MINIMAL REGULAR GRAPHS OF GIRTHS EIGHT AND TWELVE

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In **(3)** Tutte showed that the order of a regular graph of degree d and even girth $g > 4$ is greater than or equal to

$$2\sum_{0}^{g/2-1} (d-1)^i.$$

Here the *girth* of a graph is the length of the shortest circuit. It was shown in **(2)** that this lower bound cannot be attained for regular graphs of degree >2 for $g \neq 6$, 8, or 12. When this lower bound is attained, the graph is called *minimal.* In a group-theoretic setting a similar situation arose and it was noticed by Gleason that minimal regular graphs of girth 12 could be constructed from certain groups. Here we construct these graphs making only incidental use of group theory. Also we give what is believed to be an easier construction of minimal regular graphs of girth 8 than is given in **(2)**. These results are contained in the following two theorems.

THEOREM 1. *Let Q_4 be a non-degenerate quadric surface in projective 4-space $P(4, q)$. Define G_8 to be the graph whose nodes are the points and lines of Q_4, two nodes being joined if and only if they correspond to an incident point-line pair in Q_4. Then G_8 is a minimal regular graph of degree $q + 1$ and girth* 8.

THEOREM 2. *Let Q_6 be the quadric surface in $P(6, q)$ given by*

$$x_0^2 + x_1 x_{-1} + x_2 x_{-2} + x_3 x_{-3} = 0.$$

For each point x in Q_6, distinguish the lines in Q_6 incident with x and points y satisfying

$$x_0 y_i - x_i y_0 + x_{-j} y_{-k} - x_{-k} y_{-j} = 0$$

where (i, j, k) is a cyclic even permutation of $(1, 2, 3)$ or $(-1, -2, -3)$ and where $x = (x_i)$, $y = (y_i)$, $-3 \leqslant i \leqslant 3$. As in Theorem 1, *define G_{12} to be the graph whose nodes are the points of Q_6 and the lines of Q_6 distinguished above. Then G_{12} is a minimal regular graph of degree $q + 1$ and girth* 12.

To prove Theorem 1, we select the non-degenerate quadric surface Q_4 given by

$$x_0^2 + x_1 x_{-1} + x_2 x_{-2} = 0 \qquad (1)$$

where x in Q_4 is given by (x_i) $-2 \leqslant i \leqslant 2$. Actually we could have chosen any non-degenerate quadric surface in $P(4, q)$ since they are all essentially equivalent; cf. **(1)**.

LEMMA 1. *No set of points and lines in Q_4 form a triangle.*

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It is well known that the automorphism group of Q_4 is transitive on the lines of Q_4; cf. **(1)**. Therefore it is sufficient to show that the line L consisting of points of the form $(\lambda, \mu, 0, 0, 0)$ does not occur on any triangle whose points and lines are in Q_4. But to show this, it is sufficient to show that for a point y in Q_4 not on L, there is exactly one line in Q_4 through y which meets L. Now the points x in Q_4 that are on lines of Q_4 through y will lie on the tangent hyperplane of Q_4 through y; i.e. they will satisfy (1) and

$$2y_0 x_0 + y_1 x_{-1} + y_{-1} x_1 + y_2 x_{-2} + y_{-2} x_2 = 0. \tag{2}$$

If x is on L, (2) becomes

$$y_1 \mu + y_2 \lambda = 0 \tag{3}$$

for which there is exactly one solution in $P(4, q)$ unless $y_1 = y_2 = 0$. But in this case $y_0 = 0$ since y is in Q_4 and thus y lies on L.

To complete the proof of Theorem 1, we notice that since G_8 corresponds to alternate points and lines of Q_4, the girth of G_8 is $\geqslant 6$. Now the girth is not 6 since a circuit of length 6 in G_8 would correspond to a triangle in Q_4. It is well known that there are $q + 1$ lines of Q_4 through each point of Q_4 and that all together there are $1 + q + q^2 + q^3$ points and $1 + q + q^2 + q^3$ lines in Q_4. Thus the lower bound for a regular graph of degree $q + 1$ and girth 8 is attained by G_8 and its girth is therefore $\leqslant 8$ and the proof is complete.

To prove Theorem 2, we first note that points y on lines of Q_6 incident with a point x in Q_6 lie on the tangent hyperplane to Q_6 through x and thus satisfy

$$2x_0 y_0 + x_1 y_{-1} + x_{-1} y_1 + x_2 y_{-2} + x_{-2} y_2 + x_{-3} y_{-3} = 0. \tag{4}$$

Therefore, the distinguished lines through x satisfy (4) and the six bilinear equations given in Theorem 2.

LEMMA 2. *Every point in Q_6 has $q + 1$ distinguished lines through it. Furthermore points in Q_6 that correspond to nodes of distance $\leqslant 8$ in G_{12} from x in Q_6 are precisely the points of Q_6 that satisfy* (4).

It is known that the group which preserves (4) and the six bilinear equations of Theorem 2 is transitive on the points of Q_6 and the distinguished lines of Q_6. Therefore it is sufficient to prove the lemma for $x = (1, 0, 0, 0, 0, 0, 0)$. In this case (4) and the six bilinear equations become

$$y_{-2} = 0, \quad y_{-1} = 0, \quad y_0 = 0, \quad y_3 = 0, \tag{5}$$

which represent a plane lying in Q_6. Therefore x has $q + 1$ distinguished lines through it. To complete the proof of Lemma 2 we compute the points z on distinguished lines through points Q_6 satisfying (5). These points satisfy

$$\begin{aligned} y_1 z_{-1} + y_2 z_{-2} + y_{-3} z_3 &= 0, \\ y_1 z_0 + y_{-3} z_{-2} &= 0, \\ y_2 z_0 - y_{-3} z_{-1} &= 0, \\ y_1 z_3 &= 0, \\ y_2 z_3 &= 0, \\ y_{-3} z_0 - y_1 z_2 + y_2 z_1 &= 0, \end{aligned} \tag{6}$$

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which implies that $z_3 = 0$ unless $y_1 = y_2 = 0$, in which case $z = x$ and again $z_3 = 0$. But $z_3 = 0$ represents the tangent hyperplane to Q_6 through x. Conversely, if z is a point of Q_6 satisfying $z_3 = 0$, it will correspond to a node of distance $\leqslant 8$ from x if (6) can be solved for y satisfying (5). The determinant of the second, third, and sixth equation of (6) is $z_0(z_0^2 + z_1 z_{-1} + z_2 z_{-2}) = 0$ since z is in Q_6. Thus a solution may be found to these equations and if $z_0 \neq 0$ it may be verified directly that the first equation of (6) is also satisfied. Now if $z_0 = 0$, (6) becomes

$$(7) \qquad \begin{aligned} y_1 z_{-1} + y_2 z_{-2} &= 0, \\ -y_1 z_2 + y_2 z_1 &= 0 \end{aligned}$$

and the determinant of this system is $z_{-1} z_1 + z_{-2} z_2 = 0$, so again a solution for y may be found. Thus Lemma 2 is proved.

To complete the proof of Theorem 2, we note by (5) that the line L given by points of the form $(\lambda, 0, 0, 0, \mu, 0, 0)$ is a distinguished line. The points x in Q_6 which correspond to nodes of distance $\leqslant 8$ from points on L satisfy

$$(8) \qquad x_0^2 + x_1 + x_2 x_{-2} + x_3 x_{-3} = 0, \qquad \lambda x_3 + \mu x_{-1} = 0$$

since the last equation in (8) represents the points in the tangent hyperplane through $(\lambda, 0, 0, 0, \mu, 0, 0)$. But this equation can always be solved for λ and μ. Thus every point in Q_6 corresponds to a node in G_{12} of distance $\leqslant 8$ from some point on L. We thus have the situation shown in Figure 1, which illustrates the case $q = 2$.

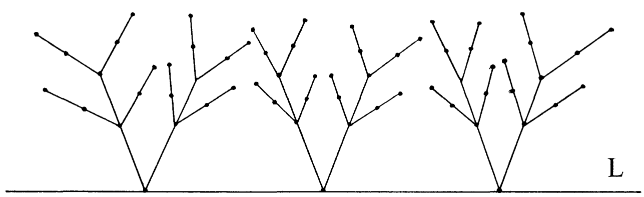

FIGURE 1.

All the points shown are different since there are $(q + 1)(1 + q^2 + q^4)$ points in Q_6 and if any two points in Figure 1 were the same, there would be less than this number of points in Q_6. Now it is clear that L could not correspond to a node of G_{12} contained in a circuit length <12 since this would make at least two of the points in Figure 1 equal. By transitivity, the same is true for any other distinguished line in G_{12} and the girth of G_{12} is $\geqslant 12$. However, the lower bound for the order of a graph of degree $q + 1$ and girth 12 is attained by G_{12} so the girth actually is equal to 12. The proofs of Theorems 1 and 2 could be carried out without using the transitivity of the groups involved, but the algebra would be more complicated.

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