, \dots, \varepsilon_n = \pm 1} \left(1 + \frac{1}{2n} \sum_{i=1}^{n-1} \varepsilon_i + \frac{n+1}{2n} \varepsilon_n \right)^{1/p} \\ &= 2^{-2/p} \sum_{\varepsilon_n = \pm 1} \left(2^{-n+1} \sum_{\varepsilon_1, \dots, \varepsilon_{n-1} = \pm 1} \left(1 + \frac{1}{2n} \sum_{i=1}^{n-1} \varepsilon_i + \frac{n+1}{2n} \varepsilon_n \right)^{1/p} \right) \\ &\leq 2^{-2/p} \sum_{\varepsilon_n = \pm 1} \left(1 + 2^{-n+1} \sum_{\varepsilon_1, \dots, \varepsilon_{n-1} = \pm 1} \frac{1}{2n} \sum_{i=1}^{n-1} \varepsilon_i + \frac{n+1}{2n} \varepsilon_n \right)^{1/p} \\ &= 2^{-2/p} \left(\left(\frac{3}{2} + \frac{1}{n} \right)^{1/p} + \left(\frac{1}{2} - \frac{1}{n} \right)^{1/p} \right) \\ &\leq 2^{-2/p} \left(\left(\frac{3}{2} \right)^{1/p} + \left(\frac{1}{2} \right)^{1/p} \right) = \frac{3^{1/p} + 1}{8^{1/p}} \end{aligned}$$

which is smaller than 1 for $p < 2.10528 \dots$

This shows that, in contrast to [1], we have to take into account the concrete definition of the u_i 's and α_i 's to be able to cover also the cases where p is close to 2. This will be done in Proposition 2.

The remaining part of the paper is the proof of Theorem 1, which follows from the upcoming Propositions 6 and 8.

2. THE RELATION OF α_i AND u_i

We begin by providing an auxiliary estimate.

LEMMA 1.

$$\frac{1 + u}{(1 + u^p)^{1/p}} \geq 1 + (2^{1-1/p} - 1)u$$

for $u \in [0, 1]$.

PROOF: Let $g(u) := (1 + u)/(1 + u^p)^{1/p}$. Note that

$$g'(u) = \frac{1 - u^{p-1}}{(1 + u^p)^{1+1/p}} \geq 0$$

while

$$g''(u) = -\frac{(p + 1)(1 - u^{p-1})u^{p-1} + (p - 1)(1 + u^p)u^{p-2}}{(1 + u^p)^{2+1/p}} \leq 0.$$

This means that g is a concave function on $[0, 1]$ and therefore $g(u) \geq g(0) + (g(1) - g(0))u$. This proves the assertion. □

PROPOSITION 2. If α_i and u_i are defined by (1) and (2), then

$$|u_i| \leq c_1 n^{-1/p} \alpha_i^{-1/p},$$

where $c_1 = \max(2^{1-1/p}, 1/(2 - 2^{1/p}))$.

PROOF: We split the proof into three cases.

FIRST CASE.

$$\alpha_i \leq \frac{1}{n}.$$

Since $c_1 \geq 1$, in this case

$$|u_i| \leq 1 \leq (n\alpha_i)^{-1/p} \leq c_1 n^{-1/p} \alpha_i^{-1/p}.$$

SECOND CASE.

$$\alpha_i > \frac{1}{n} \quad \text{and} \quad u_i \geq 0.$$

Then

$$\frac{\|y_i\|}{(2\alpha_i)^{1/p}} = \left(1 - \frac{1}{2\alpha_i n}\right)^{1/p} > \left(\frac{1}{2\alpha_i n}\right)^{1/p} = \frac{\|x\|}{(2\alpha_i)^{1/p}}$$

and it follows from the definition (2) of u_i that

$$\begin{aligned} (1 - u_i)^p &\geq \frac{(1 - u_i)^p}{1 + u_i^p} = \sigma_i = \frac{\|x - y_i\|^p}{\|x\|^p + \|y_i\|^p} = \left\| \frac{x}{(2\alpha_i)^{1/p}} - \frac{y_i}{(2\alpha_i)^{1/p}} \right\|^p \\ &\geq \left(\left(1 - \frac{1}{2\alpha_i n}\right)^{1/p} - \left(\frac{1}{2\alpha_i n}\right)^{1/p} \right)^p. \end{aligned}$$

Now, using the relations

$$1 - \frac{1}{2\alpha_i n} \leq \left(1 - \frac{1}{2\alpha_i n}\right)^{1/p} \quad \text{and} \quad \frac{1}{2\alpha_i n} \leq \left(\frac{1}{2\alpha_i n}\right)^{1/p}$$

which follow from $\alpha_i \geq 1/(2n)$, we obtain

$$u_i \leq 1 - \left(1 - \frac{1}{2\alpha_i n}\right)^{1/p} + \left(\frac{1}{2\alpha_i n}\right)^{1/p} \leq \frac{1}{2\alpha_i n} + \left(\frac{1}{2\alpha_i n}\right)^{1/p} \leq 2\left(\frac{1}{2\alpha_i n}\right)^{1/p}.$$

From this we get

$$|u_i| = u_i \leq 2^{1-1/p} n^{-1/p} \alpha_i^{-1/p}.$$

THIRD CASE.

$$\alpha_i > \frac{1}{n} \quad \text{and} \quad u_i < 0.$$

It follows from Lemma 1 for $u = -u_i$ that

$$1 - (2^{1-1/p} - 1)u_i \leq \sigma_i^{1/p} \leq \left(1 - \frac{1}{2\alpha_i n}\right)^{1/p} + \left(\frac{1}{2\alpha_i n}\right)^{1/p} \leq 1 + \left(\frac{1}{2\alpha_i n}\right)^{1/p}.$$

Finally in this case

$$|u_i| = -u_i \leq \frac{1}{2 - 2^{1/p}} n^{-1/p} \alpha_i^{-1/p}. \quad \square$$

With this proposition in hand, we can forget about the concrete nature of the α_i 's and u_i 's. All we have to show is that for given n and $\alpha_1, \dots, \alpha_n$ such that

$$\frac{1}{2n} \leq \alpha_i \leq \frac{n+1}{2n} \quad \text{and} \quad \sum_{i=1}^n \alpha_i = 1$$

the function

$$\varphi(u_1, \dots, u_n) := 2^{-n} \sum_{\varepsilon_1, \dots, \varepsilon_n = \pm 1} \left(\sum_{i=1}^n \alpha_i \frac{(1 + \varepsilon_i u_i)^p}{1 + |u_i|^p} \right)^{1/p}$$

is bigger than one as long as

$$(3) \quad |u_i| \leq c_1 n^{-1/p} \alpha_i^{-1/p}$$

and $(u_1, \dots, u_n) \neq (0, \dots, 0)$.

Since all relations on the u_i 's are symmetric and since the function φ is symmetric in u_i , we can henceforth assume that $u_i \geq 0$.

3. PROOF OF $\varphi > 1$, THE CASE OF MANY LARGE u_i 'S

COROLLARY 3.

$$\left(\sum_{i=1}^n (\alpha_i u_i)^2 \right)^{1/2} \leq c_1 n^{-1/p}.$$

PROOF: It follows from (3) that

$$\left(\sum_{i=1}^n (\alpha_i u_i)^2\right)^{1/2} \leq c_1 n^{-1/p} \left(\sum_{i=1}^n \alpha_i^{2-2/p}\right)^{1/2}.$$

Since $2 - 2/p > 1$ and $\alpha_i < 1$ we have

$$\sum_{i=1}^n \alpha_i^{2-2/p} \leq \sum_{i=1}^n \alpha_i = 1,$$

which proves the assertion. □

LEMMA 4. We have

$$(4) \quad v(u) := \frac{(1+u)^p + (1-u)^p}{2(1+u^p)} \geq 1 + c_2 u^p$$

and

$$(5) \quad w(u) := \frac{(1+u)^p - (1-u)^p}{2(1+u^p)} \leq c_3 u$$

for $u \in [0, 1]$, where $c_2 := 2^{p-2} - 1$ and $c_3 := p2^{p-1}$.

PROOF: To see (4), we let

$$g(u) := \frac{(1+u)^p + (1-u)^p - 2}{u^p}$$

and use the fact that $(1+u)^{p-1} + (1-u)^{p-1}$ is non-increasing for $p > 2$, to compute

$$g'(u) = \frac{p}{u^{p+1}} (2 - (1+u)^{p-1} - (1-u)^{p-1}) \leq 0.$$

Therefore $g(u) \geq g(1) = 2^p - 2$, which yields

$$(1+u)^p + (1-u)^p \geq 2 + (2^p - 2)u^p = 2(1+u^p) + (2^p - 4)u^p.$$

Division by $2(1+u^p)$ and $1+u^p \leq 2$ proves (4).

Since $2u/(1+u) \leq 1$, Bernoulli's inequality states

$$\frac{(1-u)^p}{(1+u)^p} = \left(1 - \frac{2u}{1+u}\right)^p \geq 1 - \frac{2pu}{1+u}.$$

It follows that

$$\frac{(1+u)^p - (1-u)^p}{1+u^p} = \frac{(1+u)^p}{1+u^p} \left(1 - \frac{(1-u)^p}{(1+u)^p}\right) \leq 2pu \frac{(1+u)^{p-1}}{1+u^p} \leq p2^p u$$

which proves (5). □

The following Lemma is known as a subgaussian tail estimate for Rademacher averages and is by now classical. A proof can be found for example, in [2, p. 90].

LEMMA 5. For a given vector $x = (\xi_1, \dots, \xi_n)$, let $\|x\|_2 := \left(\sum_{i=1}^n |\xi_i|^2\right)^{1/2}$ and

$\mathbb{B} := \left\{(\varepsilon_1, \dots, \varepsilon_n) : \sum_{i=1}^n \varepsilon_i \xi_i > t\|x\|_2\right\}$, then

$$2^{-n}|\mathbb{B}| \leq e^{-t^2/2}.$$

We are now ready to tackle the case, where ‘many’ of the u_i ’s are bigger than $1/2$.

PROPOSITION 6. There exists n_1 such that for all $n > n_1$ we have

$$\varphi(u_1, \dots, u_n) > 1$$

if $|\mathbb{A}| > n/2$, where $\mathbb{A} := \{i : u_i > 1/2\}$.

PROOF: With v and w defined as in Lemma 4, observe that

$$v(u) + \varepsilon w(u) = \frac{(1 + \varepsilon u)^p}{1 + u^p}$$

for $\varepsilon = \pm 1$. Put

$$\mathbb{B} := \left\{(\varepsilon_1, \dots, \varepsilon_n) : -\sum_{i=1}^n \alpha_i \varepsilon_i w(u_i) \leq (2 \log n)^{1/2} c_3 c_1 n^{-1/p}\right\}.$$

Since by (5) and Corollary 3

$$\left(\sum_{i=1}^n (\alpha_i w(u_i))^2\right)^{1/2} \leq c_3 \left(\sum_{i=1}^n (\alpha_i u_i)^2\right)^{1/2} \leq c_3 c_1 n^{-1/p}$$

it follows from Lemma 5 that

$$2^{-n}|\mathbb{B}| \geq 1 - \frac{1}{n}.$$

With these preliminaries we can estimate φ as follows

$$\begin{aligned} \varphi(u_1, \dots, u_n) &\geq 2^{-n} \sum_{(\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{B}} \left(\sum_{i=1}^n \alpha_i v(u_i) + \sum_{i=1}^n \alpha_i \varepsilon_i w(u_i)\right)^{1/p} \\ &\geq \left(1 - \frac{1}{n}\right) \left(\sum_{i=1}^n \alpha_i v(u_i) - (2 \log n)^{1/2} c_3 c_1 n^{-1/p}\right)^{1/p}. \end{aligned}$$

From (4) and the assumption on \mathbb{A} it follows that

$$\sum_{i=1}^n \alpha_i v(u_i) \geq \sum_{i=1}^n \alpha_i + \sum_{i \in \mathbb{A}} \alpha_i c_2 u_i^p \geq 1 + \frac{n}{2} \frac{1}{2n} c_2 2^{-p} = 1 + c_4,$$

where $c_4 := c_2 2^{-p-2}$.

Since $c_4 > 0$, we can now choose n_1 so that for all $n > n_1$

$$(2 \log n)^{1/2} c_3 c_1 n^{-1/p} < \frac{c_4}{2} \quad \text{and} \quad \left(1 - \frac{1}{n}\right) \left(1 + \frac{c_4}{2}\right)^{1/p} > \left(1 + \frac{c_4}{4}\right)^{1/p}.$$

By these assumptions on n

$$\begin{aligned} \varphi(u_1, \dots, u_n) &\geq \left(1 - \frac{1}{n}\right) \left(1 + c_4 - (2 \log n)^{1/2} c_3 c_1 n^{-1/p}\right)^{1/p} \\ &\geq \left(1 - \frac{1}{n}\right) \left(1 + \frac{c_4}{2}\right)^{1/p} \\ &\geq \left(1 + \frac{c_4}{4}\right)^{1/p}. \end{aligned}$$

This proves the assertion. □

4. PROOF OF $\varphi > 1$, THE CASE OF FEW LARGE u_i 'S

From now on, we shall only deal with the case $|\mathbf{A}| \leq n/2$. So for the rest of this section, we assume that

$$(6) \quad |\mathbf{A}| \leq \frac{n}{2}, \quad \text{where } \mathbf{A} = \{i : u_i > 1/2\}.$$

LEMMA 7. Denote

$$(7) \quad f(u) := \frac{(1 - u^2)^p}{1 + u^p} \frac{(1 + u^{p-1})^{p/(p-1)} - (1 - u^{p-1})^{p/(p-1)}}{(1 + u)^p (1 - u^{p-1})^{p/(p-1)} - (1 - u)^p (1 + u^{p-1})^{p/(p-1)}}.$$

Then $\lim_{u \rightarrow 0} f(u) = \lim_{u \rightarrow 1} f(u) = 0$ and f is bounded on $[0, 1]$.

PROOF: Note that the derivative of the function $(1 \pm u)^p (1 \mp u^{p-1})^{p/(p-1)}$ is

$$\pm p(1 \pm u)^{p-1} (1 \mp u^{p-1})^{p/(p-1)} \mp p(1 \pm u)^p (1 \mp u^{p-1})^{1/(p-1)} u^{p-2}.$$

Since $p > 2$ we therefore have

$$\lim_{u \rightarrow 0} \frac{d}{du} (1 + u)^p (1 - u^{p-1})^{p/(p-1)} - \lim_{u \rightarrow 0} \frac{d}{du} (1 - u)^p (1 + u^{p-1})^{p/(p-1)} = 2p.$$

By l'Hopital's rule

$$\begin{aligned} \lim_{u \rightarrow 0} f(u) &= \lim_{u \rightarrow 0} \frac{(1 + u^{p-1})^{p/(p-1)} - (1 - u^{p-1})^{p/(p-1)}}{(1 + u)^p (1 - u^{p-1})^{p/(p-1)} - (1 - u)^p (1 + u^{p-1})^{p/(p-1)}} \\ &= \lim_{u \rightarrow 0} \frac{p(1 + u^{p-1})^{1/(p-1)} u^{p-2} + p(1 - u^{p-1})^{1/(p-1)} u^{p-2}}{\frac{d}{du} (1 + u)^p (1 - u^{p-1})^{p/(p-1)} - \frac{d}{du} (1 - u)^p (1 + u^{p-1})^{p/(p-1)}} \\ &= 0. \end{aligned}$$

On the other hand, again by l'Hopital's rule it follows that

$$\lim_{u \rightarrow 1} \frac{(1 - u^{p-1})^{1/(p-1)}}{1 - u} = \lim_{u \rightarrow 1} \frac{u^{p-2}}{(1 - u^{p-1})^{(p-2)/(p-1)}} = +\infty.$$

Therefore

$$\begin{aligned} \lim_{u \rightarrow 1} f(u) &= \lim_{u \rightarrow 1} \frac{2^{1/(p-1)}(1-u^2)^p}{(1+u)^p(1-u^{p-1})^{p/(p-1)} - (1-u)^p(1+u^{p-1})^{p/(p-1)}} \\ &= \lim_{u \rightarrow 1} \frac{2^{1/(p-1)}}{\frac{(1-u^{p-1})^{p/(p-1)}}{(1-u)^p} - \frac{(1+u^{p-1})^{p/(p-1)}}{(1+u)^p}} \\ &= 0. \end{aligned}$$

The boundedness of f on $[0, 1]$ now follows from its continuity in $(0, 1)$ and the boundedness of the limits of $f(u)$ for $u \rightarrow 0$ and $u \rightarrow 1$. □

We can now also treat the remaining case, where only ‘few’ of the u_i ’s are bigger than $1/2$. In this case, the next proposition shows that $\varphi(u_1, \dots, u_n) > \varphi(0, \dots, 0) = 1$, provided that n is big enough. This completes the proof of Theorem 1.

PROPOSITION 8. *There exists $n_2 \geq n_1$ such that for all $n > n_2$ we have*

$$\frac{\partial \varphi}{\partial u_j}(u_1, \dots, u_n) > 0$$

for all $j = 1, \dots, n$ and all u_1, \dots, u_n satisfying (6).

PROOF: Note that

$$\frac{\partial \varphi}{\partial u_j}(u_1, \dots, u_n) = \frac{\alpha_j}{(1+u_j^p)^2} 2^{-n} \sum_{\varepsilon_1, \dots, \varepsilon_n = \pm 1} \varepsilon_j \frac{(1+\varepsilon_j u_j)^{p-1}(1-\varepsilon_j u_j^{p-1})}{\left(\sum_{i=1}^n \alpha_i \frac{(1+\varepsilon_i u_i)^p}{1+u_i^p}\right)^{1-1/p}}.$$

We shall show that for every $\varepsilon_1, \dots, \varepsilon_{j-1}, \varepsilon_{j+1}, \dots, \varepsilon_n$ the summand

$$\sum_{\varepsilon_j = \pm 1} \varepsilon_j \frac{(1+\varepsilon_j u_j)^{p-1}(1-\varepsilon_j u_j^{p-1})}{\left(\sum_{i=1}^n \alpha_i \frac{(1+\varepsilon_i u_i)^p}{1+u_i^p}\right)^{1-1/p}}$$

is positive.

To this end we denote

$$a_j(u_1, \dots, u_n) := \sum_{\substack{i=1 \\ i \neq j}}^n \alpha_i \frac{(1+\varepsilon_i u_i)^p}{1+u_i^p}$$

and show that

$$\frac{(1+u_j)^{p-1}(1-u_j^{p-1})}{\left(a_j(u_1, \dots, u_n) + \alpha_j \frac{(1+u_j)^p}{1+u_j^p}\right)^{1-1/p}} > \frac{(1-u_j)^{p-1}(1+u_j^{p-1})}{\left(a_j(u_1, \dots, u_n) + \alpha_j \frac{(1-u_j)^p}{1+u_j^p}\right)^{1-1/p}}.$$

Some manipulations show that this is equivalent to

$$a_j(u_1, \dots, u_n) > \alpha_j f(u_j),$$

where f is the function defined in (7) in Lemma 7.

Using (6), we see that

$$a_j(u_1, \dots, u_n) \geq \sum_{\substack{i \notin A \\ i \neq j}} \alpha_i \frac{(1 - u_i)^p}{1 + u_i^p} \geq \left(\frac{n}{2} - 1\right) \frac{1}{2n} \frac{2^{-p}}{1 + 2^{-p}} \geq \frac{1}{8} \frac{1}{1 + 2^p} = c_5,$$

if $n \geq 4$ and $c_5 := 1/(8 + 2^{p+3})$. It is hence enough to show that

$$(8) \quad c_5 > \alpha_j f(u_j).$$

Since $\lim_{u \rightarrow 0} f(u) = 0$ by Lemma 7, we can find $\delta > 0$ small enough such that

$$f(u) < c_5$$

for $u^p < \delta$. Since f is also bounded by Lemma 7, we can choose

$$n \geq n_2 := \max\left(\frac{c_1^p \|f\|_\infty}{c_5 \delta}, n_1, 4\right).$$

If $\alpha_j < c_5/\|f\|_\infty$ then obviously (8) holds.

If on the other hand $\alpha_j \geq c_5/\|f\|_\infty$ then

$$\alpha_j n > \frac{c_5}{\|f\|_\infty} \frac{c_1^p \|f\|_\infty}{c_5 \delta} = \frac{c_1^p}{\delta}$$

and by (3)

$$u_j^p \leq \frac{c_1^p}{n \alpha_j} < \delta.$$

Consequently

$$\alpha_j f(u_j) < \alpha_j c_5 \leq c_5,$$

since $\alpha_j \leq 1$.

This proves the assertion. □

REMARK. Using the methods developed in Sections 3 and 4, it can be shown that without Relation (3) one can prove the result of the main theorem for all $p > p_0$, where

$$p_0 := \inf\{p > 2 : g \geq 2^{(1+1/p)}\} = 2.2751 \dots$$

and

$$g(u) := \left(1 + \frac{(1 + u)^p}{1 + u^p}\right)^{1/p} + \left(1 + \frac{(1 - u)^p}{1 + u^p}\right)^{1/p}.$$

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