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Topological Free Entropy Dimensions in Nuclear C*-algebras and in Full Free Products of Unital C*-algebras

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Abstract. In the paper, we introduce a new concept, topological orbit dimension of an n-tuple of elements in a unital C*-algebra. Using this concept, we conclude that Voiculescu's topological free entropy dimension of every finite family of self-adjoint generators of a nuclear C*-algebra is less than or equal to 1. We also show that the Voiculescu's topological free entropy dimension is additive in the full free product of some unital C*-algebras. We show that the unital full free product of Blackadar and Kirchberg's unital MF algebras is also an MF algebra. As an application, we obtain that $\mathrm{Ext}(C_r^*(F_2) *_{\mathbb{C}} C_r^*(F_2))$ is not a group.

1 Introduction

The theory of free probability and free entropy was developed by Voiculescu starting in the 1980's. His theory plays a crucial role in the recent study of finite von Neumann algebras (see [5, 7, 9, 10, 12–14, 17, 21, 22, 26–28]). The notion of topological free entropy dimension of an n-tuple of elements in a unital C*-algebra, as a C*-analogue of free entropy dimension for finite von Neumann algebras, was also introduced by Voiculescu in [29], where basic properties of free entropy dimension are discussed.

We started our investigation on topological free entropy dimension in [18], where we computed the topological free entropy dimension of a self-adjoint element in a unital C*-algebra. Some estimations of topological free entropy dimensions in infinite dimensional, unital, simple C*-algebras with a unique trace, which include irrational rotation C*-algebra, UHF algebra, and $C^*_{red}(F_2) \otimes_{min} C^*_{red}(F_2)$, were also obtained in the same paper. In [19], we proved a formula for topological free entropy dimension in the orthogonal sum (or direct sum) of unital C*-algebras. As a corollary, we computed the topological free entropy dimension of every finite family of self-adjoint generators of a finite dimensional C*-algebra. In this article, we continue our investigation.

To study Voiculescu's topological free entropy dimension, it is necessary to consider the unital C*-algebras $\mathcal{A}$ having a set $\{x_1, \dots, x_n\}$ of self-adjoint generators for which the topological free entropy dimension $\delta_{\text{top}}(x_1, \dots, x_m)$ is defined. In [19] we used the terminology "approximation property" to describe such algebras. We show (Lemma 2.9.1) that these algebras are precisely the finitely generated MF algebras introduced by Blackadar and Kirchberg [1].

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We then introduce the notion $\mathfrak{K}^{(2)}_{\text{top}}$ of topological orbit dimension of an n-tuple of self-adjoint elements in a unital C*-algebra, a modification of "topological free orbit dimension" in [18] that is inspired by [17]. We prove that $\mathfrak{K}^{(2)}_{\text{top}}$ is a C*-algebra invariant. More precisely, we have the following result.

Theorem 3.2.1 Suppose that $\mathcal{A}$ is a unital MF algebra and $\{x_1, \dots, x_n\}$, $\{y_1, \dots, y_p\}$ are two families of self-adjoint generators of $\mathcal{A}$. Then

$$\mathfrak{K}^{(2)}_{\text{top}}(x_1, \dots, x_n) = \mathfrak{K}^{(2)}_{\text{top}}(y_1, \dots, y_p).$$

This allows us to unambiguously use the notation $\mathfrak{K}^{(2)}_{\text{top}}(\mathcal{A})$ for any finitely generated unital C*-algebra $\mathcal{A}$ to denote $\mathfrak{K}^{(2)}_{\text{top}}(x_1, \dots, x_n)$ for any set $\{x_1, \dots, x_n\}$ of self-adjoint generators.

Later (Definition 4.2.1) we define the orbit dimension capacity $\mathfrak{K}\mathfrak{K}^{(2)}_2(x_1, \dots, x_n)$.

Here is a list of some of our main results for finitely generated unital MF algebras $\mathcal{A}$ and $\mathcal{B}$.

(i) (Theorem 3.1.2) $\delta_{\text{top}}(x_1, \dots, x_m) \le \max\{\mathfrak{K}^{(2)}_{\text{top}}(\mathcal{A}), 1\}$, where $x_1, \dots, x_m$ a family of self-adjoint generators of $\mathcal{A}$ and $\delta_{\text{top}}(x_1, \dots, x_m)$ is Voiculescu's topological free entropy dimension.

(ii) (Theorem 3.3.3) $\mathfrak{K}^{(2)}_{\text{top}}(\mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})) \le \mathfrak{K}^{(2)}_{\text{top}}(\mathcal{A})$ for $n = 1, 2, \dots$.

(iii) (Theorem 3.4.3) $\mathfrak{K}^{(2)}_{\text{top}}(\mathcal{A} \oplus \mathcal{B}) \le \mathfrak{K}^{(2)}_{\text{top}}(\mathcal{A}) + \mathfrak{K}^{(2)}_{\text{top}}(\mathcal{B})$.

(iv) (Theorem 4.3.1) $\mathfrak{K}^{(2)}_{\text{top}}(\mathcal{A}) \le \mathfrak{K}\mathfrak{K}^{(2)}_2(x_1, \dots, x_m)$, where $x_1, \dots, x_m$ a family of self-adjoint generators of $\mathcal{A}$ and $\mathfrak{K}\mathfrak{K}^{(2)}_2(x_1, \dots, x_m)$ is the orbit dimension capacity in Definition 4.2.1.

(v) (Corollary 4.4.1) If $\mathcal{A}$ is nuclear, then

$$\mathfrak{K}^{(2)}_{\text{top}}(\mathcal{A}) = 0 \qquad \text{and} \qquad \delta_{\text{top}}(x_1, \dots, x_m) \le 1,$$

where $x_1, \dots, x_m$ a family of self-adjoint generators of $\mathcal{A}$.

The lower bound of topological free entropy dimension of a family of self-adjoint generators of a nuclear C*-algebra depends on the choice of a nuclear algebra. For example, $\delta_{\text{top}}(x_1, \dots, x_n) = 0$ if $x_1, \dots, x_n$ is a family of self-adjoint generators of the unitization of the C*-algebra of compact operators (see [18, Theorem 5.4.5]). On the other hand, $\delta_{\text{top}}(x_1, \dots, x_n) = 1$ if $x_1, \dots, x_n$ is a family of self-adjoint generators of a UHF algebra (see [18, Theorem 5.4.2]).

More applications of Theorem 4.3.1 can be found in Corollaries 4.5.1, 4.5.3, and 4.6.1.

The last part of this paper deals with free products. We show that a free product of a countable family of MF algebras is MF, and we show that, under certain conditions, the topological free entropy dimension is additive over free products (see Theorem 5.4.1). As a consequence, we show in Theorem 5.4.6 if the self-adjoint generators $x_1, \dots, x_n$ of an MF algebra are fully free (Definition 5.4.5), then

$$\delta_{\text{top}}(x_1, \dots, x_n) = \sum_{i=1}^{n} \delta_{\text{top}}(x_i) = n - \sum_{i=1}^{n} \frac{1}{n_i},$$

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where n_i is the number of elements in the spectrum of x_i in $\mathcal{A}_i$ (we use the notation $1/\infty = 0$).

The concept of MF algebras was introduced by Blackadar and Kirchberg in [1]. This class of C*-algebras plays an important role in the classification of C*-algebras and is connected to Brown, Douglas, and Fillmore's extension theory (see the striking result of Haagerup and Thorbjörnsen on $\mathrm{Ext}(C_r^*(F_2))$. We show that the unital full free product of countable collections of separable unital MF C*-algebras is again an MF algebra (See Theorem 5.1.4). Based on Haagerup and Thorbjørnsen's work on $\mathrm{Ext}(C_r^*(F_2))$, we are able to conclude that $\mathrm{Ext}(C_r^*(F_2) *_{\mathbb{C}} C_r^*(F_2))$ is not a group. This result provides us a new example of a C*-algebra whose extension semigroup is not a group.

The organization of the paper is as follows. In Section 2, we give the definitions of topological free entropy dimension and topological orbit dimension of n-tuple of elements in a unital C*-algebra. We observe that these dimensions on an n-tuple are defined precisely when the generated unital C*-algebras are MF algebras defined by Blackadar and Kirchberg. Some properties of topological orbit dimension are discussed in Section 3. In Section 4, we introduce the concept of orbit dimensional capacity and discuss its application in the computation of topological orbit dimension in finitely generated nuclear C*-algebras and several other classes of unital C*-algebras. In Section 5 we consider free products. We prove that topological free entropy dimension is additive in unital full free products of some unital C*-algebras. We also show that the unital full free product of a sequence of MF algebras is again an MF algebra, which allows us to show that $\mathrm{Ext}(C_r^*(F_2) *_{\mathbb{C}} C_r^*(F_2))$ is not a group.

2 Definitions and Preliminaries

In this section, we are going to recall Voiculescu's definition of topological free entropy dimension of an n-tuple of elements in a unital C*-algebra and give the definition of topological orbit dimension of an n-tuple of elements in a unital C*-algebra.

2.1 A Covering of a Set in a Metric Space

Suppose (X, d) is a metric space and K is a subset of X. A family of balls in X is called a *covering* of K if the union of these balls contains K and the centers of these balls lie in K.

2.2 Covering Numbers in Complex Matrix Algebra $(\mathcal{M}_k(\mathbb{C}))^n$

Let $\mathcal{M}_k(\mathbb{C})$ be the $k \times k$ full matrix algebra with entries in $\mathbb{C}$, and let τ_k be the normalized trace on $\mathcal{M}_k(\mathbb{C})$, *i.e.*, $\tau_k = \frac{1}{k}\,\mathrm{Tr}$, where Tr is the usual trace on $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{U}(k)$ denote the group of all unitary matrices in $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{M}_k(\mathbb{C})^n$ denote the direct sum of n copies of $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{M}_k^{s.a}(\mathbb{C})$ be the subset of $\mathcal{M}_k(\mathbb{C})$ consisting of all self-adjoint matrices of $\mathcal{M}_k(\mathbb{C})$. Let $(\mathcal{M}_k^{s.a}(\mathbb{C}))^n$ be the direct sum (or orthogonal sum) of n copies of $\mathcal{M}_k^{s.a}(\mathbb{C})$. Let $\|\cdot\|$ be the operator norm on $\mathcal{M}_k(\mathbb{C})^n$ defined by

$$\|(A_1, \dots, A_n)\| = \max\{\|A_1\|, \dots, \|A_n\|\}$$

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for all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$. Let $\|\cdot\|_2$ denote the trace norm induced by τ_k on $\mathcal{M}_k(\mathbb{C})^n$, *i.e.,*

$$\|(A_1, \ldots, A_n)\|_2 = \sqrt{\tau_k(A_1^* A_1) + \cdots + \tau_k(A_n^* A_n)}$$

for all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$.

For every $\omega > 0$, we define the ω-$\|\cdot\|$-ball $\mathrm{Ball}(B_1, \ldots, B_n; \omega, \|\cdot\|)$ centered at $(B_1, \ldots, B_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that

$$\|(A_1, \ldots, A_n) - (B_1, \ldots, B_n)\| < \omega.$$

Definition 2.2.1 Suppose that Σ is a subset of $\mathcal{M}_k(\mathbb{C})^n$. We define the covering number $\nu_\infty(\Sigma, \omega)$ to be the minimal number of ω-$\|\cdot\|$-balls that constitute a covering of Σ in $\mathcal{M}_k(\mathbb{C})^n$.

For every $\omega > 0$, we define the ω-$\|\cdot\|_2$-ball $\mathrm{Ball}(B_1, \ldots, B_n; \omega, \|\cdot\|_2)$ centered at $(B_1, \ldots, B_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that

$$\|(A_1, \ldots, A_n) - (B_1, \ldots, B_n)\|_2 < \omega.$$

Definition 2.2.2 Suppose that Σ is a subset of $\mathcal{M}_k(\mathbb{C})^n$. We define the covering number $\nu_2(\Sigma, \omega)$ to be the minimal number of ω-$\|\cdot\|_2$-balls that constitute a covering of Σ in $\mathcal{M}_k(\mathbb{C})^n$.

2.3 Unitary Orbits of Balls in $\mathcal{M}_k(\mathbb{C})^n$

For every $\omega > 0$, we define the ω-orbit-$\|\cdot\|_2$-ball $\mathcal{U}(B_1, \ldots, B_n; \omega, \|\cdot\|_2)$ centered at $(B_1, \ldots, B_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that there exists a unitary matrix W in $\mathcal{U}(k)$ satisfying

$$\|(A_1, \ldots, A_n) - (WB_1W^*, \ldots, WB_nW^*)\|_2 < \omega.$$

Definition 2.3.1 Suppose that Σ is a subset of $\mathcal{M}_k(\mathbb{C})^n$. We define the covering number $o_2(\Sigma, \omega)$ to be the minimal number of ω-orbit-$\|\cdot\|_2$-balls that constitute a covering of Σ in $\mathcal{M}_k(\mathbb{C})^n$.

2.4 Noncommutative Polynomials

In this article, we always assume that $\mathcal{A}$ is a unital C*-algebra. Let $x_1, \ldots, x_n$, $y_1, \ldots, y_m$ be self-adjoint elements in $\mathcal{A}$. Let $\mathbb{C}\langle X_1, \ldots, X_n, Y_1, \ldots, Y_m\rangle$ be the set of all noncommutative polynomials in the indeterminants $X_1, \ldots, X_n, Y_1, \ldots, Y_m$. Let $\mathbb{C}_{\mathbb{Q}} = \mathbb{Q} + i\mathbb{Q}$ denote the complex-rational numbers, *i.e.,* the numbers whose real and imaginary parts are rational. Then the set $\mathbb{C}_{\mathbb{Q}}\langle X_1, \ldots, X_n, Y_1, \ldots, Y_m\rangle$ of noncommutative polynomials with complex-rational coefficients is countable. Throughout this paper we write

$$\mathbb{C}_{\mathbb{Q}}\langle X_1, \ldots, X_n, Y_1, \ldots, Y_m\rangle = \{P_r : r \in \mathbb{N}\} \text{ and } \mathbb{C}_{\mathbb{Q}}\langle X_1, \ldots, X_n\rangle = \{Q_r : r \in \mathbb{N}\}.$$

Remark 2.4.1 We always assume that $1 \in \mathbb{C}\langle X_1, \ldots, X_n, Y_1, \ldots, Y_m\rangle$.

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2.5 Voiculescu's Norm-microstates Space

For all integers $r, k \geq 1$, real numbers $R, \epsilon > 0$, $r \in \mathbb{N}$, we define

$$\Gamma_R^{(\text{top})}(x_1, \ldots, x_n, y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r),$$

where $P_1, \ldots, P_r$ are the first r polynomials in $\mathbb{C}_{\mathbb{Q}}\langle X_1, \ldots, X_n, Y_1, \ldots, Y_m\rangle$ to be the subset of $(\mathcal{M}_k^{s.a}(\mathbb{C}))^{n+m}$ consisting of all these

$$(A_1, \ldots, A_n, B_1, \ldots, B_m) \in (\mathcal{M}_k^{s.a}(\mathbb{C}))^{n+m}$$

satisfying

$$\max\{\|A_1\|, \ldots, \|A_n\|, \|B_1\|, \ldots, \|B_m\|\} \leq R$$

and

$$|\|P_j(A_1, \ldots, A_n, B_1, \ldots, B_m)\| - \|P_j(x_1, \ldots, x_n, y_1, \ldots, y_m)\|| \leq \epsilon, \qquad \forall\, 1 \leq j \leq r.$$

Define the norm-microstates space of $x_1, \ldots, x_n$ in the presence of $y_1, \ldots, y_m$, denoted by

$$\Gamma_R^{(\text{top})}(x_1, \ldots, x_n : y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r),$$

to be the projection of $\Gamma_R^{(\text{top})}(x_1, \ldots, x_n, y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r)$ onto the space $(\mathcal{M}_k^{s.a}(\mathbb{C}))^n$ via the mapping

$$(A_1, \ldots, A_n, B_1, \ldots, B_m) \to (A_1, \ldots, A_n).$$

2.6 Voiculescu's Topological Free Entropy Dimension (see [29])

Recall that if n is a positive integer, $\Sigma \subset (\mathcal{M}_k^{s.a}(\mathbb{C}))^n$, and $\omega > 0$, we define $\nu_\infty(\Sigma, \omega)$ to be the covering number of the set Σ by ω-$\|\cdot\|$-balls in the metric space $(\mathcal{M}_k^{s.a}(\mathbb{C}))^n$ equipped with an operator norm.

Definition 2.6.1 Define

$$\delta_{\text{top}}(x_1, \ldots, x_n; \omega) = \sup_{R>0} \inf_{\varepsilon>0, r\in\mathbb{N}} \limsup_{k\to\infty} \frac{\log\big(\nu_\infty(\Gamma_R^{(\text{top})}(x_1, \ldots, x_n; k, \epsilon, Q_1, \ldots, Q_r), \omega)\big)}{-k^2 \log \omega}.$$

The topological free entropy dimension of $x_1, \ldots, x_n$ is defined by

$$\delta_{\text{top}}(x_1, \ldots, x_n) = \limsup_{\omega\to 0^+} \delta_{\text{top}}(x_1, \ldots, x_n; \omega)$$

Similarly, define

$$\delta_{\text{top}}(x_1, \ldots, x_n : y_1, \ldots, y_m; \omega) = \sup_{R>0} \inf_{\varepsilon>0, r\in\mathbb{N}} \limsup_{k\to\infty} \frac{\log\big(\nu_\infty(\Gamma_R^{(\text{top})}(x_1, \ldots, x_n : y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r), \omega)\big)}{-k^2 \log \omega}$$

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The topological free entropy dimension of $x_1, \ldots, x_n$ in the presence of $y_1, \ldots, y_m$ is defined by

$$\delta_{\text{top}}(x_1, \ldots, x_n : y_1, \ldots, y_m) = \limsup_{\omega \to 0^+} \delta_{\text{top}}(x_1, \ldots, x_n : y_1, \ldots, y_m; \omega)$$

Remark 2.6.2 It is clear from the definition that the supremum over $R > 0$ is unnecessary. In fact,

$$\delta_{\text{top}}(x_1, \ldots, x_n) = \limsup_{\omega \to 0^+} \inf_{\varepsilon > 0, r \in \mathbb{N}} \limsup_{k \to \infty} \frac{\log\big(\nu_\infty(\Gamma_R^{(\text{top})}(x_1, \ldots, x_n; k, \epsilon, Q_1, \ldots, Q_r), \omega)\big)}{-k^2 \log \omega}$$

whenever $R > \max\{\|x_1\|, \ldots, \|x_n\|\}$, and

$$\delta_{\text{top}}(x_1, \ldots, x_n : y_1, \ldots, y_m) = \limsup_{\omega \to 0^+} \inf_{\varepsilon > 0, r \in \mathbb{N}} \limsup_{k \to \infty} \frac{\log\big(\nu_\infty(\Gamma_R^{(\text{top})}(x_1, \ldots, x_n : y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r), \omega)\big)}{-k^2 \log \omega}$$

whenever $R > \max\{\|x_1\|, \ldots, \|x_n\|, \|y_1\|, \ldots, \|y_m\|\}$. This is because when ε is sufficiently small and r is sufficiently large, the conditions involving R are automatically satisfied.

2.7 Topological Orbit Dimension $\mathfrak{K}_{\text{top}}^{(2)}$

Recall that if n is a positive integer, $\Sigma \subset (\mathcal{M}_k^{s.a}(\mathbb{C}))^n$ and $\omega > 0$, we define $o_2(\Sigma, \omega)$ to be the covering number of the set Σ by ω-orbit-$\|\cdot\|_2$-balls in the metric space $(\mathcal{M}_k^{s.a.}(\mathbb{C}))^n$ equipped with the trace norm.

Definition 2.7.1 Define

$$\mathfrak{K}_{\text{top}}^{(2)}(x_1, \ldots, x_n; \omega) = \sup_{R > 0} \inf_{\varepsilon > 0, r \in \mathbb{N}} \limsup_{k \to \infty} \frac{\log\big(o_2(\Gamma_R^{(\text{top})}(x_1, \ldots, x_n; k, \epsilon, Q_1, \ldots, Q_r), \omega)\big)}{k^2}$$

and

$$\mathfrak{K}_{\text{top}}^{(2)}(x_1, \ldots, x_n : y_1, \ldots, y_m; \omega) = \sup_{R > 0} \inf_{\varepsilon > 0, r \in \mathbb{N}} \limsup_{k \to \infty} \frac{\log\big(o_2(\Gamma_R^{(\text{top})}(x_1, \ldots, x_n : y_1, \ldots, y_m; k, \epsilon, P_1, \ldots, P_r), \omega)\big)}{k^2}$$

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Remark 2.7.2 The values of

$$\mathfrak{K}^{(2)}_{\text{top}}(x_1,\dots,x_n;\omega) \quad \text{and} \quad \mathfrak{K}^{(2)}_{\text{top}}(x_1,\dots,x_n:y_1,\dots,y_m;\omega)$$

increase as ω decreases.

The topological orbit dimension of $x_1,\dots,x_n$ in the presence of $x_1,\dots,x_n$ is defined by

$$\mathfrak{K}^{(2)}_{\text{top}}(x_1,\dots,\ x_n) = \lim_{\omega\to 0^+} \mathfrak{K}^{(2)}_{\text{top}}(x_1,\dots,x_n;\omega)$$

The topological orbit dimension of $x_1,\dots,x_n$ in the presence of $y_1,\dots,y_m$ is defined by

$$\mathfrak{K}^{(2)}_{\text{top}}(x_1,\dots,\ x_n:y_1,\dots,y_m) = \lim_{\omega\to 0^+} \mathfrak{K}^{(2)}_{\text{top}}(x_1,\dots,x_n:y_1,\dots,y_m;\omega)$$

Remark 2.7.3 In the notation $\mathfrak{K}^{(2)}_{\text{top}}$, the subscript "top" stands for the norm-microstates space, and the superscript "(2)" stands for the using of unitary-orbit-$\|\cdot\|_2$-balls when

counting the covering numbers of the norm-microstates spaces.

Remark 2.7.4 As with the definition of topological free entropy dimension, in the definition of topological orbit dimension, the supremum over $R > 0$ is unnecessary.

2.8 C*-algebra Ultraproducts

For an introduction to ultraproducts of C*-algebras, see [11]. Suppose $\{\mathcal{M}_{k_m}(\mathbb{C})\}_{m=1}^{\infty}$ is a sequence of complex matrix algebras where k_m goes to infinity as m goes to infinity. Let γ be a free ultrafilter in $\beta(\mathbb{N})\setminus\mathbb{N}$. We can introduce a unital C*-algebra $\prod_{m=1}^{\infty}\mathcal{M}_{k_m}(\mathbb{C})$ as follows:

$$\prod_{m=1}^{\infty} M_{k_m}(\mathbb{C}) = \Big\{ (Y_m)_{m=1}^{\infty} \mid \forall\, m \geq 1,\ Y_m \in M_{k_m}(C) \text{ and } \sup_{m\geq 1}\|Y_m\| < \infty \Big\}.$$

We can also introduce a norm closed two-sided ideal $\mathcal{I}_\infty$ as follows.

$$\mathcal{I}_\infty = \Big\{ (Y_m)_{m=1}^{\infty} \in \prod_{m=1}^{\infty} M_{k_m}(\mathbb{C}) : \lim_{m\to\gamma}\|Y_m\| = 0 \Big\}.$$

Definition 2.8.1 The C*-algebra ultraproduct of $\{\mathcal{M}_{k_m}(\mathbb{C})\}_{m=1}^{\infty}$ along the ultrafilter γ, denoted by $\prod^{\gamma}\mathcal{M}_{k_m}(\mathbb{C})$, is defined to be the quotient algebra, $\prod_{m=1}^{\infty}\mathcal{M}_{k_m}(\mathbb{C})/\mathcal{I}_\infty$, of $\prod_{m=1}^{\infty}\mathcal{M}_{k_m}(\mathbb{C})$ by the ideal $\mathcal{I}_\infty$. The image of $(Y_m)_{m=1}^{\infty} \in \prod_{m=1}^{\infty}\mathcal{M}_{k_m}(\mathbb{C})$ in $\prod^{\gamma}\mathcal{M}_{k_m}(\mathbb{C})$ is denoted by $[(Y_m)_\gamma]$.

2.9 MF Algebras

In the definitions of $\delta_{\text{top}}(x_1,\dots,x_n)$ and $\mathfrak{K}^{(2)}_{\text{top}}(x_1,\dots,x_n)$, where $x_1,\dots,x_n$ are self-adjoint generators of a unital C*-algebra $\mathcal{A}$, it is necessary that a suitable number

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of the Voiculescu's norm-microstate spaces Γ_{top}-sets be nonempty. More precisely, for every family $\{P_r\}_{r=1}^m$ of noncommutative polynomials in $\mathbb{C}_{\mathbb{Q}}\langle X_1,\dots,X_n\rangle$ with rational coefficients and $R > \max\{\|x_1\|,\dots,\|x_n\|\}$, $r > 0$, and $\epsilon > 0$, there is a sequence of positive integers $k_1 < k_2 < \cdots$ such that

$$\Gamma_R^{(\text{top})}(x_1,\dots,x_n;k_s,\epsilon,P_1,\dots,P_r) \neq \varnothing, \qquad \forall\, s \geq 1.$$

In [19] we used the term *approximation property* to describe the preceding condition. However, it turns out that this property is equivalent to $\mathcal{A}$ being an MF algebra in the sense of Blackadar and Kirchberg [1], *i.e.,* there is a unital embedding from $\mathcal{A}$ into $\prod_k \mathcal{M}_{n_k}(\mathbb{C})/\sum \mathcal{M}_{n_k}(\mathbb{C})$ for a sequence $\{n_k\}_{k=1}^\infty$ of positive integers. The next lemma follows immediately from [1, Theorem 3.2.2] and [18, Lemma 5.6]. We will give a new characterization of MF algebras (Theorem 5.1.2), and use it to show that MF algebras are closed under free products (Theorem 5.1.4).

Lemma 2.9.1 *Suppose that $\mathcal{A}$ is a unital C^*-algebra and $x_1,\dots,x_n$ is a family of self-adjoint generators in $\mathcal{A}$. The following are equivalent:*

(i) *$\delta_{\text{top}}(x_1,\dots,x_n)$ is defined;*
(ii) *$\mathcal{A}$ is an MF algebra;*
(iii) *There are a sequence of positive integers $\{m_k\}_{k=1}^\infty$ and self-adjoint matrices $A_1^{(k)},\dots,A_n^{(k)}$ in $\mathcal{M}_{m_k}^{s.a.}(\mathbb{C})$ for $k = 1,2,\dots$, such that, $\forall\, P \in \mathbb{C}\langle X_1,\dots,X_n\rangle$,*

$$\lim_{k\to\infty} \|P(A_1^{(k)},\dots,A_n^{(k)})\| = \|P(x_1,\dots,x_n)\|,$$

where $\mathbb{C}\langle X_1,\dots,X_n\rangle$ is the set of all noncommutative polynomials in the indeterminants $X_1,\dots,X_n$.

In view of the preceding lemma we should only talk about $\delta_{\text{top}}(x_1,\dots,x_n)$ or $\mathfrak{K}_{\text{top}}^{(2)}(x_1,\dots,x_n)$ when the unital C*-algebra $\mathcal{A}$ generated by $x_1,\dots,x_n$ is an MF algebra.

3 Properties of Topological Orbit Dimension $\mathfrak{K}_{\text{top}}^{(2)}$

In this section, we are going to discuss properties of the topological orbit dimension $\mathfrak{K}_{\text{top}}^{(2)}$.

3.1 Some Basic Properties

The following result explains the relationship between Voiculescu's topological free entropy dimension and topological orbit dimension of n-tuple of elements in a unital C^*-algebra.

Lemma 3.1.1 *Suppose that $\mathcal{A}$ is a unital C^*-algebra and $x_1,\dots,x_n$ is a family of self-adjoint elements in $\mathcal{A}$. If $\mathfrak{K}_{\text{top}}^{(2)}(x_1,\dots,x_n) < \infty$, then $\delta_{\text{top}}(x_1,\dots,x_n) \leq 1$.*

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Proof Let $\mathbb{C}_{\mathbb{Q}}\langle X_1,\dots,X_n\rangle = \{P_r\}_{r=1}^{\infty}$ be the collection of all noncommutative polynomials $X_1,\dots,X_n$ with rational coefficients. For any

$$0<\omega<1/10, \quad R>\max\{\|x_1\|,\dots,\|x_n\|\},$$

we know from Remark 2.7.2 that

$$\inf_{r\in\mathbb{N},\ \epsilon>0}\ \limsup_{k\to\infty}\frac{\log(o_2(\Gamma_R^{(\mathrm{top})}(x_1,\dots,x_n,k,\epsilon,P_1,\dots,P_r),\omega))}{-k^2\log\omega}\le \mathfrak{K}^{(2)}_{\mathrm{top}}(x_1,\dots,x_n)\cdot\frac{1}{-\log\omega}.$$

By a result of S. Szarek's in [23], there is a family of unitary matrices $\{U_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{U}(k)$ such that

(i) $\{\mathrm{Ball}(U_\lambda;\frac{\omega}{R},\|\cdot\|)\}_{\lambda\in\Lambda}$ is a covering of $\mathcal{U}(k)$ and

(ii) the cardinality of Λ, $|\Lambda|\le (CR/\omega)^{k^2}$, where C is a constant independent of k,ω.

Thus from the relationship between covering number (see Definition 2.2.2) and the unitary orbit covering number (see Definition 2.3.1), we have

$$\begin{aligned}
&\inf_{r\in\mathbb{N},\ \epsilon>0}\ \limsup_{k\to\infty}\frac{\log(\nu_2(\Gamma_R^{(\mathrm{top})}(x_1,\dots,x_n,k,\epsilon,P_1,\dots,P_r),3\omega))}{-k^2\log\omega}\\
&\quad\le \inf_{r\in\mathbb{N},\ \epsilon>0}\ \limsup_{k\to\infty}\frac{\log(o_2(\Gamma_R^{(\mathrm{top})}(x_1,\dots,x_n,k,\epsilon,P_1,\dots,P_r),\omega)\cdot\left(\frac{CR}{\omega}\right)^{k^2})}{-k^2\log\omega}\\
&\quad\le 1+\frac{\log C+\log R}{-\log\omega}+\mathfrak{K}^{(2)}_{\mathrm{top}}(x_1,\dots,x_n)\cdot\frac{1}{-\log\omega}.
\end{aligned}$$

Now the result of the theorem follows directly from the definitions of the topological free entropy dimension and the topological orbit dimension, together with Remarks 2.6.2 and 2.7.4 and the remark in [29, Section 6] (or [18, Proposition 5.1]). ■

A direct consequence of the preceding lemma is the following theorem.

Theorem 3.1.2 *Suppose that $\mathcal{A}$ is a unital C^*-algebra and $x_1,\dots,x_n$ is a family of self-adjoint elements in $\mathcal{A}$. Then*

$$\delta_{\mathrm{top}}(x_1,\dots,x_n)\le\max\{\mathfrak{K}^{(2)}_{\mathrm{top}}(x_1,\dots,x_n),1\}.$$

In particular, if $\mathfrak{K}^{(2)}_{\mathrm{top}}(x_1,\dots,x_n)=0$, then $\delta_{\mathrm{top}}(x_1,\dots,x_n)\le 1$.

The following lemma will be needed in the proof of Theorem 3.2.1.

Lemma 3.1.3 *Let $x_1,\dots,x_n,y_1,\dots,y_p$ be self-adjoint elements in a unital C^*-algebra $\mathcal{A}$. If $y_1,\dots,y_p$ are in the C^*-subalgebra generated by $x_1,\dots,x_n$ in $\mathcal{A}$, then, for every $\omega>0$,*

$$\mathfrak{K}^{(2)}_{\mathrm{top}}(x_1,\dots,x_n;4\omega)\le\mathfrak{K}^{(2)}_{\mathrm{top}}(x_1,\dots,x_n:y_1,\dots,y_p;2\omega)\le\mathfrak{K}^{(2)}_{\mathrm{top}}(x_1,\dots,x_n;\omega).$$

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Proof It is a straightforward adaptation of the proof of [27, Prop. 1.6]. Suppose that $\{P_r\}_{r=1}^{\infty} = \mathbb{C}_{\mathbb{Q}}\langle X_1,\dots,X_n,Y_1,\dots,Y_m\rangle$, and $\{Q_s\}_{s=1}^{\infty} = \mathbb{C}_{\mathbb{Q}}\langle X_1,\dots,X_n\rangle$ respectively, are families of noncommutative polynomials in $\mathbb{C}\langle X_1,\dots,X_n,Y_1,\dots,Y_m\rangle$, and $\mathbb{C}\langle X_1,\dots,X_n\rangle$ respectively, with rational coefficients.

Given $R > \max_{1\le j\le n}\|x_j\| + \max_{1\le j\le p}\|y_j\|$, $r,s\in\mathbb{N}$ and $\epsilon>0$, we can find $r_1,s_1\in\mathbb{N}$ and $\epsilon_1>0,\epsilon_2>0$ such that, for all $k\in\mathbb{N}$,

$$\Gamma_R^{(\mathrm{top})}(x_1,\dots,x_n;k,\epsilon_1,Q_1,\dots,Q_{s_1}) \subseteq \Gamma_R^{(\mathrm{top})}(x_1,\dots,x_n:y_1,\dots,y_p;k,\epsilon,P_1,\dots,P_r),$$

$$\Gamma_R^{(\mathrm{top})}(x_1,\dots,x_n:y_1,\dots,y_p;k,\epsilon_2,P_1,\dots,P_{r_1}) \subseteq \Gamma_R^{(\mathrm{top})}(x_1,\dots,x_n;k,\epsilon,Q_1,\dots,Q_s).$$

Hence,

$$\begin{aligned}
&o_2(\Gamma_R^{(\mathrm{top})}(x_1,\dots,x_n;k,\epsilon_1,Q_1,\dots,Q_{s_1}),4\omega)\\
&\quad\le o_2(\Gamma_R^{(\mathrm{top})}(x_1,\dots,x_n:y_1,\dots,y_p;k,\epsilon,P_1,\dots,P_r),2\omega)\\
&o_2(\Gamma_R^{(\mathrm{top})}(x_1,\dots,x_n:y_1,\dots,y_p;k,\epsilon_2,P_1,\dots,P_{r_1}),2\omega)\\
&\quad\le o_2(\Gamma_R^{(\mathrm{top})}(x_1,\dots,x_n;k,\epsilon,Q_1,\dots,Q_s),\omega),
\end{aligned}$$

for all $\omega>0$. Therefore, for all $\omega>0$,

$$\begin{aligned}
&\inf_{\epsilon_1>0,s_1\in\mathbb{N}}\limsup_{k\to\infty}\frac{\log(o_2(\Gamma_R^{(\mathrm{top})}(x_1,\dots,x_n;k,\epsilon_1,Q_1,\dots,Q_{s_1}),4\omega))}{k^2}\le\\
&\qquad\limsup_{k\to\infty}\frac{\log(o_2(\Gamma_R^{(\mathrm{top})}(x_1,\dots,x_n:y_1,\dots,y_p;k,\epsilon,P_1,\dots,P_r),2\omega))}{k^2};
\end{aligned}$$

and

$$\begin{aligned}
&\inf_{\epsilon_2>0,r_1\in\mathbb{N}}\limsup_{k\to\infty}\frac{\log(o_2(\Gamma_R^{(\mathrm{top})}(x_1,\dots,x_n:y_1,\dots,y_p;k,\epsilon_2,P_1,\dots,P_{r_1}),2\omega))}{k^2}\le\\
&\qquad\limsup_{k\to\infty}\frac{\log(o_2(\Gamma_R^{(\mathrm{top})}(x_1,\dots,x_n;k,\epsilon,Q_1,\dots,Q_s),\omega))}{k^2}.
\end{aligned}$$

It follows that, for all $\omega>0$,

$$\begin{aligned}
&\inf_{\epsilon_1>0,s_1\in\mathbb{N}}\limsup_{k\to\infty}\frac{\log(o_2(\Gamma_R^{(\mathrm{top})}(x_1,\dots,x_n;k,\epsilon_1,Q_1,\dots,Q_{s_1}),4\omega))}{k^2}\le\\
&\qquad\inf_{\epsilon>0,r\in\mathbb{N}}\limsup_{k\to\infty}\frac{\log(o_2(\Gamma_R^{(\mathrm{top})}(x_1,\dots,x_n:y_1,\dots,y_p;k,\epsilon,P_1,\dots,P_r),2\omega))}{k^2};
\end{aligned}$$

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and

$$\inf_{\epsilon_2>0,r_1\in\mathbb{N}} \limsup_{k\to\infty} \frac{\log(o_2(\Gamma_R^{(\mathrm{top})}(x_1,\ldots,x_n : y_1,\ldots,y_p; k,\epsilon_2,P_1,\ldots,P_{r_1}),2\omega))}{k^2} \le \inf_{\epsilon>0,s\in\mathbb{N}} \limsup_{k\to\infty} \frac{\log(o_2(\Gamma_R^{(\mathrm{top})}(x_1,\ldots,x_n; k,\epsilon,Q_1,\ldots,Q_s),\omega))}{k^2}.$$

The rest follows from the definitions. ∎

3.2 A C*-algebra Invariant

Our next result shows that the topological orbit dimension $\mathfrak{K}_{\mathrm{top}}^{(2)}$ is in fact a C*-algebra invariant. In view of this result, we use $\mathfrak{K}_{\mathrm{top}}^{(2)}(\mathcal{A})$ to denote $\mathfrak{K}_{\mathrm{top}}^{(2)}(x_1,\ldots,x_n)$ for an arbitrary generating set $\{x_1,\ldots,x_n\}$ for $\mathcal{A}$.

Theorem 3.2.1 *Suppose that $\mathcal{A}$ is a unital C*-algebra and $\{x_1,\ldots,x_n\}$, $\{y_1,\ldots,y_p\}$ are two families of self-adjoint generators of $\mathcal{A}$. Then*

$$\mathfrak{K}_{\mathrm{top}}^{(2)}(x_1,\ldots,x_n) = \mathfrak{K}_{\mathrm{top}}^{(2)}(y_1,\ldots,y_p).$$

Proof Note that $x_1,\ldots,x_n$ are elements in $\mathcal{A}$ that generate $\mathcal{A}$ as a C*-algebra. For every $0<\omega<1$, there exists a family of noncommutative polynomials $\psi_i(x_1,\ldots,x_n)$, $1\le i\le p$, such that

$$\left(\sum_{i=1}^{p}\|y_i-\psi_i(x_1,\ldots,x_n)\|^2\right)^{1/2} < \frac{\omega}{4}.$$

For such a family of polynomials $\psi_1,\ldots,\psi_p$ and every

$$R>\max\{\|x_1\|,\ldots,\|x_n\|,\|y_1\|,\ldots,\|y_p\|\}$$

there always exists a constant $D\ge 1$, depending only on $R,\psi_1,\ldots,\psi_n$, such that

$$\left(\sum_{i=1}^{p}\|\psi_i(A_1,\ldots,A_n)-\psi_i(B_1,\ldots,B_n)\|_2^2\right)^{1/2} \le D\|(A_1,\ldots,A_n)-(B_1,\ldots,B_n)\|_2,$$

for all $(A_1,\ldots,A_n),(B_1,\ldots,B_n)$ in $\mathcal{M}_k(\mathbb{C})^n$, all $k\in\mathbb{N}$, satisfying $\|A_j\|,\|B_j\|\le R$, for $1\le j\le n$.

Suppose that $\{P_r\}_{r=1}^{\infty}$ and $\{Q_s\}_{s=1}^{\infty}$ are the families of noncommutative polynomials in $\mathbb{C}\langle Y_1,\ldots,Y_p,X_1,\ldots,X_n\rangle$, and $\mathbb{C}\langle X_1,\ldots,X_n\rangle$ respectively, with rational coefficients.

For any $s\ge 1$, $\epsilon>0$, when r is sufficiently large and ϵ' is sufficiently small, every

$$(H_1,\ldots,H_p,A_1,\ldots,A_n)$$

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in

$$\Gamma_R^{(\mathrm{top})}(y_1,\ldots,y_p,x_1,\ldots,x_n;k,\epsilon',P_1,\ldots,P_r)$$

satisfies

$$\Big(\sum_{i=1}^{p}\|H_i-\psi_i(A_1,\ldots,A_n)\|^2\Big)^{1/2}\le\frac{\omega}{4};$$

and

$$(A_1,\ldots,A_n)\in\Gamma_R^{(\mathrm{top})}(x_1,\ldots,x_n;k,\epsilon,Q_1,\ldots,Q_s).$$

On the other hand, by the definition of the orbit covering number, we know there exists a set $\{\mathcal{U}(B_1^\lambda,\ldots,B_n^\lambda;\frac{\omega}{4D},\|\cdot\|_2)\}_{\lambda\in\Lambda_k}$ of $\frac{\omega}{4D}$-orbit-$\|\cdot\|_2$-balls that cover

$$\Gamma R^{(\mathrm{top})}(x_1,\ldots,x_n;k,\epsilon,Q_1,\ldots,Q_s)$$

with the cardinality of Λ_k satisfying

$$|\Lambda_k|=o_2\Big(\Gamma_R^{(\mathrm{top})}(x_1,\ldots,x_n;k,\epsilon,Q_1,\ldots,Q_s),\frac{\omega}{4D}\Big).$$

Thus for such $(A_1,\ldots,A_n)$ in $\Gamma_R^{(\mathrm{top})}(x_1,\ldots,x_n;k,\epsilon,Q_1,\ldots,Q_s)$, there exists some $\lambda\in\Lambda_k$ and $W\in\mathcal{U}(k)$ such that

$$\|(A_1,\ldots,A_n)-(WB_1^\lambda W^*,\ldots,WB_n^\lambda W^*)\|_2\le\frac{\omega}{4D}.$$

It follows that

$$\begin{aligned}
&\Big(\sum_{i=1}^{p}\|H_i-W\psi_i(B_1^\lambda,\ldots,B_n^\lambda)W^*\|_2^2\Big)^{1/2}\\
&\quad=\Big(\sum_{i=1}^{p}\|H_i-\psi_i(WB_1^\lambda W^*,\ldots,WB_n^\lambda W^*)\|_2^2\Big)^{1/2}\\
&\quad\le\Big(\sum_{i=1}^{p}\|H_i-\psi_i(A_1,\ldots,A_n)\|_2^2\Big)^{1/2}\\
&\qquad+\Big(\sum_{i=1}^{p}\|\psi_i(A_1,\ldots,A_n)-\psi_i(WB_1^\lambda W^*,\ldots,WB_n^\lambda W^*)\|_2^2\Big)^{1/2}\\
&\quad\le\Big(\sum_{i=1}^{p}\|H_i-\psi_i(A_1,\ldots,A_n)\|^2\Big)^{1/2}+\frac{\omega}{4}\le\frac{\omega}{2},
\end{aligned}$$

for some $\lambda\in\Lambda_k$ and $W\in\mathcal{U}(k)$, *i.e.*,

$$(H_1,\ldots,H_p)\in\mathcal{U}(\psi_1(B_1^\lambda,\ldots,B_n^\lambda),\ldots,\psi_p(B_1^\lambda,\ldots,B_n^\lambda);\frac{\omega}{2}).$$

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Hence, for given $s \in \mathbb{N}$ and $\epsilon > 0$, when ϵ' is small enough and r is large enough,

$$o_2\big(\Gamma_R^{(\mathrm{top})}(y_1,\ldots,y_p : x_1,\ldots,x_n; k,\epsilon',P_1,\ldots,P_r),\omega\big) \le |\Lambda_k| = o_2(\Gamma_R^{(\mathrm{top})}\big(x_1,\ldots,x_n;k,\epsilon,Q_1,\ldots,Q_s\big), \frac{\omega}{4D}).$$

It follows that

$$\inf_{r\in\mathbb{N},\ \epsilon'>0}\lim_{k\to\infty}\frac{\log(o_2(\Gamma_R^{(\mathrm{top})}(y_1,\ldots,y_p : x_1,\ldots,x_n;k,\epsilon',P_1,\ldots,P_r),\omega))}{k^2} \le \lim_{k\to\infty}\frac{\log(o_2(\Gamma_R^{(\mathrm{top})}(x_1,\ldots,x_n;k,\epsilon,Q_1,\ldots,Q_s),\frac{\omega}{4D}))}{k^2}.$$

Therefore, by the definition of the topological orbit dimension and Remark 2.7.4, we get

$$\begin{aligned}\mathfrak{K}_{\mathrm{top}}^{(2)}(y_1,\ldots,y_p : x_1,\ldots,x_n;\omega) &= \mathfrak{K}_{\mathrm{top}}^{(2)}(y_1,\ldots,y_p : x_1,\ldots,x_n;\omega,R)\\ &= \inf_{\epsilon'>0,r\in\mathbb{N}}\lim_{k\to\infty}\frac{\log(o_2(\Gamma_R^{(\mathrm{top})}(y_1,\ldots,y_p : x_1,\ldots,x_n;k,\epsilon',P_1,\ldots,P_r),\omega))}{k^2}\\ &\le \inf_{\epsilon>0,s\in\mathbb{N}}\limsup_{k\to\infty}\frac{\log(o_2(\Gamma_R^{(\mathrm{top})}(x_1,\ldots,x_n;k,\epsilon,Q_1,\ldots,Q_s),\frac{\omega}{4D}))}{k^2}\\ &\le \mathfrak{K}_{\mathrm{top}}^{(2)}(x_1,\ldots,x_n),\end{aligned}$$

where the last inequality follows from the fact that $\mathfrak{K}_{\mathrm{top}}^{(2)}(x_1,\ldots,x_n;\omega)$ increases as ω decreases. Thus, by Lemma 3.1.3, we get

$$\mathfrak{K}_{\mathrm{top}}^{(2)}(y_1,\ldots,y_p) \le \mathfrak{K}_{\mathrm{top}}^{(2)}(x_1,\ldots,x_n).$$

Similarly, we have

$$\mathfrak{K}_{\mathrm{top}}^{(2)}(x_1,\ldots,x_n) \le \mathfrak{K}_{\mathrm{top}}^{(2)}(y_1,\ldots,y_p),$$

which completes the proof. ■

A slight modification of the proof of Theorem 3.2.1 will prove the semicontinuity of $\mathfrak{K}_{\mathrm{top}}^{(2)}$ with respect to direct limits.

Theorem 3.2.2 *Suppose that $\mathcal{A}$ is a unital finitely generated MF C^*-algebra. Suppose $\mathcal{A}_i$, $j = 1,2,\ldots$ is an increasing sequence of unital finitely generated C^*-subalgebras of $\mathcal{A}$ such that $\bigcup_{j=1}^{\infty}\mathcal{A}_j$ is norm dense in $\mathcal{A}$. Then*

$$\mathfrak{K}_{\mathrm{top}}^{(2)}(\mathcal{A}) \le \liminf_{j\to\infty}\mathfrak{K}_{\mathrm{top}}^{(2)}(\mathcal{A}_j).$$

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Proof Suppose $x_1, \dots, x_n$ is a family of self-adjoint generators of $\mathcal{A}$, and, for each $j \geq 1$, $x_1^{(j)}, \dots, x_{n_j}^{(j)}$ is a family of self-adjoint generators of $\mathcal{A}_j$. For every $0 < \omega < 1$, there exist a positive integer j and a family of noncommutative polynomials $\psi_i(x_1^{(j)}, \dots, x_{n_j}^{(j)})$, $1 \leq i \leq p$, such that

$$\left(\sum_{i=1}^{p} \|y_i - \psi_i(x_1^{(j)}, \dots, x_{n_j}^{(j)})\|^2 \right)^{1/2} < \frac{\omega}{4}.$$

The rest of the proof is identical to the proof of Theorem 3.2.1. ∎

3.3 Tensor Products with $\mathcal{M}_n(\mathbb{C})$

Suppose $\mathcal{A}$ is a finitely generated unital C*-algebra and n is a positive integer. In this subsection, we are going to compute the topological orbit dimension in the unital C*-algebra $\mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})$.

Assume that $\{e_{st}\}_{s,t=1}^n$ is a canonical system of matrix units in $\mathcal{M}_n(\mathbb{C})$ and I_n is the identity matrix of $\mathcal{M}_n(\mathbb{C})$.

The following statement is an easy adaptation of [2, Lemma 2.3].

Lemma 3.3.1 ([2, Lemma 2.3]) *For any $\epsilon > 0$, there is a constant $\delta > 0$ so that the following holds. For any $k \in \mathbb{N}$, if $\{E_{st}\}_{s,t=1}^n$ is a family of elements in $\mathcal{M}_k(\mathbb{C})$ satisfying*

$$\|E_{s_1t_1}E_{s_2t_2} - \delta_{t_1s_2}E_{s_1t_2}\| \leq \delta, \ \|E_{s_1t_1} - E_{t_1s_1}^*\| \leq \delta, \ \left\| \sum_{i=1}^{n} E_{ii} - I_k \right\| \leq \delta, \ \forall 1 \leq s_1, s_2, t_1, t_2 \leq n$$

(where $\delta_{t_1s_2}$ is 1 if $t_1 = s_2$ and is 0 if $t_1 \neq s_2$), then $n|k$ and there is some unitary matrix W in $\mathcal{M}_k(\mathbb{C})$ such that

$$\sum_{s,t=1}^{n} \|W^*E_{st}W - I_{k/n} \otimes e_{st}\| \leq \epsilon.$$

Lemma 3.3.2 *Suppose $\mathcal{A}$ is a unital C*-algebra generated by a family of self-adjoint elements $x_1, \dots, x_m$. Suppose that $\{P_r\}_{r=1}^\infty$, and $\{Q_s\}_{s=1}^\infty$ respectively, is the family of noncommutative polynomials in $\mathbb{C}\langle X_1, \dots, X_m, \{Y_{st}\}_{s,t=1}^n \rangle$, and $\mathbb{C}\langle X_1, \dots, X_m \rangle$ respectively, with rational coefficients.*

Let $R > \max\{\|x_1\|, \dots, \|x_m\|, 1\}$. For any $\omega > 0$, $r_0 > 0$ and $\epsilon_0 > 0$, there are some $r > 0$ and $\epsilon > 0$ such that the following holds. If

$$(A_1, \dots, A_m, \{E_{st}\}_{s,t=1}^n) \in \Gamma_R^{(\text{top})}(x_1 \otimes I_n, \dots, x_m \otimes I_n, \{I_{\mathcal{A}} \otimes e_{st}\}_{s,t=1}^n, k, \epsilon, P_1, \dots, P_r) \neq \varnothing,$$

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then $n|k$; there are a unitary matrix W in $\mathcal{M}_k(\mathbb{C})$ and

$$(B_1,\ldots,B_m)\in\Gamma_R^{(\mathrm{top})}(x_1,\ldots,x_m,\frac{k}{n},\epsilon_0,Q_1,\ldots,Q_{r_0})$$

such that

$$\sum_{s,t=1}^{n}\|W^*E_{st}W-I_{k/n}\otimes e_{st}\|+\sum_{i=1}^{m}\|W^*A_iW-B_i\otimes I_n\|\le\omega.$$

Proof We will prove the result by using contradiction. Suppose, to the contrary, that the result of the lemma does not hold. There are $\omega>0$, $r_0\in\mathbb{N}$, and $\epsilon_0>0$ such that, for any $r\in\mathbb{N}$, there are $k_r\in\mathbb{N}$ and

$$(A_1^{(r)},\ldots,A_m^{(r)},\{E_{st}^{(r)}\}_{s,t=1}^{n})\in \Gamma_R^{(\mathrm{top})}(x_1\otimes I_n,\ldots,x_m\otimes I_n,\{I_{\mathcal{A}}\otimes e_{st}\}_{s,t=1}^{n},k_r,1/r,P_1,\ldots,P_r)\ne\varnothing,$$

satisfying either $n\nmid k_r$, or if W is a unitary matrix in $\mathcal{M}_{k_r}(\mathbb{C})$ and

$$(B_1,\ldots,B_m)\in\Gamma_R^{(\mathrm{top})}(x_1,\ldots,x_m,\frac{k}{n},\epsilon_0,Q_1,\ldots,Q_{r_0}),\tag{3.1}$$

then

$$\sum_{s,t=1}^{n}\|W^*E_{st}^{(r)}W-I_{k/n}\otimes e_{st}\|+\sum_{i=1}^{m}\|W^*A_i^{(r)}W-B_i\otimes I_n\|>\omega.\tag{3.2}$$

Let γ be a free ultra-filter in $\beta(\mathbb{N})\setminus\mathbb{N}$. Let $\prod_{r=1}^{\gamma}\mathcal{M}_{k_r}(\mathbb{C})$ be the C* algebra ultra-product of matrices algebras $(\mathcal{M}_{k_r}(\mathbb{C}))_{r=1}^{\infty}$ along the ultra-filter γ, *i.e.*, $\prod_{r=1}^{\gamma}\mathcal{M}_{k_r}(\mathbb{C})$ is the quotient algebra of the C*-algebra $\prod_r\mathcal{M}_{k_r}(\mathbb{C})$ by $\mathcal{I}_\infty$, where

$$\mathcal{I}_\infty=\{(Y_r)_{r=1}^{\infty}\in\prod_r\mathcal{M}_{k_r}(\mathbb{C})\mid\lim_{r\to\gamma}\|Y_r\|=0\}.$$

Let ψ be the $*$-isomorphism from the C*-algebra $\mathcal{A}\otimes\mathcal{M}_n(\mathbb{C})$ into the C*-algebra $\prod_{r=1}^{\gamma}\mathcal{M}_{k_r}(\mathbb{C})$ induced by the mapping

$$x_i\otimes I_n\to[(A_i^{(r)})_r]\in\prod_{r=1}^{\gamma}\mathcal{M}_{k_r}(\mathbb{C}),\ I_{\mathcal{A}}\otimes e_{st}\to[(E_{st}^{(r)})_r]\in\prod_{r=1}^{\gamma}M_{k_r}(C)$$
$$\forall\,1\le i\le m,1\le s,t\le n.$$

Thus $\{\psi(I_{\mathcal{A}}\otimes e_{st})\}_{s,t=1}^{n}$ is also a system of matrix units of a C*-subalgebra ($*$-isomorphic to $\mathcal{M}_n(\mathbb{C})$) in $\prod_{r=1}^{\gamma}\mathcal{M}_{k_r}(\mathbb{C})$. By the preceding lemma, without loss of generality, we can assume that $n|k_r$ and there is a sequence of unitary matrices $\{W_r\}_{r=1}^{\infty}$, where W_r is in $\mathcal{M}_{k_r}(\mathbb{C})$, such that

$$[(E_{st}^{(r)})_r]=[(W_r(I_{k_r/n}\otimes e_{st})W_r^*)_r],\qquad\forall\,1\le s,t\le n.\tag{3.3}$$

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Note that

$$[(A_i^{(r)})_r][(E_{st}^{(r)})_r] = [(E_{st}^{(r)})_r][(A_i^{(r)})_r], \qquad \forall\, 1 \le i \le m, 1 \le s,t \le n.$$

Thus by (3.3), there are $B_1^{(r)}, \dots, B_m^{(r)}$ in $\mathcal{M}_{k_r/n}(\mathbb{C})$ for each $r \ge 1$ such that

$$[(A_i^{(r)})_r] = [(W_r(B_i^{(r)} \otimes I_n)W_r^*)_r], \qquad \forall\, 1 \le i \le m,$$

which contradicts our assumptions (3.1), (3.2), and (3.3). This completes the proof of the lemma. ■

Now we are ready to prove the main result in this subsection.

Theorem 3.3.3 *Suppose that $\mathcal{A}$ is a unital C^*-algebra and n is a positive integer. Suppose that $\mathcal{B} = \mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})$. Then*

$$\mathfrak{K}_{\text{top}}^{(2)}(\mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})) \le \mathfrak{K}_{\text{top}}^{(2)}(\mathcal{A}).$$

Proof Suppose $x_1, \dots, x_m$ is a family of self-adjoint generators of $\mathcal{A}$ and $y_1, \dots, y_p$ is a family of self-adjoint generators of $\mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})$. Suppose that $\{P_r\}_{r=1}^{\infty}$ and $\{Q_s\}_{s=1}^{\infty}$ are families of noncommutative polynomials in $\mathbb{C}\langle X_1, \dots, X_m, \{Y_{st}\}_{s,t=1}^n\rangle$, and $\mathbb{C}\langle X_1, \dots, X_m\rangle$ respectively, with rational coefficients.

Let $R > \max\{\|x_1\|, \dots, \|x_m\|, 1\}$. For any $\omega > 0$, $r_0 > 0$, and $\epsilon_0 > 0$, by the preceding lemma, there is a $r > 0$ such that, $\forall\, k \in \mathbb{N}$,

$$\begin{aligned} &o_2(\Gamma_R^{(\text{top})}(x_1 \otimes I_n, \dots, x_m \otimes I_n, \{I_{\mathcal{A}} \otimes e_{st}\}_{s,t=1}^n; k, 1/r, P_1, \dots, P_r, 2\omega) \\ &\qquad \le o_2(\Gamma_R^{(\text{top})}(x_1, \dots, x_n; k, 1/r_0, Q_1, \dots, Q_{r_0}, \omega). \end{aligned}$$

Thus,

$$\begin{aligned} &\inf_{r\in\mathbb{N}} \limsup_{k\to\infty} \frac{\log(o_2(\Gamma_R^{(\text{top})}(x_1 \otimes I_n, \dots, x_m \otimes I_n, \{I_{\mathcal{A}} \otimes e_{st}\}_{s,t=1}^n; k, \frac{1}{r}, P_1, \dots, P_r, 2\omega))}{k^2} \\ &\quad \le \limsup_{k\to\infty} \frac{\log(o_2(\Gamma_R^{(\text{top})}(x_1, \dots, x_n; k, \frac{1}{r_0}, Q_1, \dots, Q_{r_0}, \omega))}{k^2}. \end{aligned}$$

So

$$\begin{aligned} &\inf_{r\in\mathbb{N}} \limsup_{k\to\infty} \frac{\log(o_2(\Gamma_R^{(\text{top})}(x_1 \otimes I_n, \dots, x_m \otimes I_n, \{I_{\mathcal{A}} \otimes e_{st}\}_{s,t=1}^n; k, \frac{1}{r}, P_1, \dots, P_r, 2\omega))}{k^2} \\ &\quad \le \inf_{r_0\in\mathbb{N}} \limsup_{k\to\infty} \frac{\log(o_2(\Gamma_R^{(\text{top})}(x_1, \dots, x_n; k, \frac{1}{r_0}, Q_1, \dots, Q_{r_0}, \omega))}{k^2}. \end{aligned}$$

It follows easily that

$$\mathfrak{K}_{\text{top}}^{(2)}(x_1 \otimes I_n, \dots, x_m \otimes I_n, \{I_{\mathcal{A}} \otimes e_{st}\}_{s,t=1}^n) \le \mathfrak{K}_{\text{top}}^{(2)}(x_1, \dots, x_m).$$

By Theorem 3.2.1, we have

$$\mathfrak{K}_{\text{top}}^{(2)}(\mathcal{A} \otimes \mathcal{M}_n(\mathbb{C}) = \mathfrak{K}_{\text{top}}^{(2)}(y_1, \dots, y_p) \le \mathfrak{K}_{\text{top}}^{(2)}(x_1, \dots, x_m) = \mathfrak{K}_{\text{top}}^{(2)}(\mathcal{A}),$$

where $y_1, \dots, y_p$ is any family of self-adjoint generators of $\mathcal{B}$. ■

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The following corollary follows directly from the preceding theorem.

Corollary 3.3.4 *Suppose that $\mathcal{A}$ is a finitely generated unital C^*-algebra with $\mathfrak{K}^{(2)}_{\text{top}}(\mathcal{A}) = 0$. Then, for every positive integer n,*

$$\mathfrak{K}^{(2)}_{\text{top}}(\mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})) = 0 \quad \text{and} \quad \delta_{\text{top}}(\mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})) \le 1.$$

Example 3.3.5 Suppose that $x_1, \dots, x_m$ is a family of self-adjoint generators of a full matrix algebra $\mathcal{M}_n(\mathbb{C})$. Then $\mathfrak{K}^{(2)}_{\text{top}}(x_1, \dots, x_m) = 0$.

3.4 Direct Products of C*-algebras

In this subsection, we assume that $\mathcal{A}$ and $\mathcal{B}$ are two unital C*-algebras and $\mathcal{A} \oplus \mathcal{B}$ is the orthogonal sum of $\mathcal{A}$ and $\mathcal{B}$. We assume $x_1, \dots, x_n$, or $y_1, \dots, y_m$, is a family of self-adjoint generators of $\mathcal{A}$, or $\mathcal{B}$ respectively. Suppose that $\{P_r\}_{r=1}^{\infty}$, and $\{Q_s\}_{s=1}^{\infty}$ respectively, is the family of noncommutative polynomials in $\mathbb{C}\langle X_1, \dots, X_n\rangle$, and $\mathbb{C}\langle Y_1, \dots, Y_m\rangle$ respectively, with rational coefficients. Suppose that $\{S_r\}_{r=1}^{\infty}$ is the family of noncommutative polynomials in $\mathbb{C}\langle X_1, \dots, X_n, Y_1, \dots, X_m\rangle$ with rational coefficients.

Let $R > \max\{\|x_1\|, \dots, \|x_n\|, \|y_1\|, \dots, \|y_m\|\}$ be a positive number. By the definition of topological orbit dimension, we have the following.

Lemma 3.4.1 *Let*

$$\alpha > \mathfrak{K}^{(2)}_{\text{top}}(x_1, \dots, x_n) \qquad \text{and} \qquad \beta > \mathfrak{K}^{(2)}_{\text{top}}(y_1, \dots, y_m).$$

(i) *For each $\omega > 0$, there is $r(\omega)$ satisfying*

$$\limsup_{k_1 \to \infty} \frac{\log(o_2(\Gamma_R^{(\text{top})}(x_1, \dots, x_n; k_1, \frac{1}{r(\omega)}, P_1, \dots, P_{r(\omega)}), \omega))}{k_1^2} < \alpha,$$

$$\limsup_{k_2 \to \infty} \frac{\log(o_2(\Gamma_R^{(\text{top})}(y_1, \dots, y_m; k_2, \frac{1}{r(\omega)}, Q_1, \dots, Q_{r(\omega)}), \omega))}{k_2^2} < \beta.$$

(ii) *Therefore, for each $\omega > 0$ and $r(\omega) \in \mathbb{N}$, there is some $K(r(\omega)) \in \mathbb{N}$ satisfying*

$$\log\Big(o_2\big(\Gamma_R^{(\text{top})}(x_1, \dots, x_n; k_1, \tfrac{1}{r(\omega)}, P_1, \dots, P_{r(\omega)}), \omega\big) \Big) < \alpha k_1^2, \quad \forall\, k_1 \ge K(r(\omega));$$

$$\log\Big(o_2\big(\Gamma_R^{(\text{top})}(y_1, \dots, y_m; k_2, \tfrac{1}{r(\omega)}, Q_1, \dots, Q_{r(\omega)}), \omega\big) \Big) < \beta k_2^2, \quad \forall\, k_2 \ge K(r(\omega)).$$

Lemma 3.4.2 *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two unital C^* algebras and $x_1, \dots, x_n$, or $y_1, \dots, y_m$ is a family of self-adjoint elements that generates $\mathcal{A}$, or $\mathcal{B}$ respectively.*

Let $R > \max\{\|x_1\|, \dots, \|x_n\|, \|y_1\|, \dots, \|y_m\|\}$ be a positive number. For any $\omega > 0$, $r_0 \in \mathbb{N}$, there is some $t > 0$ so that the following holds: $\forall\, r > t$, $\forall\, k \ge 1$, if

$$(X_1, \dots, X_n, Y_1, \dots, Y_m) \in \Gamma_R^{(\text{top})}(x_1 \oplus 0, \dots, x_n \oplus 0, 0 \oplus y_1, \dots, 0 \oplus y_m; k, \tfrac{1}{r}, S_1, \dots, S_r),$$

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then there are

$$(A_1,\ldots,A_n)\in\Gamma_R^{(\mathrm{top})}(x_1,\ldots,x_n;k_1,\tfrac{1}{r_0},P_1,\ldots,P_{r_0}),$$
$$(B_1,\ldots,B_m)\in\Gamma_R^{(\mathrm{top})}(y_1,\ldots,y_m;k_2,\tfrac{1}{r_0},Q_1,\ldots,Q_{r_0}),$$

and $U\in\mathcal{U}(k)$ *so that* $k_1+k_2=k$, *and*

$$\|(X_1,\ldots,X_n,Y_1,\ldots,Y_m)-U^*(A_1\oplus 0,\ldots,A_n\oplus 0,0\oplus B_1,\ldots,0\oplus B_m)U\|<\omega.$$

Proof The proof of this lemma is a slight modification of the one of [18, Lemma 4.2]. ■

Theorem 3.4.3 *Suppose that* $\mathcal{A}$ *and* $\mathcal{B}$ *are finitely generated unital MF* C^**-algebras. Then,*

$$\mathfrak{K}_{\mathrm{top}}^{(2)}(\mathcal{A}\oplus\mathcal{B})\le\mathfrak{K}_{\mathrm{top}}^{(2)}(\mathcal{A})+\mathfrak{K}_{\mathrm{top}}^{(2)}(\mathcal{B}).$$

Proof Suppose $\{x_1,\ldots,x_n\}$, $\{y_1,\ldots,y_m\}$, and $\{z_1,\ldots,z_p\}$ are self-adjoint generating sets for $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{A}\oplus\mathcal{B}$, respectively. Recall that $\{S_r\}_{r=1}^{\infty}$, $\{P_r\}_{r=1}^{\infty}$, and $\{Q_s\}_{s=1}^{\infty}$ respectively, are the families of noncommutative polynomials in

$$\mathbb{C}\langle X_1,\ldots,X_n,Y_1,\ldots,X_m\rangle,\quad \mathbb{C}\langle X_1,\ldots,X_n\rangle,\quad\text{and}\quad \mathbb{C}\langle Y_1,\ldots,Y_m\rangle$$

respectively, with rational coefficients.

Let

$$\alpha>\mathfrak{K}_{\mathrm{top}}^{(2)}(x_1,\ldots,x_n)\qquad\text{and}\qquad\beta>\mathfrak{K}_{\mathrm{top}}^{(2)}(y_1,\ldots,y_m),$$

and $R>\max\{\|x_1\|,\ldots,\|x_n\|y_1\|,\ldots,\|y_m\|\}$ be a positive number. By definition, the values of topological orbit dimension $\mathfrak{K}_{\mathrm{top}}^{(2)}$ can only be $-\infty$ or ≥ 0. Without loss of generality, we can assume that $\alpha>0$ and $\beta>0$. By Lemma 3.4.1, for any $\omega>0$, there are $r(\omega)\in\mathbb{N}$ and $K(r(\omega))\in\mathbb{N}$ satisfying

$$o_2(\Gamma_R^{(\mathrm{top})}(x_1,\ldots,x_n;k_1,\tfrac{1}{r(\omega)},P_1,\ldots,P_{r(\omega)}),\omega)<e^{\alpha k_1^2},\ \forall\,k_1\ge K(r(\omega)),\tag{3.4}$$
$$o_2(\Gamma_R^{(\mathrm{top})}(y_1,\ldots,y_m;k_2,\tfrac{1}{r(\omega)},Q_1,\ldots,Q_{r(\omega)}),\omega)<e^{\beta k_2^2},\ \forall\,k_2\ge K(r(\omega)).\tag{3.5}$$

On the other hand, for each $\omega>0$ and $r(\omega)\in\mathbb{N}$, it follows from Lemma 3.4.2 that there is some $t\in\mathbb{N}$ so that $\forall\,r>t$, $\forall\,k\ge 1$, if

$$(X_1,\ldots,X_n,Y_1,\ldots,Y_m)$$
$$\in\Gamma_R^{(\mathrm{top})}(x_1\oplus 0,\ldots,x_n\oplus 0,0\oplus y_1,\ldots,0\oplus y_m;k,\tfrac{1}{r},S_1,\ldots,S_r),$$

then there are

$$(A_1,\ldots,A_n)\in\Gamma_R^{(\mathrm{top})}(x_1,\ldots,x_n;k_1,\tfrac{1}{r_\omega},P_1,\ldots,P_{r_\omega}),$$
$$(B_1,\ldots,B_m)\in\Gamma_R^{(\mathrm{top})}(y_1,\ldots,y_m;k_2,\tfrac{1}{r_\omega},Q_1,\ldots,Q_{r_\omega}),$$

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and $U \in \mathcal{U}(k)$ so that $k_1 + k_2 = k$, and

$$\|(X_1, \dots, X_n, Y_1, \dots, Y_m) - U^*(A_1 \oplus 0, \dots, A_n \oplus 0, 0 \oplus B_1, \dots, 0 \oplus B_m)U\| < \omega.$$

It follows that

$$
\begin{aligned}
(3.6)\qquad & o_2(\Gamma_R^{(\text{top})}(x_1 \oplus 0, \dots, x_n \oplus 0, 0 \oplus y_1, \dots, 0 \oplus y_m; k, \tfrac{1}{r}, S_1, \dots, S_r), 3\omega) \\
&\le \sum_{k_1+k_2=k} \Big(o_2(\Gamma_R^{(\text{top})}(x_1, \dots, x_n; k_1, \tfrac{1}{r_\omega}, P_1, \dots, P_{r_\omega}), \omega) \\
&\qquad\qquad \cdot\; o_2(\Gamma_R^{(\text{top})}(y_1, \dots, y_m; k_2, \tfrac{1}{r_\omega}, Q_1, \dots, Q_{r_\omega}), \omega) \Big) \\
&= \Big(\sum_{k_1=1}^{K(r(\omega))} + \sum_{k_1=K(r(\omega))+1}^{k-K(r(\omega))-1} + \sum_{k_1=k-K(r(\omega))}^{k} \Big) \\
&\qquad \Big(o_2(\Gamma_R^{(\text{top})}(x_1, \dots, x_n; k_1, \tfrac{1}{r_\omega}, P_1, \dots, P_{r_\omega}), \omega) \\
&\qquad\qquad \cdot\; o_2(\Gamma_R^{(\text{top})}(y_1, \dots, y_m; k_2, \tfrac{1}{r_\omega}, Q_1, \dots, Q_{r_\omega}), \omega) \Big)
\end{aligned}
$$

Let

$$
\begin{aligned}
M_\omega &= \max_{1 \le k_1 \le K(r(\omega))} o_2(\Gamma_R^{(\text{top})}(x_1, \dots, x_n; k_1, \tfrac{1}{r_\omega}, P_1, \dots, P_{r_\omega}), \omega) + 1, \\
N_\omega &= \max_{1 \le k_2 \le K(r(\omega))} o_2(\Gamma_R^{(\text{top})}(y_1, \dots, y_m; k_2, \tfrac{1}{r_\omega}, Q_1, \dots, Q_{r_\omega}), \omega) + 1.
\end{aligned}
$$

By (3.4) and (3.5), we know that

$$
\begin{aligned}
(3.6) &\le K(r(\omega)) M_\omega e^{\beta k_2^2} + K(r(\omega)) N_\omega e^{\alpha k_1^2} + (k - 2K(r(\omega))) \cdot (e^{\alpha k_1^2 + \beta k_2^2} + 1) \\
&\le K(r(\omega)) M_\omega e^{\beta k^2} + K(r(\omega)) N_\omega e^{\alpha k^2} + 2k \cdot e^{(\alpha+\beta)k^2} \le 3k \cdot e^{(\alpha+\beta)k^2},
\end{aligned}
$$

when k is large enough. Now it is not hard to show that

$$\mathfrak{K}_{\text{top}}^{(2)}(x_1 \oplus 0, \dots, x_n \oplus 0, 0 \oplus y_1, \dots, 0 \oplus y_m) \le \alpha + \beta.$$

Thus, by Theorem 3.2.1, we have

$$\mathfrak{K}_{\text{top}}^{(2)}(z_1, \dots, z_p) \le \mathfrak{K}_{\text{top}}^{(2)}(x_1, \dots, x_m) + \mathfrak{K}_{\text{top}}^{(2)}(y_1, \dots, y_m),$$

where $z_1, \dots, z_p$ is any family of self-adjoint generators of $\mathcal{A} \oplus \mathcal{B}$. Hence, by Theorem 3.2.1,

$$\mathfrak{K}_{\text{top}}^{(2)}\big(\mathcal{A} \oplus \mathcal{B}\big) \le \mathfrak{K}_{\text{top}}^{(2)}(\mathcal{A}) + \mathfrak{K}_{\text{top}}^{(2)}(\mathcal{B}). \qquad \blacksquare$$

The following corollary follows directly from the preceding theorem.

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Corollary 3.4.4 *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are finitely-generated unital MF C^*-algebras. If*

$$\mathfrak{K}^{(2)}_{\rm top}(\mathcal{A}) = \mathfrak{K}^{(2)}_{\rm top}(\mathcal{B}) = 0,$$

then

$$\mathfrak{K}^{(2)}_{\rm top}\big(\mathcal{A}\oplus\mathcal{B}\big) = 0 \quad \text{and} \quad \delta_{\rm top}\big(\mathcal{A}\oplus\mathcal{B}\big) \leq 1.$$

By Example 3.3.5, Corollary 3.4.4, and Theorem 3.2.2, we have the following result. This result will be extended to nuclear C*-algebras in Corollary 4.4.1.

Corollary 3.4.5 *If $\mathcal{A}$ is an AF C^*-algebra with self-adjoint generators $x_1, \ldots, x_n$, then*

$$\mathfrak{K}^{(2)}_{\rm top}(\mathcal{A}) = 0 \quad \text{and} \quad \delta_{\rm top}(x_1, \ldots, x_n) \leq 1.$$

4 Orbit Dimension Capacity

In this section, we are going to define the concept of "orbit dimension capacity" of n-tuple of elements in a unital C*-algebra, which is an analogue of "free dimension capacity" in [29].

4.1 Modified Free Orbit Dimension in Finite von Neumann Algebras

Let $\mathcal{M}$ be a von Neumann algebra with a tracial state τ, and let $x_1, \ldots, x_n$ be self-adjoint elements in $\mathcal{M}$. For any positive R and ϵ, and any m, k in $\mathbb{N}$, let $\Gamma_R(x_1, \ldots, x_n; m, k, \epsilon; \tau)$ be the subset of $\mathcal{M}_k^{s.a}(\mathbb{C})^n$ consisting of all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k^{s.a}(\mathbb{C})^n$ such that

$$\max_{1\leq j\leq n} \|A_j\| \leq R \quad \text{and} \quad |\tau_k(A_{i_1}\cdots A_{i_q}) - \tau(x_{i_1}\cdots x_{i_q})| < \epsilon,$$

for all $1 \leq i_1, \ldots, i_q \leq n$ and $1 \leq q \leq m$.

For any $\omega > 0$, let $o_2(\Gamma_R(x_1, \ldots, x_n; m, k, \epsilon; \tau), \omega)$ be the minimal number of ω-orbit-$\|\cdot\|_2$-balls in $\mathcal{M}_k(\mathbb{C})^n$ that constitute a covering of $\Gamma_R(x_1, \ldots, x_n; m, k, \epsilon; \tau)$.

Now we define, successively,

$$\mathfrak{K}^{(2)}_2(x_1, , \ldots, x_n; \omega; \tau) = \sup_{R>0} \inf_{m\in\mathbb{N}, \epsilon>0} \limsup_{k\to\infty} \frac{\log(o_2(\Gamma_R(x_1, \ldots, x_n; m, k, \epsilon; \tau), \omega))}{k^2}$$

$$\mathfrak{K}^{(2)}_2(x_1, , \ldots, x_n; \tau) = \limsup_{\omega\to 0^+} \mathfrak{K}^{(2)}_2(x_1, \ldots, x_n; \omega; \tau),$$

where $\mathfrak{K}^{(2)}_2(x_1, \ldots, x_n; \tau)$ is called the *modified free orbit-dimension* of $x_1, \ldots, x_n$ with respect to the tracial state τ.

Remark 4.1.1 If the von Neumann algebra $\mathcal{M}$ with a tracial state τ is replaced by a unital C*-algebra $\mathcal{A}$ with a tracial state τ, then $\mathfrak{K}^{(2)}_2(x_1, \ldots, x_n; \tau)$ is still well defined.

Our next result follows directly from the previous definition.

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Lemma 4.1.2 *Suppose $x_1, \dots, x_n$ is a family of self-adjoint elements in a von Neumann algebra with a tracial state τ. Let $\mathfrak{K}_2(x_1, \dots, x_n; \tau)$ be the upper orbit dimension of $x_1, \dots, x_n$ defined in* [17, Definition 1]. *We have, if $\mathfrak{K}_2(x_1, \dots, x_n; \tau) = 0$, then $\mathfrak{K}_2^{(2)}(x_1, \dots, x_n; \tau) = 0$.*

4.2 Definition of Orbit Dimension Capacity

We are ready to give the definition of "orbit dimension capacity".

Definition 4.2.1 Suppose that $\mathcal{A}$ is a unital C*-algebra and $TS(\mathcal{A})$ is the set of all tracial states of $\mathcal{A}$. Suppose that $x_1, \dots, x_n$ is a family of self-adjoint elements in $\mathcal{A}$. Define

$$\mathfrak{K}\mathfrak{K}_2^{(2)}(x_1, \dots, x_n) = \sup_{\tau \in TS(\mathcal{A})} \mathfrak{K}_2^{(2)}(x_1, \dots, x_n; \tau)$$

to be the orbit dimension capacity of $x_1, \dots, x_n$.

4.3 Topological Orbit Dimension is Majorized by Orbit Dimension Capacity

We have the following relationship between topological orbit dimension and orbit dimension capacity.

Theorem 4.3.1 *Suppose that $\mathcal{A}$ is a unital C*-algebra and $x_1, \dots, x_n$ is a family of self-adjoint elements in $\mathcal{A}$. Then*

$$\mathfrak{K}_{\text{top}}^{(2)}(x_1, \dots, x_n) \le \mathfrak{K}\mathfrak{K}_2^{(2)}(x_1, \dots, x_n).$$

Proof The proof is a slight modification of the one in section 3 of [29]. For the sake of completeness, we also include Voiculescu's arguments here.

If $\mathfrak{K}_{\text{top}}^{(2)}(x_1, \dots, x_n) = -\infty$, there is nothing to prove. We might assume that

$$\mathfrak{K}_{\text{top}}^{(2)}(x_1, \dots, x_n) > \alpha > -\infty.$$

We will show that

$$\mathfrak{K}\mathfrak{K}_2^{(2)}(x_1, \dots, x_n) = \sup_{\tau \in TS(\mathcal{A})} \mathfrak{K}_2^{(2)}(x_1, \dots, x_n; \tau) > \alpha.$$

Let $\{P_r\}_{r=1}^{\infty}$be a family of noncommutative polynomials in $\mathbb{C}\langle X_1, \dots, X_n \rangle$ with rational coefficients. Let $R > \max\{\|x_1\|, \dots, \|x_n\|\}$. From the assumption that $\mathfrak{K}_{\text{top}}^{(2)}(x_1, \dots, x_n) > \alpha$, it follows that there exist a positive number $\omega_0 > 0$ and a sequence of positive integers $\{k_q\}_{q=1}^{\infty}$ with $k_1 < k_2 < \cdots$, so that for some $\alpha' > \alpha$,

$$\lim_{q \to \infty} \frac{\log(o_2(\Gamma_R^{(\text{top})}(x_1, \dots, x_n; k_q, \frac{1}{q}, P_1, \dots, P_q), \omega_0))}{k_q^2} > \alpha'.$$

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Let $\mathcal{A}(n)$ be the universal unital C*-algebra generated by self-adjoint elements $a_1, \ldots, a_n$ of norm R, that is the unital full free product of n copies of $C[-R, R]$. A microstate

$$\eta = (A_1, \ldots, A_n) \in \Gamma_R^{(\text{top})}(x_1, \ldots, x_n; k_q, \tfrac{1}{q}, P_1, \ldots, P_q) = \Gamma(q)$$

defines a unital $*$-homomorphism $\psi_\eta \colon \mathcal{A}(n) \to \mathcal{M}_{k_q}(\mathbb{C})$ so that $\psi_\eta(a_i) = A_i$ $(1 \leq i \leq n)$ and a tracial state $\tau_\eta \in TS(\mathcal{A}(n))$ with

$$\tau_\eta = \frac{\mathrm{Tr}_{k_q} \circ \psi_\eta}{k_q}.$$

Similarly, there is a $*$-homomorphism $\psi \colon \mathcal{A}(n) \to \mathcal{A}$ such that $\psi(a_i) = x_i$ for $1 \leq i \leq n$.

It is not hard to see that the weak topology on $\Omega = TS(\mathcal{A}(n))$ is induced by the metric

$$d(\tau_1, \tau_2) = \sum_{s=1}^{\infty} \sum_{(i_1, \ldots, i_s) \in (\{1, \ldots, n\})^s} (2Rn)^{-s} |(\tau_1 - \tau_2)(a_{i_1} \cdots a_{i_s})|.$$

Therefore, Ω is a compact metric space and

$$K_q = \{\tau_\eta \in \Omega \mid \eta \in \Gamma(q)\}$$

is a compact subset of Ω because $\eta \to \tau_\eta$ is continuous and $\Gamma(q)$ is compact. Further, let $K \subseteq \Omega$ denote the compact subset $(TS(\mathcal{A})) \circ \psi$.

Given $\epsilon > 0$, from the fact that Ω is compact it follows that there is some $L(\epsilon) > 0$ so that for each $q \geq 1$,

$$K_q = K_q^1 \cup K_q^2 \cup \cdots \cup K_q^{L(\epsilon)}$$

where each compact set K_q^j has diameter $< \epsilon$. Let

$$\Gamma(q, j) = \{\eta \in \Gamma(q) \mid \tau_\eta \in K_q^j\}.$$

We have

$$\Gamma(q) = \Gamma(q, 1) \cup \cdots \cup \Gamma(q, L(\epsilon)).$$

Further, let $\Gamma'(q)$ denote some $\Gamma(q, j)$ such that

$$o_2(\Gamma'(q), \omega_0) \geq \frac{o_2(\Gamma(q), \omega_0)}{L(\epsilon)}.$$

Thus we have

$$\lim_{q \to \infty} \frac{\log o_2(\Gamma'(q), \omega_0)}{k_q^2} > \alpha'.$$

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Giving ϵ the values $1, 1/2, 1/3, \ldots, 1/s, \ldots,$ successively we can find a subsequence $\{q_s\}_{s=1}^{\infty}$ such that the chosen set $K_{q_s}^{j_s} \subseteq K_{q_s}$ has diameter $< \frac{1}{s}$ and the corresponding set $\Gamma'(q_s)$ satisfies

$$\lim_{s\to\infty} \frac{\log o_2(\Gamma'(q_s), \omega_0)}{k_{q_s}^2} > \alpha'.$$

Without loss of generality, we can assume that τ is the weak limit of some sequence $(\tau_{\eta(q_s)})_{s=1}^{\infty}$. Then $\tau \in K$. In fact,

$$\begin{aligned}
|\tau(Q(a_1, \ldots, a_n))| &= \lim_{s\to\infty} |\tau_{\eta(q_s)}(Q(a_1, \ldots, a_n))| \\
&\leq \limsup_{s\to\infty} ||\psi_{\eta(q_s)}(Q(a_1, \ldots, a_n))|| \\
&\leq \lim_{s\to\infty} (\tfrac{1}{s} + \|Q(x_1, \ldots, x_n)\|) \\
&= \|Q(x_1, \ldots, x_n)\| = \|\psi(Q(a_1, \ldots, a_n))\|.
\end{aligned}$$

Now it follows from the density of the polynomials Q in $\mathcal{A}(n)$ that $\tau \in K$.

We can further assume that there is a subsequence $\{q_{s(t)}\}_{t=1}^{\infty}$ of $\{q_s\}_{s=1}^{\infty}$ so that the chosen set $K_{q_{s(t)}}^{j_{s(t)}} \subseteq K_{q_{s(t)}}$ is $\subseteq B(\tau, 1/t)$, the ball of radius $1/t$ and center τ. Therefore, for any $m \in \mathbb{N}$ and $\epsilon > 0$, we have

$$\Gamma'(q_{s(t)}) \subseteq \Gamma_R(x_1, \ldots, x_n; k_{q_{s(t)}}, m, \epsilon; \tau)$$

when t is large enough. Thus

$$\mathfrak{K}_2^{(2)}(x_1, \ldots, x_n; \tau) \geq \mathfrak{K}_2^{(2)}(x_1, \ldots, x_n; \omega_0; \tau) \geq \lim_{t\to\infty} \frac{\log o_2(\Gamma'(q_{s(t)}), \omega_0)}{k_{q_{s(t)}}^2} > \alpha',$$

and hence

$$\mathfrak{K}\mathfrak{K}_2^{(2)}(x_1, \ldots, x_n) = \sup_{\tau \in TS(\mathcal{A})} \mathfrak{K}_2^{(2)}(x_1, \ldots, x_n; \tau) \geq \mathfrak{K}_{\text{top}}^{(2)}(x_1, \ldots, x_n). \qquad \blacksquare$$

4.4 Nuclear C*-algebras

Now we can compute the topological free entropy dimension of a family of self-adjoint generators in a unital nuclear C*-algebra.

Corollary 4.4.1 *Suppose $\mathcal{A}$ is an MF unital nuclear C*-algebra with a family of self-adjoint generators $x_1, \ldots, x_n$. We have that*

$$\mathfrak{K}_{\text{top}}^{(2)}(\mathcal{A}) = 0 \qquad \textit{and} \qquad \delta_{\text{top}}(x_1, \ldots, x_n) \leq 1.$$

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Proof It is known that every representation of a nuclear C*-algebra yields an injective von Neumann algebra. From Lemma 4.1.2 and [17, Theorem 2], it follows that

$$\mathfrak{K}_2^{(2)}(x_1,\dots,x_n;\tau)=0,\qquad \forall\,\tau\in TS(A),$$

where $TS(\mathcal{A})$ is the set of all tracial states of $\mathcal{A}$. Then, by Theorem 4.3.1 we know that

$$\mathfrak{K}_{\text{top}}^{(2)}(x_1,\dots,x_n)=0.$$

By Theorem 3.1.2, $\delta_{\text{top}}(x_1,\dots,x_n)\le 1$. ■

Remark 4.4.2 By [1] , we know that a nuclear C*-algebra $\mathcal{A}$ is an MF algebra if and only if $\mathcal{A}$ is an NF algebra with a finite family of generators. Thus Corollary 4.4.1 can be restated as follows: *If $\mathcal{A}$ is an NF algebra with a family of self-adjoint generators $x_1,\dots,x_n$, then*

$$\mathfrak{K}_{\text{top}}^{(2)}(\mathcal{A})=0\qquad\textit{and}\qquad\delta_{\text{top}}(x_1,\dots,x_n)\le 1.$$

4.5 Tensor Products

In this subsection, we are going to prove the following result.

Corollary 4.5.1 *Suppose that $\mathcal{B}_1$ and $\mathcal{B}_2$ are two unital C*-algebras, and $\mathcal{A}_1,\mathcal{A}_2$ with $1\in\mathcal{A}_1\subseteq\mathcal{B}_1, 1\in\mathcal{A}_2\subseteq\mathcal{B}_2$ are infinite dimensional, unital, simple C*-subalgebras with a unique tracial state. Suppose that $\mathcal{B}=\mathcal{B}_1\otimes_\nu\mathcal{B}_2$ is the C*-tensor product of $\mathcal{B}_1$ and $\mathcal{B}_2$ with respect to a cross norm $\|\cdot\|_\nu$. If $\mathcal{B}$ is an MF algebra, then*

$$\mathfrak{K}_{\text{top}}^{(2)}(x_1,\dots,x_n)=0\quad\textit{and}\quad\delta_{\text{top}}(x_1,\dots,x_n)=1,$$

where $x_1,\dots,x_n$ is any family of self-adjoint generators of $\mathcal{B}$.

Proof Assume that τ is a tracial state of $\mathcal{B}$ and $\mathcal{H}$ is the Hilbert $L^2(\mathcal{B},\tau)$. Let ψ be the GNS representation of $\mathcal{B}$ on $\mathcal{H}$. Note that $\mathcal{A}_1,\mathcal{A}_2$ are infinite dimensional, unital simple C*-algebras with a unique tracial state. It is not hard to see that both $\psi(\mathcal{A}_1)$ and $\psi(\mathcal{A}_2)$ generate diffuse finite von Neumann algebras on $\mathcal{H}$. Thus both $\psi(\mathcal{B}_1)$ and $\psi(\mathcal{B}_2)$ generate diffuse finite von Neumann algebras on $\mathcal{H}$. Moreover, $\psi(\mathcal{B}_1)$ and $\psi(\mathcal{B}_2)$ commute with each other. Thus, by [17, Corollary 4], we have that

$$\mathfrak{K}_2(\psi(x_1),\dots,\psi(x_n);\tau)=0,$$

where $\mathfrak{K}_2(\psi(x_1),\dots,\psi(x_n);\tau)$ is the upper free orbit dimension of $\psi(x_1),\dots,\psi(x_n)$ (with respect to τ) defined in [17]. By Lemma 3.1.1, we have

$$\mathfrak{K}_2^{(2)}(\psi(x_1),\dots,\psi(x_n);\tau)=0.$$

Since τ is an arbitrary tracial state of $\mathcal{B}$, we have $\mathfrak{K}\mathfrak{K}_2^{(2)}(x_1,\dots,x_n)=0$. By Theorem 3.1.2, we have $\mathfrak{K}_{\text{top}}^{(2)}(x_1,\dots,x_n)=0$. Therefore, $\delta_{\text{top}}(x_1,\dots,x_n)\le 1$. On the other hand, a consequence of [18, Theorem 5.2] says that

$$\delta_{\text{top}}(x_1,\dots,x_n)\ge 1,$$

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if $\mathcal{B}$ is an MF algebra. Hence,

$$\mathfrak{K}^{(2)}_{\text{top}}(x_1,\ldots,x_n)=0 \quad \text{and} \quad \delta_{\text{top}}(x_1,\ldots,x_n)=1,$$

where $x_1,\ldots,x_n$ is any family of self-adjoint generators of $\mathcal{B}$. ∎

Example 4.5.2 Assume that $\mathcal{A}$ is a finite generated unital MF algebra. By Theorem 5.1.4, and [20, Proposition 3.1], we know that $(C^*_r(F_2) *_{\mathbb{C}} \mathcal{A}) \otimes_{min} C^*_r(F_2)$ is an MF algebra, where $C^*_r(F_2)$ is the reduced C*-algebra of free group F_2 . Thus, by Corollary 4.5.1, we know that, for any family of self-adjoint generators $x_1,\ldots,x_n$ of $(C^*_r(F_2) *_{\mathbb{C}} \mathcal{A}) \otimes_{min} C^*_r(F_2)$,

$$\delta_{\text{top}}(x_1,\ldots,x_n)=1.$$

An argument similar to the one used in the proof of the preceding corollary shows the following result.

Corollary 4.5.3 *Suppose that $\mathcal{B}_1$ is a unital C*-algebra and $\mathcal{B}_2$ is an infinite dimensional, unital simple C*-algebra with a unique tracial state τ. Suppose that $\mathcal{B} = \mathcal{B}_1 \otimes_\nu \mathcal{B}_2$ is the C*-tensor product of $\mathcal{B}_1$ and $\mathcal{B}_2$ with respect to a cross norm $\|\cdot\|_\nu$. Suppose that $z_1,\ldots,z_p$ is a family of self-adjoint generators of $\mathcal{B}_2$ and $x_1,\ldots,x_n$ is a family of self-adjoint generators of $\mathcal{B}$. If $\mathcal{B}$ is an MF algebra and $\mathfrak{K}^{(2)}_2(z_1,\ldots,z_p;\tau)=0$, then*

$$\mathfrak{K}^{(2)}_{\text{top}}(x_1,\ldots,x_n)=0 \quad \textit{and} \quad \delta_{\text{top}}(x_1,\ldots,x_n)=1.$$

Example 4.5.4 Assume that $\mathcal{A}$ is a UHF algebra, or an irrational rotation algebra, or $C^*_r(F_2) \otimes_{min} C^*_r(F_2)$ and $\mathcal{B}$ is a finitely generated unital MF algebra. Then, by [1] or [20], $\mathcal{A} \otimes_{min} \mathcal{B}$ is an MF algebra. Suppose that $x_1,\ldots,x_n$ is a family of self-adjoint generators of $\mathcal{A} \otimes_{min} \mathcal{B}$. Then $\delta_{\text{top}}(x_1,\ldots,x_n)=1$.

Example 4.5.5 By [20, Theorem 3.1], $C^*(F_2) \otimes_{max} (C^*_r(F_2) \otimes_{min} C^*_r(F_2))$ is an MF algebra, where $C^*(F_2)$ is the full C*-algebra of the free group F_2. Suppose that $x_1,\ldots,x_n$ is a family of self-adjoint generators of $C^*(F_2) \otimes_{max} (C^*_r(F_2) \otimes_{min} C^*_r(F_2))$. Then $\delta_{\text{top}}(x_1,\ldots,x_n)=1$.

4.6 Crossed Products

In this subsection, we are going to prove the following result.

Corollary 4.6.1 *Suppose that $\mathcal{A}$ is an infinite dimensional unital simple C*-algebra with a unique tracial state τ. Suppose G is a countable group of actions $\{\alpha_g\}_{g\in G}$ on $\mathcal{A}$. Suppose that $\mathcal{D} = \mathcal{A} \rtimes G$ is either a full or reduced crossed product of $\mathcal{A}$ by the actions of G. Suppose that $z_1,\ldots,z_p$ is a family of self-adjoint generators of $\mathcal{A}$, and $x_1,\ldots,x_n$ is a family of self-adjoint generators of $\mathcal{D}$. If $\mathcal{D}$ is an MF algebra and $\mathfrak{K}^{(2)}_2(z_1,\ldots,z_p;\tau) = 0$, then*

$$\mathfrak{K}^{(2)}_{\text{top}}(x_1,\ldots,x_n)=0 \quad \textit{and} \quad \delta_{\text{top}}(x_1,\ldots,x_n)=1.$$

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Proof Assume that τ_1 is a tracial state of $\mathcal{D}$ and $\mathcal{H}$ is the Hilbert $L^2(\mathcal{D}, \tau_1)$. Let ψ be the GNS representation of $\mathcal{D}$ on $\mathcal{H}$. Note that $\mathcal{A}$ is an infinite dimensional unital simple C*-algebra with a unique tracial state τ. Thus $\tau_1|_{\mathcal{A}} = \tau$. It is not hard to see that $\psi(\mathcal{A})$ generates a diffuse finite von Neumann algebra on $\mathcal{H}$. Moreover, for any $g \in G$, $\psi(g^{-1})\psi(\mathcal{A})\psi(g) \subseteq \psi(\mathcal{A})$. It follows from the fact that $\mathfrak{K}_2^{(2)}(z_1, \ldots, z_p; \tau) = 0$ and Theorem 4 in [17], that

$$\mathfrak{K}_2^{(2)}(\psi(x_1), \ldots, \psi(x_n); \tau_1) = 0.$$

Since τ_1 is an arbitrary tracial state of $\mathcal{D}$, we have $\mathfrak{K}\mathfrak{K}_2^{(2)}(x_1, \ldots, x_n) = 0$. By Theorem 3.1.2, we have $\mathfrak{K}_{\text{top}}^{(2)}(x_1, \ldots, x_n) = 0$. Therefore, $\delta_{\text{top}}(x_1, \ldots, x_n) \le 1$. On the other hand, a consequence of [18, Theorem 5.2] says that

$$\delta_{\text{top}}(x_1, \ldots, x_n) \ge 1,$$

if $\mathcal{D}$ is an MF algebra. Hence,

$$\mathfrak{K}_{\text{top}}^{(2)}(x_1, \ldots, x_n) = 0 \quad \text{and} \quad \delta_{\text{top}}(x_1, \ldots, x_n) = 1,$$

where $x_1, \ldots, x_n$ is any family of self-adjoint generators of $\mathcal{D}$. ■

Example 4.6.2 Let $C_r^*(F_2) \otimes_{min} C_r^*(F_2)$ be the reduced C*-algebra of the group $F_2 \times F_2$. Let u_1, u_2, or v_1, v_2 respectively, be the canonical unitary generators of the left copy, or the right copy respectively, of $C_r^*(F_2)$, and let $0 < \theta < 1$ be a positive number. Let α be a homomorphism from $\mathbb{Z}$ into $Aut(C_r^*(F_2) \otimes_{min} C_r^*(F_2))$ induced by the following mapping: $\forall\, n \in \mathbb{Z}$, $j = 1, 2$

$$\alpha(n)(u_j) = e^{2n\pi\theta \cdot i} u_j \qquad \textit{and} \qquad \alpha(n)(v_j) = e^{2n\pi\theta \cdot i} v_j.$$

Then, by [20, Theorem 4.2], $(C_r^*(F_2) \otimes_{min} C_r^*(F_2)) \rtimes_\alpha \mathbb{Z}$ is an MF algebra. Therefore, by Corollary 4.6.1, we have $\delta_{\text{top}}(x_1, \ldots, x_n) = 1$, where $x_1, \ldots, x_n$ is any family of self-adjoint generators of $(C_r^*(F_2) \otimes_{min} C_r^*(F_2)) \rtimes_\alpha \mathbb{Z}$.

5 Full Free Products of Unital C*-algebras

Assume that $\{\mathcal{A}_i : i \in I\}$ is a family of unital C*-algebras. Recall the definition of unital full free product of the $\mathcal{A}_i$'s as follows.

Definition 5.0.3 The unital full free product of a family $\{\mathcal{A}_i : i \in I\}$ of unital C*-algebras is a unital C*-algebra $\mathcal{D}$ equipped with unital embeddings $\sigma_i \colon \mathcal{A}_i \to \mathcal{D}$ for each $i \in I$, such that: (i) $\mathcal{D} = C^*(\bigcup_{i \in I} \sigma_i(\mathcal{A}_i))$, and (ii) if ϕ_i is a unital $*$-homomorphism from $\mathcal{A}_i$ into a unital C*-algebra $\mathcal{B}$ for each $i \in I$, then there is a unital $*$-homomorphisms ψ from $\mathcal{D}$ to $\mathcal{B}$ satisfying $\phi_i = \psi \circ \sigma_i$, for each $i \in I$. Since we can identify each $\mathcal{A}_i$ with $\rho(\mathcal{A}_i)$, we usually assume that $\mathcal{A}_i \subset \mathcal{D}$ for each $i \in I$.

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5.1 Full Free Products of MF Algebras

The concept of MF algebras was introduced by Blackadar and Kirchberg in [1]. This class of C*-algebras is of interest for many reasons. For example, it plays an important role in the classification of C*-algebras and is connected to the question of whether or not the extension semigroup (in the sense of Brown, Douglas, and Fillmore) of a unital C*-algebra is a group (see the striking result of Haagerup and Thorbørnsen on $\mathrm{Ext}(C_r^*(F_2))$. Thanks to Voiculescu's result in [25], we know that every quasidiagonal C*-algebra is an MF algebra. Many properties of MF algebras have been discussed in [1]. For example, it was shown there that the inductive limit of MF algebras is an MF algebra, and every subalgebra of an MF algebra is an MF algebra. In this subsection, we will prove that unital full free product of a countable family of unital separable MF algebras is, again, an MF algebra.

Let us fix notation first. We always assume that $\mathcal{H}$ is a separable complex Hilbert space and $B(\mathcal{H})$ is the set of all bounded operators on $\mathcal{H}$. Suppose $\{x, x_k\}_{k=1}^{\infty}$ is a family of elements in $B(\mathcal{H})$. We say $x_k \to x$ in $*$-SOT ($*$-strong operator topology) if and only if $x_k \to x$ in SOT and $x_k^* \to x^*$ in SOT. Suppose $\{x_1, \dots, x_n\}$ and $\{x_1^{(k)}, \dots, x_n^{(k)}\}_{k=1}^{\infty}$ are families of elements in $B(\mathcal{H})$. We say

$$\langle x_1^{(k)}, \dots, x_n^{(k)} \rangle \to \langle x_1, \dots, x_n \rangle, \ in{*}\text{-SOT, as } k \to \infty$$

if and only if

$$x_i^{(k)} \to x_i \ in\ {*}\text{-SOT, as } k \to \infty, \ \ \forall\, 1 \le i \le n.$$

Suppose $\{\mathcal{A}_k\}_{k=1}^{\infty}$ is a family of unital C*-algebras. Let $\prod \mathcal{A}_k$ be C*-direct product of the $\mathcal{A}_k$, *i.e.*, the set of bounded sequences $(x_k)_{k=1}^{\infty}$, with $x_k \in \mathcal{A}_k$, with pointwise operations and sup norm; and let $\sum \mathcal{A}_k$ be the C*-direct sum, the set of sequences converging to zero in norm. Then $\prod \mathcal{A}_k$ is a C*-algebra and $\sum \mathcal{A}_k$ is a closed two-sided ideal. Let π be the quotient map from $\prod \mathcal{A}_k$ to $\prod \mathcal{A}_k / \sum \mathcal{A}_k$. Then $\prod \mathcal{A}_k / \sum \mathcal{A}_k$ is a unital C*-algebra. If we denote $\pi((x_k)_{k=1}^{\infty})$ by $[(x_k)_k]$ for any $(x_k)_{k=1}^{\infty}$ in $\prod \mathcal{A}_k$, then

$$\|[(x_k)_k]\| = \limsup_{k \to \infty} \|x_k\|.$$

Suppose $\mathcal{A}$ is a separable unital C*-algebra on a Hilbert space $\mathcal{H}$. Let $\mathcal{H}^{\infty} = \bigoplus_{\mathbb{N}} \mathcal{H}$, and for any $x \in \mathcal{A}$, let x^{∞} be the element $\bigoplus_{\mathbb{N}} x = (x, x, x, \dots)$ in $\prod \mathcal{A}^{(k)} \subset B(\mathcal{H}^{\infty})$, where $\mathcal{A}^{(k)}$ is the k-th copy of $\mathcal{A}$.

In [24], Voiculescu shows the following result:

> Let $\mathcal{A}$ be a separable unital C*-algebra and $\pi_i \colon \mathcal{A} \to B(\mathcal{H}_i)$ be unital faithful $*$-representations for $i = 1, 2$ satisfying $\pi_i(\mathcal{A}) \cap \mathcal{K}(\mathcal{H}_i) = 0$ for $i = 1, 2$, where $\mathcal{K}(\mathcal{H}_i)$ is the set of compact operators on $\mathcal{H}_i$. Then π_1 is approximately unitary equivalent to π_2.

The following statement is a variation of Voiculescu's result and can be found in [16, Theorem 5.1].

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Proposition 5.1.1 *Suppose $\mathcal{A}$ is a separable unital C^*-algebra, and $\mathcal{H}_1, \mathcal{H}_2$ are separable infinite dimensional Hilbert spaces. Let $\pi_i \colon \mathcal{A} \to B(\mathcal{H}_i)$ be unital $*$-representations for $i = 1, 2$. If, for every $x \in \mathcal{A}$,*

$$\operatorname{rank}(\pi_1(x)) \leq \operatorname{rank}(\pi_2(x)),$$

then there is a sequence $\{U_n\}_{n=1}^{\infty}$ of unitary operators, with $U_n \colon \mathcal{H}_1 \to \mathcal{H}_2$, such that, for every $x \in \mathcal{A}$,

$$U_n^* \pi_2(x) U_n \to \pi_1(x) \text{ $*$-SOT as } n \to \infty.$$

In particular, if $\mathcal{A} \subset B(H_2)$, $\mathcal{A} \cap \mathcal{K}(\mathcal{H}_2) = 0$, and $\pi \colon \mathcal{A} \to B(\mathcal{H}_1)$ is any unital $*$-homomorphism, then there is a sequence $\{U_n\}_{n=1}^{\infty}$ of unitary operators, with $U_n \colon \mathcal{H}_1 \to \mathcal{H}_2$,, such that, for every $x \in \mathcal{A}$,

$$U_n^* x U_n \to \pi(x) \text{ $*$-SOT as } n \to \infty.$$

Theorem 5.1.2 *Suppose that $\mathcal{A}$ is a unital C^*-algebra generated by a sequence of self-adjoint elements $x_1, x_2, \dots,$ in $\mathcal{A}$. Then the following are equivalent:*

(i) *$\mathcal{A}$ is an MF algebra*

(ii) *If $\pi \colon \mathcal{A} \to B(\mathcal{H})$ is a faithful $*$-representation of $\mathcal{A}$ on an infinite dimensional separable complex Hilbert space $\mathcal{H}$, then there are a sequence of positive integers $\{m_k\}_{k=1}^{\infty}$, families of self-adjoint matrices $\{A_1^{(k)}, A_2^{(k)}, \dots\}$ in $\mathcal{M}_{m_k}^{s.a.}(\mathbb{C})$ for $k = 1, 2, \dots,$ and unitary operators $U_k : \mathcal{H} \to (\mathbb{C}^{m_k})^{\infty}$ for $k = 1, 2, \dots,$ such that*

(a)

$$\lim_{k\to\infty} \|P(A_1^{(k)}, A_2^{(k)}, \dots)\| = \|P(x_1, x_2, \dots)\|, \ \forall\, P \in \mathbb{C}\langle X_1, X_2 \dots\rangle,$$

where $\mathbb{C}\langle X_1, X_2 \dots\rangle = \bigcup_{m=1}^{\infty} \mathbb{C}\langle X_1, X_2 \dots, X_m\rangle$

(b)

$$U_k^* \, (A_n^{(k)})^{\infty} \, U_k \to \pi(x_n) \text{ in $*$-SOT as } k \to \infty, \quad for\ 1 \leq i \leq n,$$

$$\text{where } (A_n^{(k)})^{\infty} = A_n^{(k)} \oplus A_n^{(k)} \oplus A_n^{(k)} \cdots \in B((\mathbb{C}^{m_k})^{\infty}).$$

Proof (ii) $\Rightarrow$ (i) Suppose (ii) is true. It follows from part (a) of statement (ii) that the map

$$x_n \mapsto \left[\left(A_n^{(k)}\right)_k\right]$$

extends to a faithful unital $*$-homomorphism from $\mathcal{A}$ into $\prod B(\mathbb{C}^{m_k}) / \sum B(\mathbb{C}^{m_k})$. Thus $\mathcal{A}$ is an MF algebra.

(i) $\Rightarrow$ (ii) Suppose $\mathcal{A}$ is an MF algebra, and suppose $\pi \colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a faithful unital $*$-homomorphism on an infinite dimensional separable Hilbert space $\mathcal{H}$. Let $\mathbb{C}_{\mathbb{Q}}\langle X_1, X_2, \dots\rangle = \{P_r\}_{r=1}^{\infty}$. We can assume that each P_N only involves the variables $X_1, \dots, X_N$. We also assume that $\max_{n\geq 1} \|x_n\| < R < \infty$.

Since $\mathcal{A}$ is an MF algebra, so is every subalgebra. Hence, by Lemma 2.9.1, for every $N \in \mathbb{N}$ there is

$$\left(A_{N,1}, \dots, A_{N,N}\right) \in \Gamma_R^{\text{top}}(x_1, \dots, x_N, P_1, \dots, P_N, \frac{1}{N}, k_N)$$

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for some $k_N \in \mathbb{N}$, and we can assume that $k_1 \le k_2 \le \cdots$.

We define $A_{N,n} = 0 \in \mathcal{M}_{k_N}(\mathbb{C})$ when $n > N$. We then have, for every $P \in \mathbb{C}_{\mathbb{Q}}\langle X_1, X_2, \dots \rangle$,

$$\left| \|P(x_1, x_2, \dots)\| - \left\| P\left(A_{N,1}, A_{N,2}, \dots\right) \right\| \right| \to 0$$

as $N \to \infty$.

If $m < s$ are positive integers, we define

$$T_{m,n} = \left(A_{m,n}\right)^{(\infty)} \oplus \left(A_{m+1,n}\right)^{(\infty)} \oplus \cdots \quad \text{and} \quad T_{m,s,n} = \left(A_{m,n}\right)^{(\infty)} \oplus \cdots \oplus \left(A_{s,n}\right)^{(\infty)}$$

for $n \in \mathbb{N}$.

Suppose $B \in \mathcal{M}_{k_m}(\mathbb{C})$. Note that, for every $n \ge 1$, we have

$$T_{m,s,n} \oplus B^{(\infty)} \oplus B^{(\infty)} \oplus \cdots \to T_{m,n} \text{ in the } *\text{-SOT as } s \to \infty,$$

since the first $s - m$ summands of each operator are the same. If we let $B = A_{m,n}$, then $T_{m,s,n} \oplus B^{(\infty)} \oplus B^{(\infty)} \oplus \cdots$ is unitarily equivalent to $T_{m,s,n}$. Thus, for each $s \in \mathbb{N}$, there is a unitary operator $U_{m,s}$ (not depending on n) such that

$$U_{m,s}^* T_{m,s,n} U_{m,s} \to T_{m,n} \ *\text{-SOT as } s \to \infty$$

for every $n \in \mathbb{N}$.

Next suppose $m \in \mathbb{N}$ and $P \in \mathbb{C}\langle X_1, X_2, \dots \rangle$. We then have

$$\left\| P\left(T_{m,1}, T_{m,2}, \dots\right) \right\| \ge \lim_{N \to \infty} \left\| P\left(A_{N,1}, A_{N,2}, \dots\right) \right\| = \|P(x_1, x_2, \dots)\|.$$

Hence there is a unital $*$-homomorphism π_m from $C^*\left(T_{m,1}, T_{m,2}, \dots\right)$ to $\pi(\mathcal{A})$ such that

$$\pi_m(T_{m,n}) = \pi(x_n) \quad \text{for } n \in \mathbb{N}.$$

If $T \in C^*\left(T_{m,1}, T_{m,2}, \dots\right)$ and $T \neq 0$, then clearly $\operatorname{rank}(T) = \infty$. Thus,

$$\operatorname{rank}(T) \ge \operatorname{rank}(\pi_m(T)) \quad \text{for every } T \in C^*\left(T_{m,1}, T_{m,2}, \dots\right).$$

It follows from Proposition 5.1.1 that there is a sequence $\{V_{m,t}\}$ of unitary operators such that, for every $T \in C^*\left(T_{m,1}, T_{m,2}, \dots\right)$,

$$V_{m,t} T V_{m,t}^* \to \pi_m(T) \ *\text{-SOT as } t \to \infty.$$

Suppose $\{e_1, e_2, \dots\}$ is an orthonormal basis for $\mathcal{H}$. For each $N \in \mathbb{N}$, let $m_N \ge N$ be a positive integer. Then there are $s_N, t_N \in \mathbb{N}$ such that

$$\left\| \left[V_{m_N,t_N} U_{m_N,s_N}^* T_{m_N,s_N,n} U_{m_N,s_N} V_{m_N,t_N}^* - \pi(x_n) \right] e_j \right\| < \tfrac{1}{N}$$

and

$$\left\| \left[V_{m_N,t_N} U_{m_N,s_N}^* T_{m_N,s_N,n} U_{m_N,s_N} V_{m_N,t_N}^* - \pi(x_n) \right]^* e_j \right\| < \tfrac{1}{N}$$

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for $1 \leq n, j \leq N$. It follows that

$$V_{m_N,t_N} U^*_{m_N,s_N} T_{m_N,s_N,n} U_{m_N,s_N} V^*_{m_N,t_N} \to \pi(x_n) \ *\text{-SOT as } N \to \infty,$$

for $n \geq 1$.

Let

$$A_n^{(N)} = A_{m_N,n} \oplus \cdots \oplus A_{s_N,n} \quad for\ n \in \mathbb{N}.$$

Then there is a sequence $\{W_N\}_{N=1}^{\infty}$ of unitary operators such that

$$W_N (A_n^{(N)})^{\infty} W_N{}^* = T_{m_N,s_N,n} \qquad for\ n \in \mathbb{N}.$$

Putting everything together, we now have

$$\lim_{N\to\infty} \|P(A_1^{(N)}, A_2^{(N)}, \dots)\| = \|P(x_1, x_2, \dots)\|, \quad \forall\, P \in \mathbb{C}_{\mathbb{Q}}\langle X_1, X_2 \dots\rangle,$$

and

$$V_{m_N,t_N} U^*_{m_N,s_N} W_N (A_n^{(N)})^{\infty} W_N{}^* U_{m_N,s_N} V^*_{m_N,t_N} \to \pi(x_n) \ *\text{-SOT}$$

as $N \to \infty$, for $n \geq 1$. Hence statement (ii) is true. ∎

For the proof of the main theorem of this subsection we need the following well-known lemma. For completeness we outline a proof.

Lemma 5.1.3 *Suppose Q is a finite rank projection in $B(\mathcal{H})$, and suppose $\{Q_n\}$ is a sequence of projections converging to 1 in SOT. Then there is a sequence $\{W_n\}$ of unitary operators and an $N \in \mathbb{N}$, such that*

(i) $\|1 - W_n\| \to 0$, *and*
(ii) $Q \leq W_n^* Q_n W_n$ *for all* $n \geq N$.

Proof Define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(t) = 0$ if $t \leq 1/3$, $g(t) = 3t - 1$ if $1/3 \leq t \leq 2/3$ and $g(t) = 1$ if $2/3 < t$. Let $A_n = Q + (1 - Q)\, Q_n\, (1 - Q)$. Then $\|A_n - Q_n\| \to 0$, so $\left\|A_n^2 - A_n\right\| \to 0$, which implies that $\sigma(A_n) \subset \left\{t : \max\left(|t|, |1-t|\right) \leq 1/\sqrt{n}\right\}$. Thus $P_n = g(A_n)$ is eventually a projection and $\|P_n - Q_n\| \to 0$ and $Q \leq P_n$ for every n. Also $S_n = P_n Q_n + (1 - P_n)(1 - Q_n) \to 1$, and $W_n = \left(S_n S_n^*\right)^{-1/2} S_n \to 1$ and W_n is unitary. Also $P_n S_n = S_n Q_n$ implies that $P_n \left(S_n S_n^*\right) = \left(S_n S_n^*\right) P_n$, which implies $P_n W_n = W_n Q_n$, or $P_n = W_n^* Q_n W_n$. ∎

In [8], Exel and Loring showed that the unital full free product of two residually finite dimensional C*-algebra is residually finite dimensional, which extends an earlier result by Choi in [6]. In [3], Boca showed that the unital full free product of two quasidiagonal C*-algebras is also quasidiagonal. Our next result provides the analogue of the preceding results from Choi, Exel, and Loring, and Boca in the context of MF algebras.

Theorem 5.1.4 *Suppose $\{\mathcal{A}_i : i \in I\}$ is a countable family of separable MF C^*-algebras. Then the free product $\mathcal{A} = \ast_{\mathbb{C}, i\in I} \mathcal{A}_i$ is an MF algebra.*

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Proof We can assume that $I \subset \mathbb{N}$. In fact we can assume that $I = \mathbb{N}$, because if I is finite, we can let $\mathcal{A}_i = \mathbb{C}$ when $i \in \mathbb{N}\backslash I$, and the free product stays the same.

Assume that $\mathcal{A}_i$ is generated by $\{x_{i,n} : n \in \mathbb{N}\}$. We can assume that $\mathcal{A}_i \subset \mathcal{A}$ for each $i \in I$ by identifying $\mathcal{A}_i$ with $\sigma_i(\mathcal{A}_i)$. Thus $\mathcal{A}$ is generated by $\{x_{i,n} : i \in I, n \in \mathbb{N}\}$. Write $\mathbb{C}_{\mathbb{Q}}\langle X_1, X_2, \dots\rangle = \{P_1, P_2, \dots\}$ in such a way that each P_N only involves the variables $X_1, \dots, X_N$.

Suppose $\pi\colon \mathcal{A} \to B(\mathcal{H})$ is a faithful $*$-homomorphism with $\mathcal{H}$ separable and infinite dimensional. Then the restriction of π to each $\mathcal{A}_i$ is also faithful.

Suppose $\{e_1, e_2, \dots\}$ is an orthonormal basis for $\mathcal{H}$. Suppose $N \in \mathbb{N}$. Note that each $\mathcal{A}_i$ is an MF algebra for $i \in \mathbb{N}$. It follows from Lemma 5.1.3 that for $1 \leq i \leq N$, there are a positive integer $m_{N,i}$ and a unitary operator $U_{N,i}$ and operators $A_{N,i,1}, \dots, A_{N,i,N} \in \mathcal{M}^{s.a.}_{m_{N,i}}(\mathbb{C})$ such that

$$\big|\|P_j(A_{N,i,1}, \dots, A_{N,i,N}, 0, 0, \dots)\| - \|P_j(x_{i,1}, x_{i,2}, \dots)\|\big| < \tfrac{1}{N}$$

and

$$\left\|\left[\pi\left(x_{i,n}\right) - U^*_{N,i}A^{(\infty)}_{N,i,n}U_{N,i}\right] e_k\right\| + \left\|\left[\pi\left(x_{i,n}\right) - U^*_{N,i}A^{(\infty)}_{N,i,n}U_{N,i}\right]^* e_k\right\| < \tfrac{1}{N} \tag{5.1}$$

for $1 \leq i, j, k \leq N$. If $i > N$ or $n > N$, we define $A_{N,i,n} = 0 \in \mathcal{M}^{s.a.}_{m_{N,i}}(\mathbb{C})$.

With respect to the decomposition $A^{(\infty)}_{N,i,n} = A_{N,i,n} \oplus A_{N,i,n} \oplus \cdots$, define projections $E_{N,i,1} = 1 \oplus 0 \oplus 0 \cdots$, $E_{N,i,2} = 1 \oplus 1 \oplus 0 \oplus \cdots, \dots$. Clearly $U^*_{N,i}E_{N,i,s}U_{N,i} \to 1$ in SOT as $s \to \infty$, so it follows from Lemma 5.1.3 that we can re-choose $U_{N,1}$ so that (5.1) still holds and there is a positive integers s_1 such that $e_1, \dots, e_N$ are in the range of $U^*_{N,1}E_{N,1,s_1}U_{N,1}$. Similarly, we can re-choose $U_{N,2}$ and find an $s_2 > s_1$ so that (5.1) still holds and $U^*_{N,1}E_{N,1,s_1}U_{N,1} \leq U^*_{N,2}E_{N,2,s_2}U_{N,2}$. We can proceed inductively to redefine the rest of the $U_{N,i}$'s so that (5.1) still holds and such that

$$U^*_{N,i}E_{N,i,s_i}U_{N,i} \leq U^*_{N,i+1}E_{N,i+1,s_{i+1}}U_{N,i+1}$$

for $1 \leq i < N$.

For each $1 \leq i \leq N$, let $\mathcal{H}_{N,i}$ be the range of the projection $U^*_{N,i}E_{N,i,s_i}U_{N,i}$. It is clear that $\mathcal{H}_{N,i}$ reduces each of the operators $U^*_{N,i}A^{(\infty)}_{N,i,n}U_{N,i}$ for $n \in \mathbb{N}$, and that the restriction of $U^*_{N,i}A^{(\infty)}_{N,i,n}U_{N,i}$ to $\mathcal{H}_{N,i}$ is unitarily equivalent to a direct sum of s_i copies of $A_{N,i,n}$. Since $e_1, \dots, e_N \in \mathcal{H}_{N,i}$, we can change the restriction of $U^*_{N,i}A^{(\infty)}_{N,i,n}U_{N,i}$ to $\mathcal{H}^{\perp}_{N,i}$ without affecting (5.1).

Next we choose a finite-dimensional subspace $\mathcal{H}_N$ of $\mathcal{H}$ containing $\mathcal{H}_{N,N}$ such that $d_N = \dim \mathcal{H}_N$ is the product of the dimensions of the $\mathcal{H}_{N,i}$'s for $1 \leq i \leq N$. For each $1 \leq i \leq N$, we can write $\mathcal{H}_N \ominus \mathcal{H}_{N,i}$ as a direct sum of finitely many copies of $\mathcal{H}_{N,i}$. Thus, for each $1 \leq i \leq N$, we can find a unitary operator $V_{N,i} : \mathcal{H}_N \to (\mathbb{C}^{m_{N,i}})^{d_N/m_{N,i}}$ such that, for $n \geq 1$, and every $\xi \in \mathcal{H}_{N,i}$,

$$V^*_{N,i}A^{(d_N/m_{N,i})}_{N,i,n}V_{N,i}\xi = U^*_{N,i}A^{(\infty)}_{N,i,n}U_{N,i}\xi.$$

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We can identify $\mathcal{H}_N$ with $\mathbb{C}^{d_N}$, and let $B_{N,i,n} = V_{N,i}^* A_{N,i,n}^{(d_N/m_{N,i})} V_{N,i} \in \mathcal{M}_{d_N}(\mathbb{C})$. Since $\mathcal{H}$ is infinite-dimensional, we can view $\mathcal{H} \ominus \mathcal{H}_N$ as an infinite direct sum of copies of $\mathcal{H}_N$. Thus there is a unitary operator $W_N : \mathcal{H} \to \mathcal{H}_N^\infty$ such that

$$W_N^* B_{N,i,n}^{(\infty)} W_N|_{\mathcal{H}_N} = B_{N,i,n}$$

for all $1 \le i, n \le N$.

It follows that, for $1 \le i, j, k \le N$,

$$\big| \|P_j(B_{N,i,1}, B_{N,i,2}, \dots)\| - \|P_j(x_{i,1}, x_{i,2}, \dots)\| \big| = \big| \|P_j(A_{N,i,1}, A_{N,i,2}, \dots)\| - \|P_j(x_{i,1}, x_{i,2}, \dots)\| \big| < \tfrac{1}{N}$$

and

$$\left\| \left[\pi\left(x_{i,n}\right) - W_N^* B_{N,i,n}^{(\infty)} W_N \right] e_k \right\| + \left\| \left[\pi\left(x_{i,n}\right) - W_N^* B_{N,i,n}^{(\infty)} W_N \right]^* e_k \right\| < \tfrac{1}{N}$$

Therefore,

$$W_N^* B_{N,i,n}^{(\infty)} W_N \to \pi\left(x_{i,n}\right) \text{ in } *\text{-SOT as } N \to \infty,$$

for all $n, i \in \mathbb{N}$. If $P \in \mathbb{C}_{\mathbb{Q}}\langle X_{i,n} : i, n \in \mathbb{N}\rangle$, we have

$$P((W_N^* B_{N,i,n}^{(\infty)} W_N)_{i,n\in\mathbb{N}}) \to P((\pi(x_{i,n}))_{i,n\in\mathbb{N}}) \qquad \text{in } *\text{-SOT, as } N \to \infty.$$

Thus we have

$$\|P((x_{i,n})_{i,n\in\mathbb{N}})\| = \|P((\pi(x_{i,n}))_{i,n\in\mathbb{N}})\| \le \liminf_{N\to\infty} \|P((W_N^* B_{N,i,n}^{(\infty)} W_N)_{i,n\in\mathbb{N}})\|.$$

On the other hand, for each $i, n \in \mathbb{N}$, the map

$$x_{i,n} \mapsto [(W_N^* B_{N,i,n}^{(\infty)} W_N)_N]$$

extends to a unital $*$-homomorphism $\rho_i \colon \mathcal{A}_i \to \prod B(\mathcal{H}_N)/\sum B(\mathcal{H}_N)$. From the definition of full free product, there must be a unital $*$-homomorphism $\rho : \mathcal{A} \to \prod B(\mathcal{H}_N)/\sum B(\mathcal{H}_N)$ such that $\rho|_{\mathcal{A}_i} = \rho_i$ for each i. Hence

$$\begin{aligned} \|P((x_{i,n})_{i,n\in\mathbb{N}})\| &\ge \|\rho(P((x_{i,n})_{i,n\in\mathbb{N}}))\| = \limsup_{N\to\infty} \|P((W_N^* B_{N,i,n}^{(\infty)} W_N)_{i,n\in\mathbb{N}})\| \\ &\ge \liminf_{N\to\infty} \|P((W_N^* B_{N,i,n}^{(\infty)} W_N)_{i,n\in\mathbb{N}})\| \ge \|P((x_{i,n})_{i,n\in\mathbb{N}})\|. \end{aligned}$$

Hence ρ is an isometry, which shows that $\mathcal{A}$ is MF. ■

Recall that $\mathcal{A} \subset B(\mathcal{H})$ is a separable quasidiagonal C*-algebra if there is an increasing sequence of finite-rank projections $\{E_i\}_{i=1}^\infty$ on H tending strongly to the identity such that $\|xE_i - E_i x\| \to 0$ as $i \to \infty$ for any $x \in \mathcal{A}$. The examples of quasidiagonal C*-algebras include all abelian C*-algebras and all finite dimensional

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C*-algebras. An abstract separable C*-algebra $\mathcal{A}$ is quasidiagonal if there is a faithful $*$-representation $\pi : \mathcal{A} \to B(\mathcal{H})$ such that $\pi(\mathcal{A}) \subset B(\mathcal{H})$ is quasidiagonal.

In [15], Haagerup and Thorbjørnsen showed $C_r^*(F_2)$ is an MF algebra. Combining this with Voiculescu's discussion in [25], they were able to conclude a striking result that $\mathrm{Ext}(C_r^*(F_2))$ is not a group. Also based on [25], Brown showed in [4] that if $\mathcal{A}$ is an MF algebra and $\mathrm{Ext}(\mathcal{A})$ is a group, then $\mathcal{A}$ is a quasidiagonal C*-algebra. It is a well-known fact that $C_r^*(F_2)$ is not a quasidiagonal C*-algebra and any subalgebra of a quasidiagonal C*-algebra is again quasidiagonal. The next corollary now follows from Haagerup and Thorbjørnsen's result on $C_r^*(F_2)$ and our Theorem 5.4.6.

Corollary 5.1.5 *Suppose that $\mathcal{B}$ is a unital separable MF algebra. Then $C_r^*(F_2) *_{\mathbb{C}} \mathcal{B}$ is an MF algebra. Moreover,* $\mathrm{Ext}(C_r^*(F_2) *_{\mathbb{C}} \mathcal{B})$ *is not a group.*

5.2 Topological Free Entropy Dimension in Full Free Products of Unital C*-algebras

In this subsection, given a family of positive integers $n_1, \dots, n_m$, we will let $\{X_j^{(i)}\}_{1\le i\le m; 1\le j\le n_i}$ be a family of indeterminants and let $\{P_r\}_{r=1}^{\infty}$ be a family of non-commutative polynomials in $\mathbb{C}\langle X_j^{(i)} : 1 \le i \le m; 1 \le j \le n_i\rangle$ with rational coefficients. For each $1 \le i \le m$ and $j \ge 1$, let $P_j^{(i)}$ be a polynomial in $X_1^{(i)}, \dots, X_{n_i}^{(i)}$ defined by

$$P_j^{(i)}(X_1^{(i)}, \dots, X_{n_i}^{(i)}) = P_j(0, \dots, 0, X_1^{(i)}, \dots, X_{n_i}^{(i)}, 0, \dots, 0).$$

Lemma 5.2.1 *Suppose $\{\mathcal{A}_i\}_{i=1}^m$ $(m \ge 2)$ is a family of unital C*-subalgebras of a C*-algebra $\mathcal{D}$. Suppose $\{x_j^{(i)}\}_{j=1}^{n_i}$ is a family of self-adjoint generators of $\mathcal{A}_i$ for $1 \le i \le m$. Then*

$$\delta_{\mathrm{top}}(x_1^{(1)}, \dots, x_{n_1}^{(1)}, \dots, x_1^{(m)}, \dots, x_{n_m}^{(m)}) \le \sum_{i=1}^m \delta_{\mathrm{top}}(x_1^{(i)}, \dots, x_{n_i}^{(i)}).$$

Proof Let $R > \max_{1\le i\le n, 1\le j\le n_i} \|x_j^{(i)}\|$ be a positive number. For any $r \in \mathbb{N}$ and $\epsilon > 0$, there is a positive integer r_1 such that

$$\begin{aligned}
&\Gamma_R^{(\mathrm{top})}(\sigma_1(x_1^{(1)}), \dots, \sigma_1(x_{n_1}^{(1)}), \dots, \sigma_m(x_1^{(m)}), \dots, \sigma_m(x_{n_m}^{(m)}); k, \epsilon, P_1, \dots, P_{r_1})\\
&\quad \subseteq \Gamma_R^{(\mathrm{top})}(\sigma_1(x_1^{(1)}), \dots, \sigma_1(x_{n_1}^{(1)}); k, \epsilon, P_1^{(1)}, \dots, P_r^{(1)}) \oplus \cdots\\
&\quad\quad \cdots \oplus \Gamma_R^{(\mathrm{top})}(\sigma_m(x_1^{(m)}), \dots, \sigma_m(x_{n_m}^{(m)}); k, \epsilon, P_1^{(m)}, \dots, P_r^{(m)})\\
&\quad = \Gamma_R^{(\mathrm{top})}(x_1^{(1)}, \dots, x_{n_1}^{(1)}; k, \epsilon, P_1^{(1)}, \dots, P_r^{(1)}) \oplus \cdots\\
&\quad\quad \cdots \oplus \Gamma_R^{(\mathrm{top})}(x_1^{(m)}, \dots, x_{n_m}^{(m)}; k, \epsilon, P_1^{(m)}, \dots, P_r^{(m)}),
\end{aligned}$$

where

$$P_j^{(i)}(\sigma_i(x_1^{(i)}), \dots, \sigma_i(x_{n_i}^{(i)})) = P_j(0, \dots, 0, \sigma_i(x_1^{(i)}), \dots, \sigma_i(x_{n_i}^{(i)}), 0, \dots, 0),$$

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$1 \le j \le r, 1 \le i \le m$. By the definition of topological free entropy dimension, we get that

$$\delta_{\mathrm{top}}(\sigma_1(x_1^{(1)}), \dots, \sigma_1(x_{n_1}^{(1)}), \dots, \sigma_m(x_1^{(m)}), \dots, \sigma_m(x_{n_m}^{(m)})) \le \sum_{i=1}^{m} \delta_{\mathrm{top}}(x_1^{(i)}, \dots, x_{n_i}^{(i)}). \quad \blacksquare$$

5.3 Voiculescu's Semi-microstates

Suppose that $\mathcal{A}$ is a unital C*-algebra and $x_1, \dots, x_n$ is a family of self-adjoint elements in $\mathcal{A}$. Recall Voiculescu's *semi-microstates* as follows. Suppose that

$$\mathbb{C}_{\mathbb{Q}}\langle X_1, \dots, X_n\rangle = \{Q_r : 1 \le r < \infty\}$$

is the family of noncommutative polynomials in $\mathbb{C}\langle X_1, \dots, X_n\rangle$ with rational coefficients. Let $R, \epsilon > 0$, $r, k \in \mathbb{N}$. Define

$$\Gamma_R^{(\mathrm{top}\ 1/2)}(x_1, \dots, x_n; k, \epsilon, Q_1, \dots, Q_r)$$

to be the subset of $(\mathcal{M}_k^{s.a}(\mathbb{C}))^n$ consisting of all $(A_1, \dots, A_n) \in (\mathcal{M}_k^{s.a}(C))^n$ satisfying $\max\{\|A_1\|, \dots, \|A_n\|\} \le R$ and

$$\|Q_j(A_1, \dots, A_n)\| \le \|Q_j(x_1, \dots, x_n)\| + \epsilon, \qquad \forall\, 1 \le j \le r.$$

It is easy to see that

$$\Gamma_R^{(\mathrm{top})}(x_1, \dots, x_n; k, \epsilon, Q_1, \dots, Q_r) \subseteq \Gamma_R^{(\mathrm{top}\ 1/2)}(x_1, \dots, x_n; k, \epsilon, Q_1, \dots, Q_r).$$

It was shown by Voiculescu [29] that

$$\delta_{\mathrm{top}}(x_1, \dots, x_n; \omega) = \limsup_{\omega \to 0} \inf_{\varepsilon > 0, r \in \mathbb{N}} \limsup_{k \to \infty} \frac{\log\Big(\nu_\infty(\Gamma_R^{(\mathrm{top}\ 1/2)}(x_1, \dots, x_n; k, \epsilon, Q_1, \dots, Q_r), \omega)\Big)}{-k^2 \log \omega}.$$

Lemma 5.3.1 *Suppose $\{\mathcal{A}_i\}_{i=1}^m$ ($m \ge 2$) is a family of unital C*-algebras and $\mathcal{D}$ is the full free product of the unital C*-algebras $\{\mathcal{A}_i\}_{i=1}^m$ equipped with the unital embedding $\{\sigma_i : \mathcal{A}_i \to \mathcal{D}\}_{i=1}^m$. Suppose $\{x_j^{(i)}\}_{j=1}^{n_i}$ is a family of self-adjoint generators of $\mathcal{A}_i$ for $1 \le i \le m$. Let $R > \max\{\|x_j^{(i)}\|,\ 1 \le i \le m, 1 \le j \le n_i\}$ be a positive number. For any $r_0 \in \mathbb{N}$ and $\epsilon_0 > 0$, there are $r_1 \in \mathbb{N}$ and $\epsilon_1 > 0$ such that, for any $k \in \mathbb{N}$, if*

$$(A_1^{(i)}, \dots, A_{n_i}^{(i)}) \in \Gamma_R^{(\mathrm{top}\ 1/2)}(x_1^{(i)}, \dots, x_{n_i}^{(i)}; k, \epsilon_1, P_1^{(i)}, \dots, P_{r_1}^{(i)}), \ \text{for } 1 \le i \le m,$$

where $P_1^{(i)}, \dots, P_r^{(i)}$ are defined as in Lemma 5.2.1, then

$$(A_1^{(1)}, \dots, A_{n_1}^{(1)}, \dots, A_1^{(m)}, \dots, A_{n_m}^{(m)}) \in \Gamma_R^{(\mathrm{top}\ 1/2)}(\sigma_1(x_1^{(1)}), \dots, \sigma_1(x_{n_1}^{(1)}), \cdots, \sigma_m(x_1^{(m)}), \dots, \sigma_m(x_{n_m}^{(m)}); k, \epsilon_0, P_1, \dots, P_{r_0}).$$

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Proof We will prove the result by contradiction. Suppose, to the contrary, the result does not hold. Then there are some $r_0 \in \mathbb{N}$ and $\epsilon_0 > 0$ so that the following holds:

(i) for any $r \in \mathbb{N}$, there are $k_r \in \mathbb{N}$ and

$$(A_1^{(i,r)}, \dots, A_{n_i}^{(i,r)}) \in \Gamma_R^{(\text{top }1/2)}(x_1^{(i)}, \dots, x_{n_i}^{(i)}; k_r, 1/r, P_1^{(i)}, \dots, P_r^{(i)}), \text{ for } 1 \le i \le m,$$

satisfying

(5.2)
$$(A_1^{(1,r)}, \dots, A_{n_1}^{(1,r)}, \dots, A_1^{(m,r)}, \dots, A_{n_m}^{(m,r)}) \notin \Gamma_R^{(\text{top }1/2)}(\sigma_1(x_1^{(1)}), \dots, \sigma_1(x_{n_1}^{(1)}), \cdots, \sigma_m(x_1^{(m)}), \dots, \sigma_m(x_{n_m}^{(m)}); k_r, \epsilon_0, P_1, \dots, P_{r_0}).$$

Let γ be a free ultra-filter in $\beta(\mathbb{N}) \setminus \mathbb{N}$. Let $\prod_{r=1}^{\gamma} \mathcal{M}_{k_r}(\mathbb{C})$ be the C* algebra ultra-product of matrices algebras $(\mathcal{M}_{k_r}(\mathbb{C}))_{r=1}^{\infty}$ along the ultra-filter γ, *i.e.*, $\prod_{r=1}^{\gamma} \mathcal{M}_{k_r}(\mathbb{C})$ is the quotient algebra of the unital C*-algebra $\prod_r^{\infty} \mathcal{M}_{k_r}(\mathbb{C})$ by $\mathcal{I}_\infty$, where $\mathcal{I}_\infty = \{(Y_r)_{r=1}^{\infty} \in \prod_r \mathcal{M}_{k_r}(\mathbb{C}) \mid \lim_{r\to\gamma} \|Y_r\| = 0\}$.

Let ϕ_i be the unital $*$-homomorphism from the C*-algebra $\mathcal{A}_i$ into the C*-algebra $\prod_{r=1}^{\gamma} \mathcal{M}_{k_r}(\mathbb{C})$, induced by the mapping

$$x_j^{(i)} \to [(A_j^{(i,r)})_r] \in \prod_{r=1}^{\gamma} \mathcal{M}_{k_r}(\mathbb{C}), \qquad \forall\, 1 \le j \le n_i,$$

where $[(A_j^{(i,r)})_r]$ is the image of $(A_j^{(i,r)})_{r=1}^{\infty}$ in the quotient algebra $\prod_{r=1}^{\gamma} \mathcal{M}_{k_r}(\mathbb{C})$.

By the definition of full free product, we know that there is a unital $*$-homomorphism ψ from $\mathcal{D}$ into $\prod_{r=1}^{\gamma} \mathcal{M}_{k_r}(\mathbb{C})$ so that $\phi_i = \psi \circ \sigma_i$. Hence,

$$\lim_{r\to\gamma} \|P_t(A_1^{(1,r)}, \dots, A_{n_1}^{(1,r)}, \dots, A_1^{(m,r)}, \dots, A_{n_m}^{(m,r)})\| \le \|P_t(\sigma_1(x_1^{(1)}), \dots, \sigma_1(x_{n_1}^{(1)}), \dots, \sigma_m(x_1^{(m)}), \dots, \sigma_m(x_{n_m}^{(m)}))\|, \qquad \forall\, 1 \le t \le r_0.$$

This contradicts equation (5.2). This completes the proof of the lemma. ■

Recall the definition of a stable family of elements in a unital C*-algebra in [19] as follows.

Definition 5.3.2 Suppose that $\mathcal{A}$ is a unital C*-algebra and $x_1, \dots, x_n$ is a family of self-adjoint elements in $\mathcal{A}$. Let $\{Q_i\}_{i=1}^{\infty}$ be the collection of all noncommutative polynomials in $\mathbb{C}\langle X_1, \dots, X_n\rangle$ with rational complex coefficients. The family of elements $x_1, \dots, x_n$ is called stable if for any $\alpha < \delta_{\text{top}}(x_1, \dots, x_n)$ and $R > \max\{\|x_1\|, \dots, \|x_n\|\}$ there is a positive number $C > 0$ satisfying: *for any* $r \in \mathbb{N}, \omega > 0$ *there is a* $k_0 \in \mathbb{N}$ *such that*

$$\nu_\infty\left(\Gamma_R^{(\text{top})}(x_1, \dots, x_n; q \cdot k_0, \tfrac{1}{r}, Q_1, \dots, Q_r), \omega\right) \ge C^{(q\cdot k_0)^2} \left(\tfrac{1}{\omega}\right)^{\alpha\cdot(q\cdot k_0)^2}, \forall\, q \in \mathbb{N}.$$

Example 5.3.3 Any family of self-adjoint generators $x_1, \dots, x_n$ of a finite dimensional C*-algebra is stable. A self-adjoint element x in a unital C*-algebra is stable (see [19]).

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We now define a slight generalization of stability.

Definition 5.3.4 Suppose that $\mathcal{A}$ is a unital C*-algebra and $x_1,\dots,x_n$ is a family of self-adjoint elements in $\mathcal{A}$. Suppose that $\mathbb{C}_{\mathbb{Q}}\langle X_1,\dots,X_n\rangle = \{Q_r\}_{r=1}^{\infty}$. Suppose $\{k_m\}_{k=1}^{\infty}$ is a strictly increasing sequence of positive integers. The family of elements $x_1,\dots,x_n$ is *supported on* $\{k_s\}_{s=1}^{\infty}$ if

$$\delta_{\text{top}}(x_1,\dots,\ x_n) = \liminf_{\omega\to 0^+}\sup_{R>0}\inf_{\varepsilon>0,r\in\mathbb{N}}\liminf_{s\to\infty}\frac{\log\left(\nu_\infty(\Gamma_R^{(\text{top})}(x_1,\dots,x_n;k_m,\epsilon,Q_1,\dots,Q_r),\omega)\right)}{-k_s^2\log\omega}.$$

5.4 Main Result in this Section

Now we are ready to show the additivity of topological free entropy dimension in the full free products of some unital C*-algebras.

Theorem 5.4.1 *Suppose that $\{\mathcal{A}_i\}_{i=1}^m$ $(m\geq 2)$ is a family of unital MF C*-algebras whose free product is $\mathcal{D}$. We assume $\mathcal{A}_i\subset\mathcal{D}$ for $1\leq i\leq m$. Suppose $\{k_m\}_{m=1}^{\infty}$ is a strictly increasing sequence of positive integers, and suppose, for $1\leq i\leq m$, that $\{x_j^{(i)}\}_{j=1}^{n_i}$ is a family of self-adjoint generators of $\mathcal{A}_i$ that is supported on $\{k_s\}_{s=1}^{\infty}$. Then*

$$\delta_{\text{top}}(x_1^{(1)},\dots,x_{n_1}^{(1)},\dots,x_1^{(m)},\dots,x_{n_m}^{(m)}) = \sum_{i=1}^m \delta_{\text{top}}(x_1^{(i)},\dots,x_{n_i}^{(i)}).$$

Proof Suppose that $\{P_r\}_{r=1}^{\infty} = \mathbb{C}_{\mathbb{Q}}\langle X_1^{(1)},\dots,X_{n_1}^{(1)},\dots,X_1^{(m)},\dots,X_{n_m}^{(m)}\rangle$, and, let $P_j^{(i)}$ be a polynomial in $X_1^{(i)},\dots,X_{n_i}^{(i)}$ defined as in Lemma 5.2.1 for $1\leq i\leq m$, and $j\in\mathbb{N}$. Choose

$$R>\max\{\|x_j^{(i)}\| : 1\leq j\leq n_i, 1\leq i\leq m\}.$$

Suppose $r_0\in\mathbb{N}$, $\varepsilon_0>0,\omega_0>0$. It follows from Lemma 5.3.1 that there is an $r_1\in\mathbb{N}$ and an $\varepsilon_1>0$ such that, for all $0<\varepsilon<\varepsilon_1$, $r\geq r_1$, $m\in\mathbb{N}$, we have

$$\Gamma_R^{(\text{top }1/2)}(x_1^{(1)},\dots,x_{n_1}^{(1)},\dots,x_1^{(m)},\dots,x_{n_m}^{(m)};k_s,\epsilon_0,P_1,\dots,P_{r_0}) \supseteq \prod_{j=1}^m \Gamma_R^{(\text{top})}(x_1^{(j)},\dots,x_{n_j}^{(j)};k_s,\epsilon,P_1^{(i)},\dots,P_r^{(i)}).$$

It follows that, for all $\omega>0$, we have

$$\begin{aligned}
&\log\Big(\nu_\infty\big(\Gamma_R^{(\text{top }1/2)}(x_1^{(1)},\dots,x_{n_1}^{(1)},\dots,x_1^{(m)},\dots,x_{n_m}^{(m)};k_s,\epsilon_0,P_1,\dots,P_{r_0},\omega\big)\Big)\\
&\geq \log\Big(\prod_{j=1}^m \nu_\infty\big(\Gamma_R^{(\text{top})}(x_1^{(j)},\dots,x_{n_j}^{(j)};k_s,\epsilon,P_1^{(i)},\dots,P_r^{(i)}),\omega\big)\Big)\\
&= \sum_{j=1}^m \log\Big(\nu_\infty\big(\Gamma_R^{(\text{top})}(x_1^{(j)},\dots,x_{n_j}^{(j)};k_s,\epsilon,P_1^{(i)},\dots,P_r^{(i)}),\omega\big)\Big).
\end{aligned}$$

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Therefore we conclude that, for $0 < \varepsilon < \varepsilon_1$ and $r \geq r_1$, we have

$$\begin{aligned}
&\limsup_{k\to\infty} \frac{\log\Big(\nu_\infty\big(\Gamma_R^{(\text{top }1/2)}(x_1^{(1)},\dots,x_{n_1}^{(1)},\dots,x_1^{(m)},\dots,x_{n_m}^{(m)};k,\epsilon_0,P_1,\dots,P_{r_0}),\omega\big)\Big)}{-k^2\log\omega}\\
&\geq \liminf_{s\to\infty} \frac{\log\Big(\nu_\infty\big(\Gamma_R^{(\text{top }1/2)}(x_1^{(1)},\dots,x_{n_1}^{(1)},\dots,x_1^{(m)},\dots,x_{n_m}^{(m)};k_s,\epsilon_0,P_1,\dots,P_{r_0}),\omega\big)\Big)}{-k_s^2\log\omega}\\
&\geq \sum_{j=1}^m \liminf_{s\to\infty} \frac{\log\Big(\nu_\infty\big(\Gamma_R^{(\text{top})}(x_1^{(j)},\dots,x_{n_j}^{(j)};k_s,\epsilon,P_1^{(i)},\dots,P_r^{(i)}),\omega\big)\Big)}{-k_s^2\log\omega},
\end{aligned}$$

since, for all sequence $\{\alpha_m\}, \{\beta_m\}$, we have

$$\liminf_{m\to\infty}(\alpha_m+\beta_m) \geq \liminf_{m\to\infty}\alpha_m + \liminf_{s\to\infty}\beta_m.$$

It follows that

$$\begin{aligned}
&\geq \inf_{r\in\mathbb{N},\varepsilon>0} \sum_{j=1}^m \liminf_{s\to\infty} \frac{\log\Big(\nu_\infty\big(\Gamma_R^{(\text{top})}(x_1^{(j)},\dots,x_{n_j}^{(j)};k_s,\epsilon,P_1^{(i)},\dots,P_r^{(i)}),\omega\big)\Big)}{-k_s^2\log\omega}\\
&\geq \sum_{j=1}^m \inf_{r\in\mathbb{N},\varepsilon>0} \liminf_{s\to\infty} \frac{\log\Big(\nu_\infty\big(\Gamma_R^{(\text{top})}(x_1^{(j)},\dots,x_{n_j}^{(j)};k_s,\epsilon,P_1^{(i)},\dots,P_r^{(i)}),\omega\big)\Big)}{-k_s^2\log\omega}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\delta_{\text{top}}(x_1^{(1)},\dots,x_{n_1}^{(1)},\dots,x_1^{(m)},\dots,x_{n_m}^{(m)})\\
&\geq \limsup_{\omega\to0^+} \sum_{j=1}^m \inf_{r\in\mathbb{N},\varepsilon>0} \liminf_{s\to\infty} \frac{\log\Big(\nu_\infty\big(\Gamma_R^{(\text{top})}(x_1^{(j)},\dots,x_{n_j}^{(j)};k_s,\epsilon,P_1^{(i)},\dots,P_r^{(i)}),\omega\big)\Big)}{-k_s^2\log\omega}\\
&\geq \sum_{j=1}^m \liminf_{\omega\to0^+} \inf_{r\in\mathbb{N},\varepsilon>0} \liminf_{s\to\infty} \frac{\log\Big(\nu_\infty\big(\Gamma_R^{(\text{top})}(x_1^{(j)},\dots,x_{n_j}^{(j)};k_s,\epsilon,P_1^{(i)},\dots,P_r^{(i)}),\omega\big)\Big)}{-k_s^2\log\omega}\\
&= \sum_{i=1}^m \delta_{\text{top}}(x_1^{(i)},\dots,x_{n_i}^{(i)}),
\end{aligned}$$

since each $\{x_1^{(i)},\dots,x_{n_i}^{(i)}\}$ is supported on $\{k_s\}_{s=0}^\infty$. The inequality

$$\delta_{\text{top}}(x_1^{(1)},\dots,x_{n_1}^{(1)},\dots,x_1^{(m)},\dots,x_{n_m}^{(m)}) \leq \sum_{i=1}^m \delta_{\text{top}}(x_1^{(i)},\dots,x_{n_i}^{(i)})$$

follows from Lemma 5.2.1. ∎

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Corollary 5.4.2 *Suppose that* $\{\mathcal{A}_i\}_{i=1}^m$ $(m \geq 2)$ *is a family of unital* C^**-algebras. Suppose that* $\{x_j^{(i)}\}_{j=1}^{n_i}$ *is a family of self-adjoint generators of* $\mathcal{A}_i$ *for* $i = 1, 2, \dots, m$*. Suppose* $\{x_j^{(i)}\}_{j=1}^{n_i}$ *is a stable family in the sense of Definition* 5.3.2 *for* $1 \leq i \leq m$*. Let the unital* C^**-algebra* $\mathcal{D}$ *be the full free product of* $\{\mathcal{A}_i\}_{i=1}^m$ *equipped with the unital embedding* $\{\sigma_i \colon \mathcal{A}_i \to \mathcal{D}\}_{i=1}^m$*. Then*

$$\delta_{\text{top}}(\sigma_1(x_1^{(1)}), \dots, \sigma_1(x_{n_1}^{(1)}), \dots, \sigma_m(x_1^{(m)}), \dots, \sigma_m(x_{n_m}^{(m)})) = \sum_{i=1}^m \delta_{\text{top}}(x_1^{(i)}, \dots, x_{n_i}^{(i)}).$$

If we identify each $x_j^{(i)}$ *in* $\mathcal{A}_i$ *with its image* $\sigma_i(x_j^{(i)})$ *in* $\mathcal{D}$ *when no confusion arises, then*

$$\delta_{\text{top}}(x_1^{(1)}, \dots, x_{n_1}^{(1)}, \dots, x_1^{(m)}, \dots, x_{n_m}^{(m)}) = \sum_{i=1}^m \delta_{\text{top}}(x_1^{(i)}, \dots, x_{n_i}^{(i)}).$$

As a corollary, we have the following result.

Corollary 5.4.3 *Suppose that* $\mathcal{A}_i$ $(i = 1, 2, \dots, m)$ *is a unital* C^* *algebra generated by a self-adjoint element* x_i *in* $\mathcal{A}_i$*. Let* $\mathcal{D}$ *be the full free product of* $\mathcal{A}_1, \dots, \mathcal{A}_n$ *equipped with unital embedding from each* $\mathcal{A}_i$ *into* $\mathcal{D}$*. Identify the element* x_i *in* $\mathcal{A}_i$ *with its image in* $\mathcal{D}$*. Then*

$$\delta_{\text{top}}(x_1, \dots, x_n) = \sum_{i=1}^n \delta_{\text{top}}(x_i) = n - \sum_{i=1}^n \tfrac{1}{n_i},$$

where n_i *is the number of elements in the spectrum of* x_i *in* $\mathcal{A}_i$*. (We use the notation* $1/\infty = 0$*.)*

Proof It follows from Example 5.3.3, Theorem 5.4.1, and the results in [18]. ■

Corollary 5.4.4 *Suppose that* $\mathcal{A}_i$ *is a finite dimensional* C^**-algebra generated by a family of self-adjoint element* $\{x_j^{(i)}\}_{1 \leq j \leq n_i}$ *for* $1 \leq i \leq m$*. Let* $\mathcal{D}$ *be the full free product of* $\mathcal{A}_1, \dots \mathcal{A}_m$ *equipped with unital embedding from each* $\mathcal{A}_i$ *into* $\mathcal{D}$*. Identify the element* $\{x_j^{(i)}\}$ *in* $\mathcal{A}_i$ *with its image in* $\mathcal{D}$*. Then*

$$\delta_{\text{top}}(\{x_j^{(i)}\}_{1 \leq j \leq n_i, 1 \leq i \leq m}) = \sum_{i=1}^m \delta_{\text{top}}(\{x_j^{(i)}\}_{1 \leq j \leq n_i}) = m - \sum_{i=1}^m \frac{1}{dim_{\mathbb{C}} \mathcal{A}_i},$$

where $dim_{\mathbb{C}} \mathcal{A}_i$ *is the complex dimension of* $\mathcal{A}_i$*.*

Proof It follows from Example 5.3.3, Theorem 5.4.1, and the results in [19]. ■

It is worth noting that we can define the notation of "full freeness" so as to state Theorem 5.4.1 in a form similar to Voiculescu's free additivity theorem for free entropy dimension [26].

Definition 5.4.5 Suppose $\mathcal{A}$ is a unital C*-algebra and $\{x_j^{(i)}\}_{j=1}^{n_i}$ is a collection of self-adjoint elements of $\mathcal{A}$ for $1 \leq i \leq m$. We say these collections are *fully free* if the inclusion maps from each $C^*(\{x_j^{(i)}\}_{j=1}^{n_i})$ extend to an *isometric* unital $*$-homomorphism from the full free product of the $C^*(\{x_j^{(i)}\}_{j=1}^{n_i})$'s into $\mathcal{A}$.

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We can restate the theorem using the new terminology.

Theorem 5.4.6 *Suppose $\mathcal{A}$ is a unital MF algebra and $\{x_j^{(i)}\}_{j=1}^{n_i}$ ($1 \le i \le m$) are fully free collections of self-adjoint elements of $\mathcal{A}$ all supported on a common sequence $\{k_s\}_{s=1}^{\infty}$. Then*

$$\delta_{\text{top}}(x_1^{(1)}, \ldots, x_{n_1}^{(1)}, \ldots, x_1^{(m)}, \ldots, x_{n_m}^{(m)}) = \sum_{i=1}^{m} \delta_{\text{top}}(x_1^{(i)}, \ldots, x_{n_i}^{(i)}).$$

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