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DISPROOF OF A COEFFICIENT ESTIMATE RELATED TO BAZILEVIC FUNCTIONS

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A coefficient estimate for powers of a class of Bazilevic functions obtained by M.M. Elhosh, is disproved and some sharp bounds are given.

Suppose that m is a positive integer. For functions $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ analytic in the open unit disk $\mathrm{U} = \{z\colon |z| < 1\}$ and for $\phi(z) = z/(1-z)$ we write (see also[6] and [7])

$$(1) \qquad \{f(z^m)\}^{1/m} = \sum_{n=0}^{\infty} a_n(m) z^{mn+1} \text{ and } \{\phi(z^m)\}^{1/m} = \sum_{n=0}^{\infty} b_n(m) z^{mn+1}$$

so that

$$(2) \qquad \begin{aligned} &a_0(m) = 1,\ a_1(m) = \frac{1}{m} a_2,\ a_2(m) = \frac{1}{m}\left(a_3 - \frac{m-1}{2m} a_2^2\right), \\ &b_0(m) = 1 \text{ and} \\ &b_n(m) = \frac{(1+m)(1+2m)\cdots(1+(n-1)m)}{(n!)m^n};\ n = 1, 2, \cdots. \end{aligned}$$

For $0 < \alpha < \infty$ let $B(\alpha)$, called Bazilevic of type α, denote the class of functions $f(z) = z + a_2 z^2 + \cdots$ analytic in U and satisfying

$$(3) \qquad f(z) = \left\{\alpha \int_0^z g^{\alpha}(t)\, p(t) t^{-1} dt\right\}^{1/\alpha}$$

where $g(z)$ is starlike, that is $\mathrm{Re}\{zg'(z)/g(z)\} > 0$ and $p(z)$ is of positive real part with $p(0) = 1$.

It has been proved by many authors including Bazilevic [1] that the functions in $B(\alpha)$ are univalent. (See Bernardi [2].) The coefficient problem for $f(z)$ in $B(\alpha)$ has been settled by Leach [8] (also see Math. Rev. 83C: 30015). For more references on the coefficients of Bazilevic functions see [2].

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Elhosh [4] considered the subclass $B_1(\alpha)$ of $B(\alpha)$ for which $p(z) \equiv 1$. Note that $B_1(1)$ is the class of convex functions, that is $f(z)$ is convex if and only if $zf'(z)$ is starlike. (See [5] p.115.) Elhosh ([4], Theorem 4) stated the following: Let $f(z) \in B_1(\alpha)$, let $0 < \alpha \leqslant 1$, and let $F(z) = \{f(z^2)\}^{1/2} = z + c_3 z^3 + c_5 z^5 + \cdots$. Then for $n \geqslant 1$, we have

$$|C_{2n+1}| \leqslant \frac{1}{2n+1}. \tag{4}$$

In this paper we show that (4) is not true when $1/2 < \alpha \leqslant 1$.

COUNTER–EXAMPLE 1. Let $\alpha = 1$. Consider the extremal function $\phi(z) = z/(1-z) \in B_1(1)$. Then for $F(z) = \{\phi(z^2)\}^{1/2}$ we obtain from (2) that

$$b_n(2) = \frac{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n-1)}{(n!)2^n}.$$

It is easy to see that $b_n(2) > 1/(2n+1)$.

COUNTER–EXAMPLE 2. Suppose that $f(z) = z + a_2 z^2 + \cdots$ belongs to $B_1(\alpha)$; $0 < \alpha < \infty$. Then for $F(z) = \{f(z^m)\}^{1/m}$ given by (1) we have the sharp bounds

$$|a_1(m)| \leqslant \frac{2\alpha}{m(\alpha+1)} \tag{5}$$

and

$$|a_2(m)| \leqslant \frac{\alpha}{m(\alpha+2)} \left[1 + \frac{2\alpha(\alpha+m+2)}{m(\alpha+1)^2}\right]. \tag{6}$$

For $m = 2$, (5) and (6) are sharper than (4) when $0 < \alpha < 1/2$ and disprove (4) when $1/2 < \alpha \leqslant 1$.

For $f(z) \in B_1(\alpha)$ and for the starlike functions $g(z)$ we have

$$f(z) = \{\alpha \int_0^z g^\alpha(t) t^{-1} dt\}^{1/\alpha}. \tag{7}$$

Write

$$\frac{zg'(z)}{g(z)} = p(z) = 1 + p_1 z + p_2 z^2 + \cdots \tag{8}$$

where $\operatorname{Re} p(z) > 0$. Note that $|p_n| \leqslant 2$. (See [5], p.80.)

Let $F(z) = \{f(z^m)\}^{1/m}$. We obtain from (8), (7) and (1) that

$$a_1(m) = \frac{\alpha}{m(\alpha+1)} p_1 \tag{9}$$

and

$$a_2(m) = \frac{\alpha}{2m(\alpha+2)}\left[p_2 + \frac{\alpha(m+\alpha+2)}{m(\alpha+1)^2} p_1^2\right]. \tag{10}$$

Now (5) follows from (9) upon noting that $|p_1| \leqslant 2$. For (6) we obtain from (10) that

$$\begin{aligned} |a_2(m)| &= \frac{\alpha}{2m(\alpha+2)}\left|p_2 + \frac{\alpha(m+\alpha+2)}{m(\alpha+1)^2} p_1^2\right| \\ &\leqslant \frac{\alpha}{2m(\alpha+2)}\left[|p_2| + \frac{\alpha(m+\alpha+2)}{m(\alpha+1)^2} |p_1|^2\right] \\ &\leqslant \frac{\alpha}{m(\alpha+2)}\left[1 + \frac{2\alpha(m+\alpha+2)}{m(\alpha+1)^2}\right] \end{aligned}$$

where we used the fact that $|p_1| \leqslant 2$ and $|p_2| \leqslant 2$. To show that (5) and (6) are sharp we let $p_1 = p_2 = 2$ in (9) and (10).

REMARK 1. The functions in $B_1(\alpha)$ are related to β-convex functions; $\alpha = 1/\beta$ (see [**3**], p.5), which was first introduced by Mocanu [**9**].

REMARK 2. Using Remark 1, with a little manipulation, we can obtain the estimates (5) and (6) from [**3**] and [**10**].

REMARK 3. Szynal and Wajler [**11**] obtained sharp bounds for the fourth coefficients of β-convex functions which can be used for $a_3(m)$.

REMARK 4. Finding sharp bounds for $a_n(m)$ when $n \geqslant 4$ is more difficult and is an open problem.

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