DIRECT PRODUCTS AND THE HOPF PROPERTY

Dedicated to the memory of Hanna Neumann

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1. Introduction

In [1] Neumann and Dey prove that the free product of two finitely generated Hopf groups is Hopf and ask whether a similar result holds for direct products. It is the purpose of this paper to show that this is not the case. We prove

THEOREM A. *There exists a finitely generated group G satisfying the following conditions*:

(i) *G is isomorphic to a proper direct factor of itself*;
(ii) *G is the direct product of two Hopf groups.*

The method of proof can easily be adapted to give the following result:

THEOREM B. *There exists a non-trivial finitely generated group G_0 isomorphic to its own direct square.*

Theorem B provides an answer to a question raised by J. Wiegold and K. W. Gruenberg in lectures, for G_0 is a finitely generated group having the property that there is a bound on the number of generators of the direct power G_0^n as n varies through the natural numbers.

The construction of G proceeds as follows: We first select a suitable member S of the family of simple groups constructed by Camm in [2], and form the cartesian power $C = S^N$ where N denotes the natural numbers. It is then possible to choose a finitely generated subgroup G of C in such a way that conditions (i) and (ii) of Theorem A are satisfied. The proof of (ii) is very long, so we merely sketch the method of proof here, referring the reader to [3] for full details.

The construction of G_0 is very similar to that of G, except that, in this case we use an HNN extension of one of Camm's groups as the 'base group' S_0.

It should be noted, however, that it is possible to use various other groups in place of S_0: in particular, the description of the 'base group' may be considerably simplified at the expense of increasing the number of generators of G_0. The possibility of such a simplification was pointed out to me by Miller, and his example is to be found in [3].

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2. The proof of Theorem A

We begin with a brief description of Camm's simple groups. Let $D = \langle a, p\rangle$ and $E = \langle b, q\rangle$ be two two-generator free groups. Let I be the set $\{\pm 1, \pm 2, \pm 3, \cdots\}$ and let ρ, σ and τ be permutations of I. We put $g_i = a^i p^{\rho(i)}$ and $h_i = b^i q^{\sigma(i)}$ for each $i \in I$. The elements g_i then freely generate a subgroup U of D, and the elements h_i freely generate an isomorphic subgroup V of E. We form the free product $S_{\rho,\sigma,\tau}$ of D and E amalgamating U with V under the isomorphism ϕ defined by

$$\phi(g_i) = h_{\tau(i)} \qquad \text{for each } i \in I.$$

In [2] Camm shows that for suitable choices of ρ, σ and τ the groups $S_{\rho,\sigma,\tau}$ are all simple. She does not define ρ, σ and τ completely. The definitions she stipulates all map positive integers to positive integers, and other definitions can be made arbitrarily, subject to only one condition, which states essentially that $|\rho(n)|$, $|\sigma(n)|$ and $|\tau(n)|$ are 'not much bigger' than $|n|$. It is easy to satisfy this condition by defining ρ, σ and τ on the negative integers in such a way that each of them fixes each block $\{-20r, -20r+1, -20r+2, \cdots, -20r+19\}$ $(r \in N)$ set-wise.

We now construct our 'base group' S. This is to be a group $S_{\rho,\sigma,\tau}$ with ρ, σ and τ defined in the following way:

(I) ρ, σ and τ are defined as in [2] wherever these definitions are made. In particular

$$\rho(1) = \sigma(1) = \tau(1) = 1.$$

(II) The definitions of ρ, σ and τ on the positive integers are not completely determined by (I). We make further definitions as follows:

$$\rho(11) = \sigma(11) = \tau(11) = 11$$

$$\rho(12) = \sigma(12) = \tau(12) = 12$$

$$\rho(13) = \tau(13) = 13, \ \sigma(13) = 14.$$

All other definitions on the positive integers are made arbitrarily, subject to the condition stipulated in [2].

(III) The following definitions are made on the negative integers:

Let $$P_1 = \{2^r\colon r > 0,\ r \in Z\},$$
$$P_2 = N \backslash P_1, \qquad \text{and let}$$
$$\Delta_t = \{n \in Z\colon -20t \leqq n \leqq -20t + 19\}.$$

For each $t \in P_2$, ρ, σ and τ fix Δ_t point-wise. For each $t \in P_1$, ρ, σ and τ fix Δ_t set-wise, and induce on Δ_t permutations ρ_t, σ_t and τ_t defined (in the usual notation, as products of disjoint cycles) by

$$\rho_t = (-20t + 9, -20t + 10),$$
$$\sigma_t = (-20t, -20t + 7, -20t + 4, -20t + 3, -20t + 1) \cdot (-20t + 2, -20t + 5, -20t + 6),$$
$$\tau_t = (-20t + 2, -20t + 3).$$

Let C be the cartesian power S^N, and let D be the subgroup of C consisting of all functions of finite support. An element f of C will be written as an infinite vector $(f_1, f_2, f_3, \cdots)$, where the ith entry f_i denotes the value of f at i. Then G is the subgroup of C generated by

$$x = (b^{-1}a, b^{-1}a, b^{-1}a, \cdots)$$
$$y = (p, p, p, \cdots)$$
$$z = (p^{20}ap^{-19}, p^{40}ap^{-39}, p^{60}ap^{-59}, \cdots).$$

We define S_i to be the subgroup $\{f \in C\colon f(j) = 1 \text{ for all } j \neq i\}$ of C, so that S_i is isomorphic to S for each $i \in N$.

Proposition 1. $G \cong S \times G$.∎

Proof. We first prove that G intersects S_1 non-trivially. Thus we wish to find some element $w \neq 1$ of S such that

$$(w, 1, 1, 1, \cdots, 1, \cdots) \in G.$$

We prove that the element

$$w = (y^{-31} zy^{31})^{-1} x(y^{-31}zy^{31})x^{-1}$$

of G has this form.

The value of w at $t \in N$ is

$$(p^{20t-31}ap^{-20t+32})^{-1}(b^{-1}a)(p^{20t-31}ap^{-20t+32})(b^{-1}a)^{-1}$$
$$= (p^{20(t-1)-11}ap^{-20(t-1)+12})^{-1}(b^{-1}a)(p^{20(t-1)-11}ap^{-20(t-1)+12}) \cdot \cdot (b^{-1}a)^{-1}.$$

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Now by definitions (III), ρ, σ and τ each fix $-20t+11$, $-20t+12$ and $-20t+13$ for each $t>0$. Also, by definitions (I), ρ, σ and τ each fix 1, so for $t>1$,

$$
\begin{aligned}
&(b^{-1}a)(p^{20(t-1)-11}ap^{-20(t-1)+12})(b^{-1}a)^{-1}\\
&\quad = b^{-1}(ap)(a^{-20(t-1)+12}p^{-20(t-1)+12})^{-1}(a^{-20(t-1)+13}p^{-20(t-1)+13})(p^{-1}a^{-1})b\\
&\quad = q(b^{-20(t-1)+12}q^{-20(t-1)+12})^{-1}(b^{-20(t-1)+13}q^{-20(t-1)+13})q^{-1}\\
&\quad = q^{20(t-1)-11}bq^{-20(t-1)+12}\\
&\quad = (b^{-20(t-1)+11}q^{-20(t-1)+11})^{-1}(b^{-20(t-1)+12}q^{-20(t-1)+12})\\
&\quad = (a^{-20(t-1)+11}p^{-20(t-1)+11})^{-1}(a^{-20(t-1)+12}p^{-20(t-1)+12})\\
&\quad = p^{20(t-1)-11}ap^{-20(t-1)+12}.
\end{aligned}
$$

Thus the value of w at t is 1 for all $t>1$. But the value of w at 1 is

$$(p^{-11}ap^{12})^{-1}(b^{-1}a)(p^{-11}ap^{12})(b^{-1}a)^{-1},$$

and, by definitions (II),

$$
\begin{aligned}
&(b^{-1}a)(p^{-11}ap^{12})(b^{-1}a)^{-1} = b^{-1}(ap)(a^{12}p^{12})^{-1}(a^{13}p^{13})(p^{-1}a^{-1})b\\
&\quad = q(b^{12}q^{12})^{-1}(b^{13}q^{14})q^{-1}\\
&\quad = q^{-11}bq^{13} = (b^{11}q^{11})^{-1}(b^{12}q^{12})q = (a^{11}p^{11})^{-1}(a^{12}p^{12})q = p^{-11}ap^{12}q.
\end{aligned}
$$

Thus the value of w at 1 is q, and consequently G intersects S_1. But S is simple, and the set $\{f(1): f \in G\}$ is the whole of S. Thus $G \cap S_1 = S_1$, showing that S_1 is a direct factor of G. Let $\alpha: C \to C$ be the monomorphism given by

$$(f\alpha)(i) = \begin{cases} f(i-1) & \text{for } i>1\\ 1 & \text{for } i=1, \end{cases}$$

and let $G_1 = G\alpha$. Then G_1 is generated by

$$
\begin{aligned}
x_1 &= (1, b^{-1}a, b^{-1}a, b^{-1}a, \cdots)\\
y_1 &= (1, p, p, p, \cdots)\\
z_1 &= (1, p^{20}ap^{-19}, p^{40}ap^{-39}, p^{60}ap^{-59}, \cdots)
\end{aligned}
$$

and is clearly a direct complement for S_1 in G. Moreover $\alpha|_G$ is an isomorphism from G onto G_1. Consequently $G \cong S \times G$, as required.

Let C_1 and C_2 denote the cartesian powers S^{P_1} and S^{P_2}, regarded in the obvious way as subgroups of C. For each $f \in G$ we define functions $f_A, f_B \in C$ as follows:

$$f_A(i) = \begin{cases} f(i) & \text{if } i \in P_1\\ 1 & \text{otherwise} \end{cases}$$

$$f_B(i) = \begin{cases} f(i) & \text{if } i \in P_2 \\ 1 & \text{otherwise.} \end{cases}$$

Let A and B denote the subgroups $\{f_A : f \in G\}$ and $\{f_B : f \in G\}$ of C respectively. In the usual notation we denote a commutator $a^{-1}b^{-1}ab$ by $[a, b]$.

PROPOSITION 2. $G = A \times B$.

PROOF. We need only show that $A \leqq G$. Using definitions (III), we see that the following relations hold in S:

$$(b^{-1}a)(p^{20t}ap^{-20t+1})(a^{-1}b) = p^{20t}ap^{-20t+1} \qquad \text{if } t \in P_2, \text{ and}$$

$$(b^{-1}a)(p^{20t}ap^{-20t+1})(a^{-1}b) = p^{20t+1}a^2p^{-20t+1} \qquad \text{if } t \in P_1.$$

Consequently

$$\begin{aligned}[x^{-1}, z^{-1}] &= (p^{21}ap^{-20}, p^{41}ap^{-40}, 1, p^{81}ap^{-80}, 1, 1, \cdots) \\ &= yz_Ay^{-1} \qquad \text{(where } z_A \text{ is defined as above),}\end{aligned}$$

which shows that z_A and all its conjugates by powers of y lie in G. But we also have

$$(b^{-1}a)(p^{20t-3}ap^{-20t+4})(a^{-1}b) = p^{20t-6}ap^{-20t+7}q \qquad \text{for all } t \in P_1,$$

and so

$$x(y^{-3}z_Ay^3)x^{-1} = (y^{-6}z_Ay^6)(x_Ay_A),$$

which shows that x_Ay_A lies in G. Further

$$(b^{-1}a)(p^{20t-10}ap^{-20t+11})(a^{-1}b) = p^{20t-9}ap^{-20t+11} \qquad \text{for all } t \in P_1,$$

and so

$$x(y^{-10}z_Ay^{10})x^{-1} = (y^{-9}z_Ay^9)y_A,$$

which shows that y_A lies in G. Thus $A = \langle x_A, y_A, z_A \rangle \leqq G$, and the result follows.

By Proposition 1, G is non-Hopf, and by Proposition 2, $G = A \times B$. Thus we need only show that A and B are Hopf. We give sketches of the proofs, referring the reader to [3] for details.

PROPOSITION 3. *A is Hopf.*

PROOF. We must prove that every ependomorphism of A is an automorphism. To this end, we factor out a characteristic subgroup of A, and look at the ependomorphisms of the corresponding factor group.

Let D_1 be the subgroup of C_1 consisting of all functions of finite support. An easy extension of the proof of Proposition 1 shows that $D_1 \leqq A$. In fact D_1 is the direct product of the minimal normal subgroups of A, for it is easily seen that any normal subgroup N of A intersects all those S_i $(i \in P_1)$ for which there is some $n \in N$ such that $n(i) \neq 1$. Consequently, for any ependomorphism ϕ of A we have $\phi(D_1) \leqq D_1$. We investigate the ependomorphisms of the factor group $\bar{A} = A/D_1$, denoting the images of x_A, y_A and z_A in $\bar{A}$ by x, y and z. We put $xy = t$, $y^3zy^{-3} = u$, and prove

LEMMA 3.1. *$L = \langle y, u\rangle$ and $M = \langle t, u\rangle$ are both two-generator free groups. Let*

$$H = \langle y^{-k}uy^k (k \in Z, k \neq 11, 12, 13)\ y^{-11}uy^{12}, y^{-12}uy^{13}, y^{-13}uy^{11}\rangle \leqq L,$$

$$K = \langle t^{-k}ut^k (k \in Z, k \notin \{2, 3, 4, \cdots, 10\}),\ t^{-2}ut^9, t^{-3}ut^7, t^{-4}ut^5,$$
$$t^{-5}ut^6, t^{-6}ut^8, t^{-7}ut^{10}, t^{-8}ut^3, t^{-9}ut^4, t^{-10}ut^2\rangle \leqq M,$$

and let θ be the map from H to K defined by

$$\theta(y^{-k}uy^k) = t^{-k}ut^k \qquad \text{for all } k \geqq 14, \text{ and for all } k \leqq 1,$$
$$\theta(y^{-2}uy^2) = t^{-2}ut^9, \quad \theta(y^{-3}uy^3) = t^{-10}ut^2, \quad \theta(y^{-4}uy^4) = t^{-3}ut^{-1}ut^3,$$
$$\theta(y^{-5}uy^5) = t^{-3}u^{-1}t^8, \quad \theta(y^{-6}uy^6) = t^{-8}ut^{-1}ut^5, \quad \theta(y^{-7}uy^7) = t^{-6}ut^8,$$
$$\theta(y^{-8}uy^8) = t^{-9}ut^4, \quad \theta(y^{-9}uy^9) = t^{-5}ut^6, \quad \theta(y^{-10}uy^{10}) = t^{-7}ut^{10},$$
$$\theta(y^{-11}uy^{12}) = t^{-11}ut^{11}, \quad \theta(y^{-12}uy^{13}) = t^{-13}ut^{13}, \quad \theta(y^{-13}uy^{11}) = t^{-12}ut^{12}.$$

Then θ is an isomorphism, and $\bar{A} = *(L, M; H, K, \theta)$, the free product of L and M amalgamating H with K under the isomorphism θ.

PROOF. A relation $w(x, y, z) = 1$ holds in $\bar{A}$ if and only if the corresponding word $w_A = w(x_A, y_A, z_A)$ of A lies in D_1, that is, if and only if the value $w_A(i) = w(b^{-1}a, p, p^{20i}ap^{-20i+1})$ of w_A at i is trivial for all but finitely many integers $i \in P_1$. Thus we may find relations in $\bar{A}$ by making calculations within S.

As the subgroups $\langle p, p^{20i+3}ap^{-20i-2}\rangle$ and $\langle q, q^{20i+3}bq^{-20i-2}\rangle$ of S are free for each $i \in P_1$, the above reasoning shows that the subgroups $\langle y, u\rangle$ and $\langle t, u\rangle$ of $\bar{A}$ are free. It follows from cancellation arguments in the free groups L and M that H and K are free on the stated generators, so that θ is indeed an isomorphism. Moreover the elements $\{y^i : i \in Z\}$ and $\{t^i : i \in Z\}$ form complete sets of coset representatives for H in L and K in M respectively. (An argument similar to that required here may be found in [2], Lemma 3.) Using definitions (III), we may check that the relations determined by θ all hold in $\bar{A}$, so that $\bar{A}$ is a homomorphic image of $R = *(L, M; H, K, \theta)$.

If $\bar{A}$ were a proper homomorphic image of R, there would be an element w of normal form length greater than one in R, such that w has trivial image in $\bar{A}$. Let w be a word of length n in R, say $w = hs_1s_2 \cdots s_n$ where $h \in H$ and for each index i the factor s_i is a power of y or of t, taken alternately. We note that in S the powers of p form a transversal for U in D and the powers of q form a transversal for V in E. We note further that for each generator g of H the value of g_A

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at i lies in the amalgamated subgroup U of S for all but finitely many $i \in P_1$. Now h is a word involving only finitely many of the generators of H, so it follows that the value of h_A at i lies in the amalgamated subgroup U of S for all but finitely many $i \in P_1$. Moreover the values of y_A and t_A at i are p and q respectively. Consequently, for all but finitely many $i \in P_1$, the value of w_A at i is reduced as written and has normal form length n in S, so that the image of w in $\bar{A}$ is not equal to 1. Thus $\bar{A} = R$, as required.

Having determined the structure of $\bar{A}$, we are now in a position to find its ependomorphisms. We shall need the following two theorems, both to be found in [4] (Theorem 4.5, p. 209 and Corollary N4, p. 169 respectively).

THEOREM 1. *Let $G = *(A, B; H, K, \phi)$, and suppose that $x, y \in G$ are such that $xy = yx$. Then either*

(i) *x or y is in a conjugate of H, or*

(ii) *if neither x nor y is in a conjugate of H, but x is in a conjugate of a factor, then y is in that same conjugate of a factor, or*

(iii) *if neither x nor y is in a conjugate of a factor, then $x = ghg^{-1} \cdot W^j$ and $y = gh'g^{-1} \cdot W^k$, where $g, W \in G$, $h, h' \in H$ and ghg^{-1}, $gh'g^{-1}$ and W commute in pairs.*

THEOREM 2. *Let $Aut(G)$ be the automorphism group of a group G, and let $Inn(G)$ be the normal subgroup of $Aut(G)$ consisting of inner automorphisms. Let F_2 denote a free group of rank two and A_2 a free abelian group of rank two. Then*

$$\frac{Aut(F_2)}{Inn(F_2)} \cong Aut(A_2).$$

Moreover $Aut(A_2)$ is isomorphic to $GL(2, Z)$, that is, the multiplicative group of 2×2 matrices with integer entries, and determinant ± 1.

We now sketch a proof of

LEMMA 3.2. *Every ependomorphism of $\bar{A}$ is an inner automorphism.*

PROOF. Let ϕ be an ependomorphism of $\bar{A}$. Using the relations given in Lemma 3.1, the words $x(y^i z y^{-i})x^{-1}$ of $\bar{A}$ may be evaluated for each $i \in Z$. We find that x commutes with $y^i z y^{-i}$ if and only if $i \geqq 3$ or $i \leqq -11$. Thus $\phi(x)$ commutes with $\phi(y^i z y^{-i})$ for these values of i, and we may apply Theorem 1 to establish the normal form structure of the images under ϕ of elements of $\bar{A}$. Without loss of generality we may assume that $\phi(x)$ is cyclically reduced. Suppose first that $\phi(x)$ lies in a factor, but not in a conjugate of H. By Theorem 1, $\phi(y^3 z y^{-3})$ and $\phi(y^4 z y^{-4})$ both lie in that same factor or in a conjugate of H. A length argument shows that even in the latter case they must both lie in the same factor as $\phi(x)$. Now the factors are free, and so it follows that $\phi(x)$, $\phi(y^3 z y^{-3})$ and $\phi(y^4 z y^{-4})$ are all powers of the same element. Thus, in particular,

$$\phi([y^3zy^{-3}, y^4zy^{-4}]) = 1.$$

We may conjugate this relation by powers of $\phi(y)$ and by $\phi(x)$. Using the fact that $\phi(x)$ commutes with $\phi(y^izy^{-i})$ for all $i \geqq 3$ or $\leqq -11$, and using the relation

$$x(y^{-10}zy^{10})x^{-1} = y^{-9}zy^{10}$$

which holds in $\bar{A}$, we may now deduce that $\phi(y) = 1$. Having established this, it is not difficult to deduce from the relations of $\bar{A}$ that $\phi(z) = 1$ also, and consequently that ϕ is not an ependomorphism. A similar argument deals with the case where $\phi(y^3zy^{-3})$ lies in a conjugate of a factor, but not in a conjugate of H.

We next consider the case when both $\phi(x)$ and $\phi(y^3zy^{-3})$ lie in (possibly different) conjugates of H. This case can also be eliminated. The argument is basically the same as that used in the above case, but the proof is longer and the normal form theorem, with associated length and cancellation arguments, is used more strongly. We refer the reader to [3] for details.

Next we assume that $\phi(y^3zy^{-3})$ lies in a conjugate of H. Without loss of generality we may assume that $\phi(x)$ is cyclically reduced. The above results show that $\phi(x)$ cannot lie in a factor, so $\phi(x)$ has normal form length greater than one. An easy length argument then shows that $\phi(y^izy^{-i})$ lies in H for all $i \geqq 3$ or $\leqq -11$. The next step is to establish the structure of $\phi(y)$. It is easy to eliminate the case in which $\phi(y) \in H$: if this were true, we would have $\phi(y^izy^{-i}) \in H$ for each $i \in Z$, and then applying ϕ to the relation

$$x(y^{-3}zy^3)x^{-1} = y^{-6}zy^6(xy)$$

of $\bar{A}$ would yield an equation in which the right-hand side is cyclically reduced of length greater than one, but in which the left-hand side is not cyclically reduced unless it lies in a factor. A similar argument eliminates the case where $\phi(y)$ does not lie in a conjugate of a factor, and yet another argument of the same nature shows that we may assume that $\phi(y)$ lies within a factor. (A conjugation may be necessary to establish this, but it does not affect the fact that $\phi(x)$ is cyclically reduced.)

The proof continues in a similar manner (see [3]) until we obtain the following situation:

There exists an element $h \in H$ and elements c and d belonging to different factors $F(c)$, $F(d)$ of $\bar{A}$ such that

$$\begin{aligned} \phi(y^3zy^{-3}) &= h \\ \phi(y) &= c \\ \phi(x) &= dc^{-1} \end{aligned}$$

and such that h and c together generate $F(c)$ and h and d together generate $F(d)$.

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We consider first the case in which $F(c) = L$ and $F(d) = M$. Now h is equal to a word w in the generators of H. Since h and c generate L, h is a primitive element of the two-generator free group L, so we may apply Theorem 2. This shows that the sum of the exponents of y^3zy^{-3} and the sum of the exponents of y appearing in w must be coprime. Suppose that the total sum of the exponents of all generators $y^i(y^3zy^{-3})y^{-i}$ $(i \neq -11, -12, -13)$ appearing in w is α_0, and suppose that $y^{-11}(y^3zy^{-3})y^{12}$ appears to total exponent α_1, that $y^{-12}(y^3zy^{-3})y^{13}$ appears to total exponent α_2 and that $y^{-13}(y^3zy^{-3})y^{11}$ appears to total exponent α_3. Suppose further that $\alpha_1 + \alpha_2 - 2\alpha_3 \neq 0$. Now h and c generate L, and $c^{-i}hc^i \in H$ for all but finitely many $i \in Z$. It is not difficult to deduce from this information that $y^{-i}hy^i \in H$ for all but finitely many $i \in Z$. But this cannot be the case if $\alpha_1 + \alpha_2 - 2\alpha_3 \neq 0$. Thus $\alpha_1 + \alpha_2 - 2\alpha_3 = 0$, so applying Theorem 2, we see that $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = \pm 1$. We now use the full strength of Theorem 2, in the form

$$\frac{\operatorname{Aut}(F_2)}{\operatorname{Inn}(F_2)} \cong \operatorname{Aut}(A_2) \cong GL(2, Z).$$

Since L is a free group of rank two we may deduce that there exists an element e of L and integers r, $\varepsilon_1 = \pm 1$, and $\varepsilon_2 = \pm 1$ such that

$$\begin{aligned} \phi(y^3zy^{-3}) &= e(y^3zy^{-3})^{\varepsilon_1}e^{-1} \\ \phi(y) &= e(y^{\varepsilon_2}(y^3zy^{-3})^r)e^{-1}. \end{aligned}$$

But $e = h'y^j$ for some $h' \in H$ and some integer j. Thus, modulo an inner automorphism by h', we have

$$\begin{aligned} \phi(y^3zy^{-3}) &= (y^{3+j}zy^{-(3+j)})^{\varepsilon_1} \in H \\ \phi(y) &= y^{\varepsilon_2}(y^{3+j}zy^{-(3+j)})^{r}. \end{aligned}$$

Now $\phi(y^3zy^{-3})$ and $\phi(xy)$ together generate M, and so, by a further application of Theorem 2, we obtain

$$\phi(xy) = (y^{3+j}zy^{-(3+j)})^l(xy)^{\varepsilon_3}(y^{3+j}zy^{-(3+j)})^m$$

for some integers l, m and $\varepsilon_3 = \pm 1$.

Having found specific values for $\phi(x)$, $\phi(y)$ and $\phi(z)$, we may now check through the defining relations of $\bar{A}$ to see whether any of the above maps are, in fact, ependomorphisms of $\bar{A}$. We discover that the only case in which ϕ is an ependomorphism of $\bar{A}$ is that in which $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$ and $j = r = l = m = 0$. Consequently, modulo an inner automorphism, ϕ is the identity map. It follows that ϕ is an inner automorphism of $\bar{A}$. The case $F(c) = M$, $F(d) = L$ can be dealt with similarly, and eliminated.

The only remaining case is that in which $\phi(y^3zy^{-3})$ has cyclically reduced length greater than one. This case may also be eliminated. The proof is long, but very similar to that for the case in which $\phi(y^3zy^{-3})$ lies in a conjugate of H. We refer the reader to [3] for details. This completes the proof of Lemma 3.2.

We are now in a position to complete the proof of Proposition 3. Let ϕ be an ependomorphism of A. As noted at the beginning of the proof of Proposition 3, we have $\phi(D_1) \leqq D_1$ and so ϕ induces on $\bar{A}$ an ependomorphism, which, by Lemma 3.2, is an inner automorphism. Consequently there exist an element g of A, and elements ξ, η and ζ of D_1 such that

$$\phi(x_A) = gx_Ag^{-1}\xi$$
$$\phi(y_A) = gy_Ag^{-1}\eta$$
$$\phi(z_A) = gz_Ag^{-1}\zeta.$$

We choose an integer r such that ξ, η and ζ all lie in $T = \{f \in D_1 : f(i) = 1$ for all $i > 2^r\}$. It is now clear that for any $n > r$, we have $\phi(S_{2^n}) = S_{2^n}$, and that $\ker(\phi) \leqq T$. Further ϕ maps D_1 onto D_1 and T onto T. But T is a direct product of finitely many simple groups and, as such, is Hopf. Consequently $\ker(\phi)$ is trivial, so that ϕ is an automorphism of A. Thus A is Hopf, as required.

We now turn our attention to the group B. We recall the definition of x_B, y_B and z_B made just before the statement of Proposition 2.

PROPOSITION 4. *B is Hopf.*

PROOF. Let D_2 be the subgroup of C_2 consisting of all functions of finite support. Using the method of Proposition 3, we see that B contains D_2 and that D_2 is the direct product of the minimal normal subgroups of B. We put $\bar{B} = B/D_2$ and consider the structure of $\bar{B}$.

Let $\bar{C}$ be the cartesian power $\bar{C} = \bar{A}^N$, and let $\bar{D}$ be the subgroup of $\bar{C}$ consisting of all functions of finite support. Let T be the subgroup of $\bar{C}$ generated by

$$X = (x, x, x, \cdots)$$
$$Y = (y, y, y, \cdots)$$
$$Z = (y^{20}zy^{-20}, y^{40}zy^{-40}, y^{60}zy^{-60}, \cdots).$$

We prove

LEMMA 4.1. *T contains $\bar{D}$. Further the map ω from B to $\bar{C}$ defined by*

$$\omega(x_B) = X$$
$$\omega(y_B) = Y$$
$$\omega(z_B) = Z$$

is a homomorphism of B onto T with kernel D_2. (Consequently $\bar{B}$ is isomorphic to T, and may be identified with it under ω.)

PROOF. Consideration of the elements $[X^{-1},(Y^{-20t}ZY^{20t})^{-1}]$ $(t \in N)$ of T shows, by an argument similar to that used in the proof of Proposition 2, that $\bar{D} \leqq T$. Let ω be the map from B onto T defined above. Now a relation $R(x_B,y_B,z_B) = 1$ holds in B if and only if

$$R(x_B,y_B,z_B)(i) = 1$$

for each $i \in P_2$. Similarly a relation $S(X,Y,Z) = 1$ holds in T if and only if

$$S(X,Y,Z)(i) = 1$$

for each $i \in N$, that is if and only if the relation

$$S(x,y,y^{20i}zy^{-20i}) = 1$$

holds in $\bar{A}$ for each $i \in N$. Now $\bar{A} = A/D_1$, so that $S(x,y,y^{20i}zy^{-20i}) = 1$ in $\bar{A}$ if and only if the relation

$$S(x_A,y_A,y_A^{20i}\,z_Ay_A^{-20i})(t) = 1$$

holds in S for all but finitely many $t \in P_1$. With this in mind it is not difficult to show that ω is a homomorphism. The fact that the kernel of ω is equal to D_2 may be proved in a similar way. However, the argument is rather more complicated, and it is necessary in the course of the proof to use the normal form theorem in S. We refer the reader to [3] for details. This shows that ω is an isomorphism between $B/D_2 = \bar{B}$ and T, so we may identify $\bar{B}$ with T under ω. From now on, we shall make this identification without further comment. This completes the proof of Lemma 4.1.

We next consider $\bar{\bar{B}} = \bar{B}/\bar{D}$, and, by abuse of notation, we use x, y and z to denote the images of X, Y and Z in $\bar{\bar{B}}$.

LEMMA 4.2. *$E = \langle y, y^3zy^{-3}\rangle$ and $F = \langle xy, y^3zy^{-3}\rangle$ are both two-generator free groups. Let*

$$H = \langle y^i(y^3zy^{-3})y^{-i} \colon i \in Z\rangle \leqq E,$$

$$K = \langle (xy)^i(y^3zy^{-3})(xy)^{-i} \colon i \in Z\rangle \leqq F,$$

and let θ be the isomorphism between H and K defined by

$$\theta(y^i(y^3zy^{-3})y^{-i}) = (xy)^i(y^3zy^{-3})(xy)^{-i}$$

for each $i \in Z$. Then $\bar{\bar{B}} = *(E,F;H,K,\theta)$.

PROOF. A word $w = w(x,y,z)$ is equal to 1 in $\bar{\bar{B}}$ if and only if the corresponding word $w_B = w(X,Y,Z)$ of $\bar{B}$ lies in $\bar{D}$. Hence $w = 1$ in $\bar{\bar{B}}$ if and only if the value $w_B(i)$ of w_B at i is trivial for all but finitely many $i \in N$. The proof is now completed analogously with that of Lemma 3.1, using the normal form theorem in $\bar{A}$.

We wish to prove that every ependomorphism of B is an automorphism. Proceeding as in the proof of Proposition 3, we have shown that every ependomorphism of B induces one of $\bar{B}$. We must now show that every ependomorphism of $\bar{B}$ induces one of $\bar{\bar{B}}$. To this end we prove

LEMMA 4.3. *$\bar{\bar{B}}$ is residually finite.*

PROOF. We wish to prove that given any non-trivial element g of $\bar{\bar{B}}$ there is a normal subgroup N_g of finite index in $\bar{\bar{B}}$ such that $g \notin N_g$. Suppose first that $g \notin H = K$. Now H is normal in $\bar{\bar{B}}$ and $\bar{\bar{B}}/H$ is a free group of rank two, so the result follows, in this case, from the fact that free groups are residually finite. (See, e.g., [4], p. 144, problem 14.)

Now suppose that $g \in H$. Since E is free, we may find a normal subgroup M_1 of finite index in E such that $g \notin M_1$. Let M_2 be the image of M_1 under the map $\chi: E \to F$ given by

$$\chi(y^3zy^{-3}) = y^3zy^{-3}$$
$$\chi(y) = xy.$$

Since the restriction of χ to H is equal to the amalgamation map θ, it follows that $g \notin M_2$. If M is the normal closure in $\bar{\bar{B}}$ of the subgroup of $\bar{\bar{B}}$ generated by M_1 and M_2, then

$$\bar{\bar{B}}/M \cong *(E/M_1, F/M_2; HM_1/M_1, KM_2/M_2, \pi)$$

where π is the induced amalgamation map from HM_1/M_1 to KM_2/M_2. Thus $\bar{\bar{B}}/M$ is the generalised free product of two finite groups, and so there is a finite homomorphic image (the permutational product) of $\bar{\bar{B}}/M$ embedding E/M_1 and F/M_2. (See [5]). Since g has non-trivial image in E/M_1, the result follows.

LEMMA 4.4. *$\bar{D}$ is the intersection of all the normal subgroups of finite index in $\bar{B}$.*

PROOF. Let R be the intersection of the normal subgroups of finite index in $\bar{B}$. Since $\bar{B}/\bar{D}$ is residually finite, it follows that $R \leqq \bar{D}$. For each $i \in N$, we define $\bar{A}_i$ to be $\{f \in \bar{C}: f(j) = 1 \text{ for all } j \neq i\}$. Then for each $i \in N$, we have $\bar{A}_i \leqq \bar{D}$ and $\bar{A}_i \cong \bar{A}$. If Q is any normal subgroup of finite index in $\bar{B}$, then $Q \cap \bar{A}_i$ is of finite index in $\bar{A}_i$. However, it is easy to prove that for any integer t, the adjunction of the relation $y^t = 1$ to the relations of $\bar{A}$ causes $\bar{A}$ to collapse onto the identity. Consequently $\bar{A}$ has no normal subgroups of finite index. Thus $Q \cap \bar{A}_i = \bar{A}_i$ for each $i \in N$ and so $\bar{D} \leqq Q$. This is true for each such Q, and so $\bar{D} \leqq R$. Thus $\bar{D} = R$, as required.

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Now $\bar{B}$ is finitely generated, and so it follows from Lemma 4.4 that $\phi(\bar{D}) = \bar{D}$ for every ependomorphism ϕ of $\bar{B}$. We now find the ependomorphisms of $\bar{\bar{B}}$ and see which of these are induced from ependomorphisms of $\bar{B}$. Finally, we investigate which of these are induced from ependomorphisms of B.

LEMMA 4.5. *Every ependomorphism ψ of $\bar{\bar{B}}$ may be written, modulo an inner automorphism, in one of the following forms, where $\varepsilon_i = \pm 1$ for each $i = 1, 2, 3$, and k, r and s are integers.*

(i) $\psi(y) = y^{\varepsilon_1}(y^k z y^{-k})^s x^r$, $\psi(x) = x^{\varepsilon_2}$, $\psi(y^3 z y^{-3}) = (y^k z y^{-k})^{\varepsilon_3}$

(ii) $\psi(y) = y^{\varepsilon_1}(y^k z y^{-k})^s x^r$, $\psi(x) = (y^k z y^{-k})^{\varepsilon_2}$, $\psi(y^3 z y^{-3}) = x^{\varepsilon_3}$.

PROOF. Since $\bar{\bar{B}}$ is finitely generated and residually finite it is Hopf. (See, e.g., [4], p. 415.) Thus every ependomorphism of $\bar{\bar{B}}$ is an automorphism. Further, it is easy to check from the defining relations of $\bar{\bar{B}}$ that

$$x(y^i z y^{-i})x^{-1} = y^i z y^{-i} \qquad \text{for each } i \in Z.$$

Consequently $\psi(x)$ commutes with $\psi(y^i z y^{-i})$ for each $i \in Z$, and we may use Theorem 1 to investigate the normal form structure of $\psi(x)$, $\psi(y)$ and $\psi(z)$. Suppose first that one of $\psi(x)$ and $\psi(y^3 z y^{-3})$ lies in a conjugate of a factor, but not in a conjugate of H. It follows, as in the proof of Lemma 3.2, that

$$\psi([y^3 z y^{-3}, y^4 z y^{-4}]) = 1.$$

But

$$[y^3 z y^{-3}, y^4 z y^{-4}] \neq 1,$$

so this contradicts the fact that ψ is an automorphism of $\bar{\bar{B}}$. Next suppose that $\psi(x)$ has cyclically reduced length greater than one. Without loss of generality, we may assume that $\psi(x)$ is cyclically reduced, and then, by a length argument, we see that every element of $\bar{\bar{B}}$ which commutes with $\psi(x)$ is also cyclically reduced. Now x commutes with every element of H, that is, with every element of $\langle y^3 z y^{-3} \rangle^{\bar{\bar{B}}}$, the normal closure of $y^3 z y^{-3}$ in $\bar{\bar{B}}$. Thus $\psi(x)$ commutes with every element of $\langle \psi(y^3 z y^{-3}) \rangle^{\bar{\bar{B}}}$. But if $\psi(y^3 z y^{-3})$ does not lie in a conjugate of H, then its normal closure contains some elements which are not cyclically reduced and which therefore cannot commute with $\psi(x)$. Consequently $\psi(y^3 z y^{-3})$ lies in a conjugate of H. A similar argument deals with the case in which $\psi(y^3 z y^{-3})$ has cyclically reduced length greater than one. Consequently one of $\psi(x)$ and $\psi(y^3 z y^{-3})$ lies in a conjugate of H.

We will deal with the case in which $\psi(y^3 z y^{-3})$ lies in a conjugate of H. The other case may be dealt with in a similar way. Since H is normal in $\bar{\bar{B}}$, we must have $\psi(y^3 z y^{-3}) = h$ for some $h \in H$. Further $\bar{\bar{B}}/H$ is a free group of rank two

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generated by the images of x and y under the canonical map $\bar{\bar{B}} \to \bar{\bar{B}}/H$, and ψ induces a map ψ_1 of $\bar{\bar{B}}/H$ onto $\bar{\bar{B}}/\langle h \rangle^{\bar{\bar{B}}}$. Since free groups are Hopf (see, e.g., [4], Theorem 2.13, p. 109) it follows that $\langle h \rangle^{\bar{\bar{B}}} = H$, and that ψ_1 is an automorphism of $\bar{\bar{B}}/H$. Thus Theorem 2 may be applied to $\bar{\bar{B}}/H$ and its automorphism ψ_1. We wish to find the total exponent to which y appears in $\psi(x)$. We write $\psi(x)$ in normal form

$$\psi(x) = h_1 y^{\alpha_1}(xy)^{\beta_1} y^{\alpha_2}(xy)^{\beta_2} \cdots y^{\alpha_n}(xy)^{\beta_n}$$

where $h_1 \in H$ and $\alpha_i, \beta_i \in Z$ for each $i = 1, 2, \cdots, n$, and we use the fact that $\psi(x)$ commutes with every element of $\langle h \rangle^{\bar{\bar{B}}} = H$, and the relations of $\bar{\bar{B}}$ to deduce that

$$\alpha_1 + \beta_1 + \alpha_2 + \beta_2 + \cdots + \alpha_n + \beta_n = 0$$

and that $h_1 = 1$. Theorem 2 now shows that there exist elements $g \varepsilon \langle x, y \rangle$, $h' \in H$ and integers $s, t, \varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$, such that

$$\begin{aligned} \psi(x) &= g x^{\varepsilon_2} g^{-1} \\ \psi(y) &= g x^s y^{\varepsilon_1} x^t g^{-1} h'. \end{aligned}$$

Now H is normal in $\bar{\bar{B}}$, so we may conjugate by gx^s and obtain ψ, modulo an inner automorphism, in the form

$$\psi(y^3 z y^{-3}) = h_2, \ \psi(x) = x^{\varepsilon_2}, \ \psi(y) = y^{\varepsilon_1} x^r h_3$$

for some elements h_2, $h_3 \in H$, and some integers r, $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$. We now consider $\bar{\bar{B}}/\langle x \rangle^{\bar{\bar{B}}}$. By checking the defining relations of $\bar{\bar{B}}$, we find that this is the free group of rank two generated by the images of y and z under the canonical map $\bar{\bar{B}} \to \bar{\bar{B}}/\langle x \rangle^{\bar{\bar{B}}}$. Moreover, ψ induces an automorphism of $\bar{\bar{B}}/\langle x \rangle^{\bar{\bar{B}}}$, so Theorem 2 may again be applied. Since y appears to total exponent zero in any generator of H, we deduce that there is an element g_1 of E and integers s and $\varepsilon_3 = \pm 1$ such that

$$h_2 = g_1 (y^3 z y^{-3})^{\varepsilon_3} g_1^{-1}, \ y^{\varepsilon_1} h_3 = g_1 (y^{\varepsilon_1} (y^3 z y^{-3})^s) g_1^{-1}.$$

But $g_1 = h_4 y^l$ for some $h_4 \in H$ and some integer l, so we see that, modulo an inner automorphism by h_4, ψ takes the form

$$\psi(x) = x^{\varepsilon_2}, \ \psi(y) = y^{\varepsilon_1}(y^k z y^{-k})^s x^r, \ \psi(y^3 z y^{-3}) = (y^k z y^{-k})^{\varepsilon_3}$$

for some integers $k, r, s, \varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$ and $\varepsilon_3 = \pm 1$. A similar proof deals with the case where $\psi(x)$ lies in a conjugate of H, and this yields an automorphism of type (ii).

To prove that B is Hopf, it is sufficient to show that for every ependomorphism θ of B, some power of θ is an automorphism. Now if θ induces $\bar{\theta}$ on $\bar{B}$ and $\bar{\bar{\theta}}$ on $\bar{\bar{B}}$, then θ^i induces $\bar{\theta}^i$ on $\bar{B}$ and $\bar{\bar{\theta}}^i$ on $\bar{\bar{B}}$. But Lemma 4.5 shows that modulo

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an inner automorphism of $\bar{\bar{B}}$, for every ependomorphism ψ of $\bar{\bar{B}}$, some power of ψ will take the form (i) as above with $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$. Moreover, if

$$\psi(y) = y(y^k zy^{-k})^s x^r,\ \psi(x) = x,\ \psi(y^3 zy^{-3}) = y^k zy^{-k},$$

then, modulo an inner automorphism,

$$\psi^2(y) = y(y^{2k-3}zy^{-(2k-3)})^{2s}x^{2r},\ \psi^2(x) = x,\ \psi^2(y^3zy^{-3}) = y^{2k-3}zy^{-(2k-3)}.$$

This allows us to make certain restrictions on r, s and k. We now sketch a proof of

LEMMA 4.6. *If $\bar{\theta}$ is an ependomorphism of $\bar{B}$, and $\bar{\theta}$ induces $\bar{\bar{\theta}}$ on $\bar{\bar{B}}$, where $\bar{\bar{\theta}}$ takes the form (i) above, with $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$, then $r = s = 0$, $k \geqq 3$ and $k \equiv 3$ (mod. 20).*

PROOF. We may write $\bar{\theta}$ in the form

$$\bar{\theta}(Y) = Y(Y^kZY^{-k})^sX^r\xi,\ \bar{\theta}(X) = X\eta,\ \bar{\theta}(Y^3ZY^{-3}) = Y^kZY^{-k}\zeta,$$

where ξ, η and ζ all lie in $\bar{D}$. Let n be the smallest integer such that $\xi(i) = \eta(i) = \zeta(i) = 1$ for all $i \geqq n$. Now the relations of $\bar{A}$ show that if $i \geqq -20$, or if $i < -20$ and i is congruent to 0, 1, 2, 3, 4, 5 or 6 modulo 20, then

$$[Y^i(Y^3ZY^{-3})Y^{-i}, X](t) = 1 \qquad \text{for all } t \in N.$$

Thus for each of these values of i

$$[Y^i(Y^3ZY^{-3})Y^{-i}, X] = 1$$

is a relation of $\bar{B}$.

Suppose that $r \neq 0$. By considering, if necessary, $\bar{\theta}^2$ in place of $\bar{\theta}$, we may assume that $|r| > 1$. We apply $\bar{\theta}$ to the above relations, and evaluate the expression

$$\bar{\theta}([Y^i(Y^3ZY^{-3})Y^{-i}, X])(t)$$

at a suitably chosen $t \in N$. Using the relations of $\bar{A}$, we find that if $|r| > 1$, then

$$\bar{\theta}([Y^i(Y^3ZY^{-3})Y^{-i}, X])(t) \neq 1$$

for any $t > n$ such that $20t + k > 3$, and for any $i \leqq -(20t + k - 2)$. This contradiction shows that we must have $r = 0$.

Next we consider the value of k. We have proved that $r = 0$. This property of r is preserved by taking powers of $\bar{\theta}$, so we may again consider powers of $\bar{\theta}$ in place of $\bar{\theta}$. We have already shown that

$$\bar{\theta}([Y^{-(20t+k-2)}(Y^3ZY^{-3})Y^{20t+k-2}, X])(t) \neq 1$$

whenever $t > n$ and $20t + k > 3$. Thus

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$$\bar{\theta}([Y^{-(20t+k-2)}(Y^3ZY^{-3})Y^{20t+k-2}, X]) \neq 1.$$

Consequently $-(k-2)$ is not congruent to 0, 1, 2, 3, 4, 5 or 6 modulo 20. Thus $k \equiv l \pmod{20}$ where $3 \leqq l \leqq 15$. Let π be the map $\pi: Z \to Z$ defined by

$$\pi(i) = 2i - 3.$$

Then, by considering powers of $\bar{\theta}$ in place of $\bar{\theta}$, we see that for every $j \in N$ the congruence class of $\pi^j(k)$ modulo 20 must lie between 3 and 15. This shows that $k \equiv 3$, 8 or 13 (mod. 20), so replacing $\bar{\theta}$ by a suitable power of itself, we may assume that $k \equiv 3$ (mod. 20). We observe that this property of k is invariant under the operation of taking powers of $\bar{\theta}$.

Next we consider s. If $s \neq 0$, then, as before, we may assume that $|s| > 1$ and that $s \equiv 0$ or 1 (mod. 3). Since $k \equiv 3$ (mod. 20) we may put $k = 20m + 3$, where $m \in Z$. We consider the expressions

$$\bar{\theta}([Y^{-(20(t+m)+14)}(Y^3ZY^{-3})Y^{20(t+m)+14}, X])(t)$$

and

$$\bar{\theta}([Y^{-(20(t+m)+15)}(Y^3ZY^{-3})Y^{20(t+m)+15}, X])(t)$$

where again $t > n$ and $20t + k > 3$. Since $-14 \equiv 6$ (mod. 20) and $-15 \equiv 5$ (mod. 20) these expressions both represent the trivial element of $\bar{A}$. But consideration of the relations of $\bar{A}$ shows that for $|s| > 1$ and $s \equiv 0$ or 1 (mod. 3) these two relations of $\bar{A}$ are incompatible. (See [3] for details.) Thus $s = 0$, as required.

Finally we prove that $k \geqq 3$. Suppose not. Then since $k \equiv 3$ (mod. 20) we may write k in the form $k = -20m + 3$ where $m > 0$. Let

$$c_r = [X^{-1}, (Y^{-(20r+10)}ZY^{20r+10})^{-1}].$$

Then the relations of $\bar{A}$ show that

$$c_r(t) = \begin{cases} 1 & \text{if } t \neq r \\ y^{-1} & \text{if } t = r. \end{cases}$$

However $\langle y \rangle^{\bar{A}} = \bar{A}$ and so we must have $\langle c_r \rangle^{B} = \bar{A}_r$. We now evaluate $\bar{\theta}(c_r)(t)$ for each $t \in N$, and discover that for $t > n$,

$$\bar{\theta}(c_r)(t) = \begin{cases} 1 & \text{if } t \neq r+m \\ y^{-1} & \text{if } t = r+m. \end{cases}$$

We choose r such that $r + m > n$ and consider $\bar{\theta}(\bar{A}_r) = \langle \bar{\theta}(c_r) \rangle^{B}$. If $\bar{\theta}(c_r)(t) \neq 1$ for some $t \leqq n$, then, since $\bar{B} \geqq \bar{D}$ and since $\bar{A}$ is centreless, we must have

$$\bar{\theta}(\bar{A}_r) \cap \bar{A}_t \neq \langle 1 \rangle$$

for that value of t. Thus $\bar{A}_{r+m}$ is a proper homomorphic image of $\bar{A}_r$. But this contradicts the fact that $\bar{A}$ is Hopf, and so we must have $\bar{\theta}(\bar{A}_r) = \bar{A}_{r+m}$ for each $r \in N$ such that $r + m > n$. But $\bar{\theta}(\bar{D}) = \bar{D}$, and so it follows that $m < n$ and

$$\bar{\theta}(\bar{A}_1 \times \bar{A}_2 \times \cdots \times \bar{A}_{n-m}) = \bar{A}_1 \times \bar{A}_2 \times \cdots \times \bar{A}_n.$$

Let i be any integer such that $1 \leqq i \leqq n - m$ and consider $\bar{\theta}(\bar{A}_i) = \langle \bar{\theta}(c_i) \rangle^{\bar{B}}$. Just as before, we see that if

$$\langle \bar{\theta}(c_i)(j) \rangle^{\bar{A}} = \bar{A}$$

for any j such that $1 \leqq j \leqq n$, then $\bar{\theta}(c_i)(k) = 1$ for all $k \neq j$. Since $m > 0$, it follows that there is some k $(1 \leqq k \leqq n)$ such that $\bar{A}_k \neq \bar{\theta}(\bar{A}_j)$ for any j such that $1 \leqq j \leqq n - m$. Thus $\bar{A}_k$ is generated by t $(t \geqq 2)$ proper normal subgroups $N_1, N_2, \cdots N_t$, where these subgroups commute element-wise. It is clear that at least one of $N_1, N_2, \cdots N_t$, say N_1, contains a cyclically reduced element c which does not lie in the amalgamated subgroup H_k of $\bar{A}_k$. Suppose c lies in a factor of $\bar{A}_k$. Let $d \notin H_k$ be an element of the other factor. Then, since N_1 is normal in $\bar{A}_k$, we have $dcd^{-1}c^{-1} \in N_1$, so that N_1 contains a cyclically reduced element of length greater than one. Thus in all cases N_1 contains a cyclically reduced element of length greater than one, and so it also contains elements which are not cyclically reduced, but which have cyclically reduced length greater than one. But the elements of N_i $(2 \leqq i \leqq t)$ all commute with the elements of N_1. Thus, by a length argument, we see that all the subgroups N_i $(2 \leqq i \leqq t)$ lie within H_k, so that N_1 and H_k together generate $\bar{A}_k$. It follows that N_1 contains an element, c say, which lies in a factor. But the elements of N_i $(2 \leqq i \leqq t)$ all commute with c and lie in H_k. Since the factors are free, we deduce that $\bar{A}_k/N_1$ is cyclic. But it is easy to prove, from the defining relations of $\bar{A}$, that $\bar{A}$ is perfect. Thus $N_1 = \bar{A}_k$, contradicting the fact that N_1 is a proper normal subgroup of $\bar{A}_k$. Consequently we must have $m \leqq 0$, and so $k \geqq 3$ as required.

Next we investigate which of these ependomorphisms $\bar{\theta}$ of $\bar{B}$ can be induced from an ependomorphism θ of B. We have

$$\bar{\theta}(Y) = Y\xi, \ \bar{\theta}(X) = X\eta, \ \bar{\theta}(Y^3ZY^{-3}) = Y^{20m+3}ZY^{-(20m+3)}\zeta$$

for some $m > 0$, and some elements ξ, η, ζ of $\bar{D}$. Recalling the identification of $\bar{B}$ with $T \leqq \bar{C}$ given in Lemma 4.1, we prove

LEMMA 4.7. *Let θ be an ependomorphism of B, inducing an ependomorphism $\bar{\theta}$ of $\bar{B}$, where $\bar{\theta}$ takes the above form. Let n be the smallest integer such that ξ, η and ζ all satisfy*

$$\xi(i) = \eta(i) = \zeta(i) = 1$$

for all $i > n$. Then $m = 0$, and θ induces on $\Pi_{1 \leqq i \leqq n} \bar{A}_i$ an automorphism $\bar{\theta}_n$ with the following structure:

There is a permutation π of the set $\{1, 2, \cdots, n\}$ such that for each i, $(1 \leqq i \leqq n)$,

$$\bar{\theta}_n(\bar{A}_i) = \bar{A}_{\pi(i)}$$

and if $\bar{A}_i, \bar{A}_{\pi(i)}$ are identified with $\bar{A}$ in the obvious way, then $\bar{\theta}_n$ induces an inner automorphism of $\bar{A}$.

PROOF. We write θ in the form

$$\theta(y_B) = y_B\xi\xi_1,\ \theta(x_B) = x_B\eta\eta_1,\ \theta(z_B) = y_B^{20m} z_B y_B^{-20m}\zeta\zeta_1$$

where $\xi_1, \eta_1, \zeta_1 \in D_2$ and ξ, η, ζ are elements of B such that

$$\xi(2^k + i) = \eta(2^k + i) = \zeta(2^k + i) = 1$$

whenever $2^k + i \in P_2$ and $i > n$. Let $r \in P_2$ be chosen in such a way that

$$\xi_1(i) = \eta_1(i) = \zeta_1(i) = 1$$

whenever $i > r$. Suppose $m \neq 0$. Then by Lemma 4.6, $m > 0$ and so we may choose $k > r$ in such a way that $k \neq 2^s + i$ for any $s > 0$ and any i such that $0 \leqq i \leqq n$, and such that $k + m = 2^t$ for some $t > 0$. Now the relations of S show that

$$[y_B^{-10} z_B y_B^{10}, x_B](i) = 1$$

for all $i \in P_2$. It follows that

$$[y_B^{-10} z_B y_B^{10}, x_B] = 1$$

is a relation of B. However it is easy to check from the relations of S, that when k is chosen as above we have

$$\theta([y_B^{-10} z_B y_B^{10}, x_B])(k) \neq 1.$$

This contradiction shows that we must have $m = 0$.

To prove the remaining part of Lemma 4.7, we return to the group $\bar{B}$ and its ependomorphism $\bar{\theta}$. The proof is then completed in a manner entirely analogous to the proof that $k \geqq 3$ in Lemma 4.6, so details are omitted here. The fact that every ependomorphism of $\bar{A}$ is an inner automorphism (Lemma 3.2) completes the proof.

We may now complete the proof of Proposition 4. It is sufficient to prove that for every ependomorphism θ of B, some power of θ is an automorphism. Consequently, working modulo an inner automorphism, and using Lemmas 4.5, 4.6 and 4.7, we may assume that θ takes the form

$$\theta(y_B) = y_B\xi\xi_1,\ \theta(x_B) = x_B\eta\eta_1,\ \theta(z_B) = z_B\zeta\zeta_1$$

where $\xi_1, \eta_1, \zeta_1 \in D_2$, and $r \in P_2$ is chosen in such a way that

$$\xi_1(i) = \eta_1(i) = \zeta_1(i) = 1$$

whenever $i > r$, and where ξ, η, ζ are elements of B such that

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$$\xi(2^k + i) = \eta(2^k + i) = \zeta(2^k + i) = 1$$

whenever $2^k + i \in P_2$ and $i > n$, where n is defined as in Lemma 4.7. We consider the element

$$w_t = [y_B^{-20t-11} z_B y_B^{20t+11}, x_B]$$

of B. The relations of S show that

$$w_t(k) = \begin{cases} 1 & \text{if } k > t \\ pq^{-1}p^{-1} & \text{if } k = t. \end{cases}$$

Now let k be chosen in such a way that $k \in P_2$, $k > \max(t, r)$ and $k \neq 2^s + i$ for any $s > 0$ and any i such that $1 \leqq i \leqq n$. It is easy to check from the relations of S that we then have $\theta(w_t)(k) = 1$. Moreover, if $t > r$ and $t \neq 2^s + i$ for any $s > 0$ and any i such that $1 \leqq i \leqq n$, we have $\theta(w_t)(t) \neq 1$. Now S is simple, and so we have

$$S = \langle w_t(t) \rangle^S$$

for each $t \in P_2$. But the groups S_i are exactly the minimal normal subgroups of B, so for each $i \in P_2$ we have either $\theta(S_i) = S_j$ for some $j \in P_2$ or $\theta(S_i) = 1$. It follows that if $t > r$, and $t \neq 2^s + i$ for any $s > 0$ and for any i such that $1 \leqq i \leqq n$, then we must have $\theta(S_t) = S_t$.

We now consider those $t \in P_2$ which take the form $2^s + i$ with $s > 0$ and $1 \leqq i \leqq n$. Let $x_{B,i}$, $y_{B,i}$ and $z_{B,i}$ be elements of B defined in the following way:

$$x_{B,i}(k) = y_{B,i}(k) = z_{B,i}(k) = 1$$

if $k \in P_2$ and $k \neq 2^s + i$ for any $s > 0$, and for each $s > 0$

$$\begin{aligned} x_{B,i}(2^s + i) &= b^{-1}a \\ y_{B,i}(2^s + i) &= p \\ z_{B,i}(2^s + i) &= p^{20\cdot 2^s} a p^{-20\cdot 2^s + 1}. \end{aligned}$$

The form of $\bar{\theta}_n$ shows that there are elements g_i of B and $\xi^{(i)}$, $\eta^{(i)}$ and $\zeta^{(i)}$ of D_2 such that for each i ($1 \leqq i \leqq n$),

$$\begin{aligned} \theta(x_{B,i}) &= g_i x_{B,\pi(i)} g_i^{-1} \xi^{(i)} \\ \theta(y_{B,i}) &= g_i y_{B,\pi(i)} g_i^{-1} \eta^{(i)} \\ \theta(z_{B,i}) &= g_i z_{B,\pi(i)} g_i^{-1} \zeta^{(i)}. \end{aligned}$$

We choose $r' \in P_2$ such that for each i ($1 \leqq i \leqq n$)

$$\xi^{(i)}(t) = \eta^{(i)}(t) = \zeta^{(i)}(t) = 1$$

for all $t \in P_2$ such that $t > r'$. By considering the element

$$w'_t = [y_{B,i}^{-20t-11} z_{B,i} y_{B,i}^{20t+11}, x_{B,i}]$$

of B (where $t \in P_1$), and using the same argument as before, we prove that if $k > r'$ and $k = 2^s + i$ for some $s > 0$ and some i such that $1 \leqq i \leqq n$, then we must have

$$\theta(S_k) = S_{2^s+\pi(i)}.$$

Let $r'' = \max(r, r')$. It follows from the above reasoning that

$$\ker(\theta) \leqq S_3 \times S_5 \times \cdots \times S_{r''}$$

and that

$$\theta(S_3 \times S_5 \times \cdots \times S_{r''}) = S_3 \times S_5 \times \cdots \times S_{r''}.$$

But the groups S_i are all simple. Consequently $\ker(\theta) = 1$, and so θ is an automorphism of B. Thus B is Hopf, as required.

This completes the proof of Theorem A.

3. The proof of Theorem B

We now use a similar method to construct a finitely generated group G_0 isomorphic to its own direct square. For this construction we shall need the structure known as a Higman-Neumann-Neumann (HNN) or Britton extension, and the associated normal form theorem, Britton's lemma. Details of the theorems required here may be found in [3], Chapter 1.

We begin by constructing the 'base group' S_0. We take S' to be the group $S_{\rho,\sigma,\tau}$ where ρ, σ and τ are defined on the negative integers as follows:

Let $\Delta_t = \{n \in Z\colon -20t \leqq n \leqq -20t + 19\}$. Then ρ, σ and τ fix each Δ_t set-wise, and induce on Δ_t permutations ρ_t, σ_t and τ_t defined (as products of disjoint cycles) by

$$\rho_t = (-20t + 18, -20t + 19)$$

$$\sigma_t = (-20t + 9, -20t + 10)(-20t + 12, -20t + 15, -20t + 13)(-20t + 14, -20t + 19)$$

$$\tau_t = (-20t + 12, -20t + 13)(-20t + 14, -20t + 19, -20t + 18).$$

The subgroup $\langle a, p\rangle$ of S' is free, and so we may make an HNN extension S'' of S' by adding to S' the generator c and the relations

$$c^{-1}ac = a$$

$$c^{-1}pc = p^2.$$

An application of Britton's lemma shows that the subgroup $\langle b^{-1}a, c\rangle$ of S'' is free. Let H be a group with presentation

$$H = \langle d, e, f\colon ede^{-1} = d^2, dfd^{-1} = f^2\rangle.$$

H is the result of two HNN extensions, first of the free group $\langle f\rangle$ by the generator d, according to the relation $dfd^{-1} = f^2$, and then of the resulting group by the generator e according to the relation $ede^{-1} = d^2$. It follows from Britton's lemma that the subgroups $\langle e\rangle$ and $\langle f\rangle$ of H generate their free product, which is a free group of rank two. We may, therefore, form the free product of S'' and H amalgamating the subgroups $\langle b^{-1}a, c\rangle$ of S'' and $\langle e, f\rangle$ of H according to the relations

$$b^{-1}a = e$$
$$c = f.$$

The resulting group is

$$S_0 = \langle a, p, b, q, c, d\colon a^i p^{\rho(i)} = b^{\tau(i)} q^{\sigma\tau(i)} \text{ for each } i \in I,\ c^{-1}ac = a,$$
$$c^{-1}pc = p^2,\ (b^{-1}a)d(b^{-1}a)^{-1} = d^2,\ dcd^{-1} = c^2\rangle.$$

Let C be the cartesian power $C = S_0^N$, and let G_0 be the subgroup of C generated by

$$\begin{aligned} x &= (b^{-1}a, b^{-1}a, b^{-1}a, \cdots) \\ y &= (p, p, p, \cdots) \\ z &= (p^{10}ap^{-9}, p^{20}ap^{-19}, p^{30}ap^{-29}, \cdots) \\ t &= (c, c, c, \cdots) \\ u &= (d, d, d, \cdots). \end{aligned}$$

We define P_1 to be the subset of N consisting of all odd numbers, P_2 to be the corresponding set of even numbers, and C_1, C_2 to be the cartesian powers $S_0^{P_1}$, $S_0^{P_2}$ respectively, considered in the natural way as subgroups of C. For each $f \in G_0$, we define functions $f_1, f_2 \in C$ as follows:

For each $j = 1, 2$

$$f_j(i) = \begin{cases} f(i) & \text{if } i \in P_j \\ 1 & \text{otherwise.} \end{cases}$$

We define

$$G_j = \{f_j\colon f \in G_0\}$$

for each $j = 1, 2$. We prove that $G_2 \leqq G_0$, from which it follows that $G_0 = G_1 \times G_2$.

Using the definitions of ρ, σ and τ given above, we see that for each $t \in N$ the following relations hold in S_0:

(1) $(b^{-1}a)(p^{20t}ap^{-20t+1})(a^{-1}b) = p^{20t}ap^{-20t+1}$

(2) $(b^{-1}a)(p^{20t-10}ap^{-20t+11})(a^{-1}b) = p^{20t-9}a^2p^{-20t+11}$

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$$(3)\quad (b^{-1}a)(p^{20t-13}ap^{-20t+14})(a^{-1}b) = p^{20t-15}a^{-4}p^{-20t+11}q$$

$$(4)\quad (b^{-1}a)(p^{20t-16}ap^{-20t+17})(a^{-1}b) = p^{20t-16}a^{-4}p^{-20t+12}q^7p^{-1}q^{-1}.$$

Consider the element $[x^{-1},z^{-1}]$ of G_0. By (1) and (2) we have

$$\begin{aligned}[x^{-1},z^{-1}] &= (p^{11}ap^{-10},1,p^{31}ap^{-30},1,p^{51}ap^{-50},1,\cdots)\\ &= yz_1y^{-1}.\end{aligned}$$

Thus G_0 contains z_1 and all its conjugates by powers of y. It now follows, in a similar way, from (3) and (4) that G_0 contains x_1y_1 and y_1. Consequently G_0 also contains x_2, y_2 and z_2. Further,

$$\begin{aligned}u^{-1}x_2u &= (1,d^{-1}(b^{-1}a)d,1,d^{-1}(b^{-1}a)d,1,\cdots)\\ &= (1,d(b^{-1}a),1,d(b^{-1}a),1,\cdots)\\ &= u_2x_2\end{aligned}$$

which shows that G_0 also contains u_2. Also

$$\begin{aligned}t^{-1}u_2t &= (1,c^{-1}dc,1,c^{-1}dc,1,\cdots)\\ &= (1,cd,1,cd,1,\cdots)\\ &= t_2u_2\end{aligned}$$

which shows that G_0 contains t_2. Thus $G_2 = \langle x_2,y_2,z_2,t_2,u_2\rangle \leqq G_0$, and so $G_0 = G_1 \times G_2$.

Let $\alpha\colon C \to C$ be the monomorphism defined by

$$(f\alpha)(i) = \begin{cases} f(i-1) & \text{for } i > 1\\ 1 & \text{for } i = 1.\end{cases}$$

Then $\alpha|_{G_1}$ is an isomorphism of G_1 onto G_2, so in order to complete the proof that $G_0 \cong G_0 \times G_0$, we need only show that $G_2 \cong G_0$. To this end, we let $\beta\colon C \to C$ be the monomorphism defined by

$$\begin{aligned}(f\beta)(2i) &= f(i)\\ (f\beta)(2i-1) &= 1.\end{aligned}$$

We investigate the image of G_0 under β. Since

$$\begin{aligned}\beta(z) &= (1,p^{10}ap^{-9},1,p^{20}ap^{-19},1,\cdots)\\ &= t_2z_2y_2^{-1}t_2^{-1}y_2\end{aligned}$$

we see that β maps G_0 onto G_2. Thus $\beta|_{G_0}$ is an isomorphism between G_0 and G_2, so that $G_1 \cong G_2 \cong G_0$, as required. This completes the proof of Theorem B.

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