10

Local supersymmetry

We know that the superpartners of SM particles must acquire SUSY breaking masses, since otherwise they would have been produced in experiments via their gauge interactions. This requires an understanding of the mechanism of supersymmetry breaking. A variety of models for supersymmetry breaking have been postulated in the literature. The general consensus seems to be that the SM superpartners cannot acquire tree-level masses via spontaneous breaking of *global supersymmetry* at the TeV scale: we have seen in Chapter 7 that this leads to phenomenological problems with tree-level sum rules which imply that some sfermions must be lighter than fermions. Within the framework of the MSSM our ignorance of the SUSY breaking mechanism is parametrized by 178 soft SUSY breaking parameters.

The MSSM is, therefore, regarded as a low energy effective theory to be derived from a theory that incorporates supersymmetry breaking. In the next chapter, we will discuss various models for the generation of soft SUSY breaking parameters that have been suggested in the literature. These models circumvent the problems with the sum rules in one of two different ways. Either the models are based on local supersymmetry, or the soft SUSY breaking parameters are generated only at the loop level. As preparation for a discussion of the first of these classes of models, in this chapter we present a short discussion of locally supersymmetric theories where the parameters of SUSY transformations depend on the spacetime co-ordinates. Since supersymmetry is a spacetime symmetry, local supersymmetry necessarily involves gravitation. Local supersymmetry is, therefore, also referred to as supergravity. Supergravity is a large and complex subject in its own right, and its elaboration is beyond the scope of this book. Our purpose here is only to provide the reader with the basic ideas so as to facilitate the development of particle physics models based on it. We begin by reviewing general relativity, the classical theory of gravitation whose supersymmetric extension naturally leads to supergravity.

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10.1 Review of General Relativity

Before proceeding to discuss supergravity, it will be useful to review the classical theory of gravitation, as embodied in Einstein's General Relativity (GR). In GR, physics is formulated on a curved four-dimensional spacetime manifold, and gravitation is a manifestation of this curvature.

The principle of *special* relativity states that the laws of physics are the same for all *inertial* observers. This is bothersome obviously because we can evidently discern the laws of physics, even though we live on Earth in an accelerating frame. Einstein generalized the principle of special relativity to include *all* observers, including those in accelerating frames.

Einstein was deeply impressed by demonstrations such as the Eötvös experiment that gravitational and inertial mass were equal to very high precision. He reasoned that in a freely falling elevator, one would not be able to discern any effects of gravitation via any experiment confined to a sufficiently small region of measurement. This led to the formulation of the *principle of equivalence*, which is one of the cornerstones of GR. It states that in an arbitrary gravitational field one can always transform co-ordinates to a freely falling (locally Lorentz) frame, where effects of gravitation are locally eliminated. In this freely falling frame, the laws of physics take their special relativistic form. Einstein described the equivalence principle as "the happiest thought of my life".

The effects of gravitation can be incorporated by starting with (local) equations that we know to hold in the absence of gravitation, and generalizing these to be form invariant under general co-ordinate transformations. This is so because the equivalence principle tells us that we can always transform to a co-ordinate system (the freely falling frame) in which the effects of gravity are locally absent. To make the equations form-invariant, we will see that we are led to introduce new "fields" (the affine connection introduced below) that incorporate the effects of gravitation. The situation is quite analogous to that in local gauge theories where, to maintain the invariance of the field equations under local gauge transformations, one is forced to introduce the vector fields and the related field strength tensors.

10.1.1 General co-ordinate transformations

General relativity requires the laws of physics to be the same for *any* observer, be they in a co-ordinate system which is rotating, accelerating, or whatever. Whether we use a co-ordinate system x^μ or $x'^\mu = x'^\mu(x)$, we should arrive at the same physical equations, except that the quantities would appear in a different co-ordinate system. This means that the equations describing the laws of physics take the *tensor* form.

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Denoting a general co-ordinate transformation (GCT) by

$$x^\mu \to x'^\mu = x'^\mu(x) \quad \text{(GCT)}, \tag{10.1}$$

the differential line element $\mathrm{d}x^\mu$ transforms under GCTs as

$$\mathrm{d}x^\mu \to \mathrm{d}x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\mathrm{d}x^\nu. \tag{10.2}$$

By the chain rule for differentiation, we note that $\partial x^\mu/\partial x'^\nu$ is the inverse of the transformation matrix $\partial x'^\nu/\partial x^\rho$ that appears in the GCT of the line element in (10.2). The differential volume element

$$\mathrm{d}^4x' = \mathrm{d}x'^0\mathrm{d}x'^1\mathrm{d}x'^2\mathrm{d}x'^3 = J\mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3, \tag{10.3}$$

where the Jacobian is the determinant of the transformation matrix $J = |\partial x'^\mu/\partial x^\nu|$.

An object is a *contravariant* vector under GCTs if its components transform as,

$$V^\mu \to V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}V^\nu. \tag{10.4a}$$

The differential line element $\mathrm{d}x^\mu$ is thus a contravariant vector. Contravariant tensors of rank n are objects with n indices whose components transform as,

$$A^{\mu_1\mu_2\ldots\mu_n} \to A'^{\mu_1\mu_2\ldots\mu_n} = \frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}}\frac{\partial x'^{\mu_2}}{\partial x^{\rho_2}}\ldots\frac{\partial x'^{\mu_n}}{\partial x^{\rho_n}}A^{\rho_1\rho_2\ldots\rho_n}, \tag{10.4b}$$

while scalars transform as

$$\phi \to \phi' = \phi. \tag{10.4c}$$

A scalar may thus be thought of as a tensor of rank zero, and a vector as a tensor of rank one.

The derivative of a scalar function $\phi(x)$, which under a GCT becomes $\phi'(x')$, transforms as

$$\frac{\partial\phi}{\partial x^\mu} \to \frac{\partial\phi'}{\partial x'^\mu} = \frac{\partial\phi}{\partial x'^\mu} = \frac{\partial\phi}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\mu}. \tag{10.5a}$$

The transforming matrix is the inverse of the transformation matrix for contravariant vectors. Objects which transform like $\partial\phi/\partial x^\mu$, i.e. as

$$V_\mu \to V'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}V_\nu \tag{10.5b}$$

are known as covariant vectors. Covariant tensors of rank n are defined to be objects with n indices whose components transform as,

$$A_{\mu_1\mu_2\ldots\mu_n} \to A'_{\mu_1\mu_2\ldots\mu_n} = \frac{\partial x^{\rho_1}}{\partial x'^{\mu_1}}\frac{\partial x^{\rho_2}}{\partial x'^{\mu_2}}\cdots\frac{\partial x^{\rho_n}}{\partial x'^{\mu_n}}A_{\rho_1\rho_2\ldots\rho_n}. \tag{10.5c}$$

Notice that the indices corresponding to contravariant components are written as superscripts, while those corresponding to covariant components are written as subscripts. In this sense, it is convenient to write $\partial\phi/\partial x^\mu$ as $\partial_\mu\phi$.

Mixed tensors with n contravariant and m covariant indices are analogously defined.

Exercise *If $A^{\mu_1\mu_2\ldots\mu_n}$ and $B^{\nu_1\nu_2\ldots\nu_m}$ are contravariant components of tensors with rank n and m, respectively, show that the entity S with $n+m$ indices defined by $S^{\mu_1\mu_2\ldots\mu_n\nu_1\nu_2\ldots\nu_m} = A^{\mu_1\mu_2\ldots\mu_n}B^{\nu_1\nu_2\ldots\nu_m}$ transforms as a contravariant tensor of rank $n+m$. An analogous result also holds for covariant as well as mixed tensors.*

Exercise *If $A^{\mu_1\mu_2\ldots\mu_n}_{\nu_1\nu_2\ldots\nu_m}$ is a mixed tensor with n contravariant and m covariant indices, show that $A^{\mu_1\mu_2\ldots\mu_n}_{\mu_1\nu_2\ldots\nu_m}$ (where the index μ_1 is summed over) is a mixed tensor with $n-1$ contravariant and $m-1$ covariant indices.*

Exercise *Verify that if a tensor is zero in one frame, it is zero in all frames. Convince yourself that this implies that tensor equations retain their form under GCTs. This is why we required that the equations of GR should take the tensorial form.*

Exercise *Let $A^{\mu\nu\ldots\sigma}B_{\alpha\beta\ldots\sigma} = T^{\mu\nu\ldots}_{\alpha\beta\ldots}$ where A and T transform as tensors of the appropriate rank. Show that B transforms as a tensor. We will use this result to show that the "metric tensor" indeed transforms as a tensor.*

10.1.2 Covariant differentiation, connection fields, and the Riemann curvature tensor

We have just seen that the derivative of a scalar function gives us a vector function. It is, therefore, reasonable to ask whether the derivative of a tensor function results in a tensor with rank higher by one. To check this, we consider how the derivative of a (first rank) tensor transforms under a GCT: $\partial V^\mu/\partial x^\nu$. Under a GCT, this transforms as

$$\begin{aligned}\frac{\partial V^\mu}{\partial x^\nu} \to \frac{\partial V'^\mu}{\partial x'^\nu} &= \frac{\partial}{\partial x'^\nu}\left(\frac{\partial x'^\mu}{\partial x^\rho}V^\rho\right)\\ &= \frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial}{\partial x^\sigma}\left(\frac{\partial x'^\mu}{\partial x^\rho}V^\rho\right)\\ &= \frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial V^\rho}{\partial x^\sigma} + \frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial^2 x'^\mu}{\partial x^\sigma\partial x^\rho}V^\rho. \qquad (10.6)\end{aligned}$$

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The presence of the second term in the last line shows that $\partial V^\mu/\partial x^\nu$ does *not* transform as a tensor.[1] The situation is reminiscent of that encountered in local gauge theory. If the field transformed according to some representation of the gauge group, the ordinary derivative of the field did not transform properly. In the same spirit, we introduce a covariant derivative,

$$D_\nu V^\mu \equiv \partial_\nu V^\mu + \Gamma^\mu_{\rho\nu} V^\rho, \tag{10.7}$$

where $\Gamma^\mu_{\rho\nu}(x)$ is a *connection* field,[2] analogous to the vector potential in the covariant derivative of gauge theories. We require that $D_\nu V^\mu$ transforms as a tensor under GCT:

$$D_\nu V^\mu \to D'_\nu V'^\mu = \frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\rho} D_\sigma V^\rho.$$

This then implies that the connection must transform as

$$\Gamma'^\mu_{\rho\nu} = \frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\tau}\frac{\partial x^\lambda}{\partial x'^\rho}\Gamma^\tau_{\lambda\sigma} - \frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial x^\tau}{\partial x'^\rho}\frac{\partial^2 x'^\mu}{\partial x^\sigma \partial x^\tau}. \tag{10.8}$$

Evidently the connection field does not transform as a tensor; its transformation property is that of an *affine connection*. If we construct the symmetric and antisymmetric parts of the affine connection under interchange of the lower indices,

$$\Gamma^\mu_{\rho\nu} = \frac{1}{2}\left(\Gamma^\mu_{\rho\nu} + \Gamma^\mu_{\nu\rho}\right) + \frac{1}{2}\left(\Gamma^\mu_{\rho\nu} - \Gamma^\mu_{\nu\rho}\right) \equiv S^\mu_{\rho\nu} + A^\mu_{\rho\nu}, \tag{10.9}$$

it is easy to see that the antisymmetric piece $A^\mu_{\rho\nu}$ transforms as a tensor. This tensor is known as the torsion tensor. The torsion tensor, usually taken to be zero in GR, does not vanish in supergravity theories when gravitinos (see below) are present.

Since the gradient of a scalar field transforms as a vector, the covariant derivative of a scalar is the same as its ordinary derivative: $\partial_\mu \phi = D_\mu \phi$. If we require that the covariant derivative satisfy the usual product rule, then

$$D_\nu \left(V^\mu W_\mu\right) = \left(D_\nu V^\mu\right) W_\mu + V^\mu \left(D_\nu W_\mu\right) = \partial_\nu \left(V^\mu W_\mu\right),$$

for any contravariant vector V^μ and any covariant vector W_μ. This is only possible if $D_\nu W_\mu = \partial_\nu W_\mu - \Gamma^\rho_{\mu\nu} W_\rho$, i.e. the connection field enters with a minus sign for derivatives of covariant vectors. Covariant derivatives of higher rank tensors can be made by simply introducing a connection field term for each index: e.g. $D_\mu A^\rho_\nu = \partial_\mu A^\rho_\nu + \Gamma^\rho_{\sigma\mu} A^\sigma_\nu - \Gamma^\sigma_{\nu\mu} A^\rho_\sigma$.

Unlike ordinary derivatives, covariant derivatives (except when they act on scalar functions) do not commute. We had already noted this when we considered gauge theories, where we had seen that the commutator of covariant derivatives yields the

[1] Note that if the transformation $x \to x'$ is linear (as is the case for special relativity), this offending second term would be absent.

[2] Manifolds on which a continuous connection field can be defined are known as *affine manifolds*.

field strength tensor $F_{\mu\nu A}$ (see Eq. (6.46)). We can perform a similar exercise in GR:

$$\left[D_\mu, D_\nu\right] V^\rho = R^\rho_{\tau\mu\nu} V^\tau + 2A^\tau_{\mu\nu} D_\tau V^\rho, \tag{10.10}$$

where

$$R^\rho_{\tau\mu\nu} = \partial_\mu \Gamma^\rho_{\tau\nu} - \partial_\nu \Gamma^\rho_{\tau\mu} + \Gamma^\rho_{\sigma\mu}\Gamma^\sigma_{\tau\nu} - \Gamma^\rho_{\sigma\nu}\Gamma^\sigma_{\tau\mu} \tag{10.11}$$

defines the Riemann curvature tensor, and $A^\tau_{\mu\nu}$ is the torsion tensor.

Exercise *This exercise illustrates the use of the equivalence principle described at the beginning of this section.*

In the absence of gravitation (and any other forces) the equation of motion for a spinless particle is

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = 0. \tag{10.12a}$$

Even in the presence of gravitation, this equation still holds true in the freely falling frame, according to the principle of equivalence. A GCT into any other (non-freely falling) frame with co-ordinates $x'^\mu(x)$ implies that

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} \to \frac{\partial x'^\mu}{\partial x^\nu}\frac{\mathrm{d}^2 x^\nu}{\mathrm{d}\tau^2} + \frac{\partial^2 x'^\mu}{\partial x^\lambda \partial x^\nu}\frac{\partial x^\nu}{\partial \tau}\frac{\partial x^\lambda}{\partial \tau}.$$

Notice that the second term (whose presence tells us that $\mathrm{d}^2 x^\mu/\mathrm{d}\tau^2$ is not a vector under GCTs) is the same as the corresponding term in the transformation (10.8). Hence deduce that the equation

$$\frac{\mathrm{d}^2 x'^\mu}{\mathrm{d}\tau^2} + \Gamma'^\mu_{\rho\nu}\frac{\mathrm{d}x'^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x'^\nu}{\mathrm{d}\tau} = 0 \tag{10.12b}$$

is covariant under GCTs. Since Γ' vanishes in the frame in which there is no gravity, this equation then reduces to (10.12a). Hence, the equivalence principle tells us that (10.12b) describes the motion of a particle in an external gravitational field.

Note that torsion makes no contribution to the motion of the particle.

10.1.3 The metric tensor

In the previous section, we have made no mention of the metric tensor in our discussion of the covariant derivative, the connection or even the curvature tensor. Even the equation of motion for a particle in a gravitational field can be stated in terms of just the connection fields. Indeed, there are non-metric theories of gravity, e.g. theories with torsion, but these violate the equivalence principle as we will show.

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From now on, we will focus our attention on standard GR where it is assumed that spacetime is a Riemannian manifold.

Riemannian manifolds, which are manifolds on which a metric (introduced below) can be defined, form a natural setting for formulating GR. On any sufficiently small patch of such a manifold, it is possible to find a Cartesian co-ordinate system for which the separation between two points is given by a Pythagorean-type law. On such a manifold, the differential line element is given by

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{10.13a}$$

and accordingly the length squared of any four vector is given by

$$V^2 = g_{\mu\nu}(x)V^\mu V^\nu. \tag{10.13b}$$

Since the left-hand side is a scalar and the line elements on the right-hand side are vectors, by one of the previous exercises the quantity $g_{\mu\nu}$ transforms as a covariant second rank tensor known as the metric tensor. The metric tensors $g_{\mu\nu}(x)$ and $g^{\mu\nu}(x)$ can be used to raise and lower indices in GR.

We will assume a four-dimensional spacetime with one time-like direction. The principle of equivalence then tells us that we can always transform to a freely falling co-ordinate frame where the metric tensor is locally flat (Minkowski), i.e.

$$g_{\mu\nu}(x) \to \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{10.14}$$

In this frame, the derivative of the metric vanishes. This can be covariantly written as:

$$D_\mu g_{\nu\lambda} = \partial_\mu g_{\nu\lambda} - \Gamma^\rho_{\mu\nu} g_{\lambda\rho} - \Gamma^\rho_{\mu\lambda} g_{\nu\rho} = 0. \tag{10.15}$$

Using the transformation property of the metric tensor together with (10.8), it is straightforward to check that

$$\Gamma^\tau_{\mu\lambda} - \frac{1}{2} g^{\nu\tau} \left(\partial_\mu g_{\nu\lambda} + \partial_\lambda g_{\mu\nu} - \partial_\nu g_{\lambda\mu} \right) \tag{10.16}$$

transforms as a tensor. The part of this tensor *symmetric* under $\mu \leftrightarrow \lambda$ vanishes in the frame where the metric is locally Minkowskian, and hence must vanish in all frames. We thus obtain,

$$\Gamma^\tau_{\mu\lambda} = \frac{1}{2} g^{\nu\tau} \left(\partial_\mu g_{\nu\lambda} + \partial_\lambda g_{\mu\nu} - \partial_\nu g_{\lambda\mu} \right), \tag{10.17}$$

for the components of the connection that are *symmetric* under $\mu \leftrightarrow \lambda$. The corresponding antisymmetric components of the connection are *not* determined by the metric, but depend on the torsion tensor introduced above.

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10.1.4 Einstein Lagrangian and field equations

To obtain the field equations of GR from an action principle, we can try to find an appropriate Lagrangian density, and vary the corresponding action $S = \int \mathcal{L} \mathrm{d}^4 x$. For $\mathcal{L}$, we can construct a scalar by performing successive contractions on the Riemann tensor:

$$R_{\nu\tau} = R^{\rho}_{\nu\rho\tau} \quad \text{(Ricci tensor)}, \quad \text{and} \tag{10.18a}$$

$$R = g^{\nu\tau} R_{\nu\tau} \quad \text{(Ricci scalar)}. \tag{10.18b}$$

The Ricci scalar R is a candidate Lagrangian density, but we also know that the measure $\mathrm{d}^4 x$ is not invariant under GCTs. However, $\sqrt{-g}\, \mathrm{d}^4 x$ is invariant, where $g = det(g_{\mu\nu})$. Thus, $\mathcal{L} = \sqrt{-g} R$ is a candidate Lagrangian density for GR, and is known as the Einstein Lagrangian. Since the Lagrangian density must have mass dimension four, it must be multiplied by a constant with dimensions of M^2. Hence, we write the Lagrangian density for the gravitational field as,

$$\mathcal{L}_G = -\frac{1}{2\kappa^2} \sqrt{-g} R \tag{10.19}$$

where κ^{-2} has dimensions of mass squared.

Exercise *Using the transformation properties of $g_{\mu\nu}$ and $\mathrm{d}^4 x$, show that $\sqrt{-g}\, \mathrm{d}^4 x$ is invariant under GCTs.*

Exercise *Show that the Ricci tensor obtained by contracting the Riemann curvature tensor is symmetric.*

Variation of the Einstein action with respect to the fields $g_{\mu\nu}$ is a lengthy calculation, but can be made simpler using the Palatini formalism wherein the connection fields $\Gamma^{\tau}_{\mu\nu}$ and their derivatives are regarded as independent fields along with $g_{\mu\nu}(x)$. Either approach leads to Einstein's field equations in a vacuum:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \tag{10.20}$$

This equation is generally covariant, and contains at most the second derivative of the metric. We could have included higher powers of R into the action but these would have led to higher derivatives in the equations of motion.

We may also add the effects of matter and/or energy to the Einstein Lagrangian. For instance, including a real scalar field ϕ with Lagrangian $\mathcal{L}_M = \sqrt{-g}(g_{\mu\nu}\partial^{\mu}\phi\partial^{\nu}\phi - m^2\phi^2)$ into the action will bring a source term involving the symmetric energy momentum tensor $T_{\mu\nu}$ into the equations of motion. Although we have illustrated this for coupling to scalar fields, the same is true for coupling

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to *all* matter fields. The constant κ that we introduced for dimensional relations determines the gravitational coupling of matter. It must be chosen to obtain agreement with Newtonian gravity in the non-relativistic, weak field limit. It turns out that $\kappa^2 = 8\pi G_N/c^4$, with G_N being Newton's constant. In natural units with $\hbar = c = 1$, the Planck mass $M_{Pl} = G_N^{-1/2}$. The reduced Planck mass is defined by $M_P = M_{Pl}/\sqrt{8\pi}$ so that $\kappa = 1/M_P$, with $M_P \simeq 2.4 \times 10^{18}$ GeV. It is common to use units in which M_P is also set to unity.

Finally, we can also include the term $\mathcal{L}_\Lambda = \frac{\sqrt{-g}}{\kappa^2}\Lambda$ into the Lagrangian density without bringing higher derivatives of the metric into the field equations. Here, Λ is known as the cosmological constant. Indeed there is evidence for a small but non-zero cosmological constant ($\Lambda \sim (3\text{ meV})^4$ in natural units) in Einstein's equations, indicative of a dark energy that pervades the Universe. Including matter as well as the cosmological constant, Einstein's field equations become,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - g_{\mu\nu}\Lambda = 8\pi G_N T_{\mu\nu}. \tag{10.21}$$

Notice that both sides of this equation are symmetric under interchange of tensor indices.

10.1.5 Spinor fields in General Relativity

The preceding formulation of GR can admit fields transforming as scalars, vectors, and tensors. In supersymmetry, we must necessarily include spinor fields as well, but there exists no generalization of spinorial Lorentz transformation rules to general co-ordinate transformations: mathematically speaking, the group $GL(4)$ has no finite dimensional spinor representations. What is done instead is to define, for every point on the curved spacetime, a tangent space with a flat Minkowski metric in which the spinors may transform. Thus, the action we construct should be invariant under GCTs $x^\mu \to x'^\mu$ on the curved manifold, *and* invariant under local Lorentz transformations (LLTs) on the flat tangent space:

$$\xi^a \to \xi'^a = \Lambda^a_b(x)\xi^b. \tag{10.22}$$

For each spacetime point, $\xi^a(x)$ define a (locally inertial) co-ordinate system in the flat tangent space. It is customary to take Greek indices $\mu = 0$–3 for objects transforming under GCTs, and Latin indices $a = 0$–3 for objects transforming under LLTs. The transformation from local Lorentz co-ordinates to general co-ordinates is given by

$$\frac{\partial \xi^a}{\partial x^\mu} \equiv e^a_\mu, \tag{10.23}$$

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where e^a_μ is known as the *vierbein*.

The vierbein transforms under GCTs as

$$e^a_\mu(x) \to e'^a_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} e^a_\nu(x), \tag{10.24}$$

and under LLTs as

$$e^a_\mu \to e'^a_\mu = \Lambda^a_b e^b_\mu(x). \tag{10.25}$$

The vierbein allows us to connect one co-ordinate system with the other. Thus, an object v^μ which transforms as a vector under GCTs can be related to an object V^a which transforms as a vector under LLTs via

$$V^a = e^a_\mu v^\mu. \tag{10.26}$$

In particular, the metric tensor in each space is related as

$$g_{\mu\nu}(x) = e^a_\mu e^b_\nu \eta_{ab}, \tag{10.27}$$

where η_{ab} is the usual Minkowski metric. From the above relation, knowledge of the vierbein completely determines the form of the metric tensor, and it is sometimes convenient to think of the vierbein as a "square root" of the metric tensor. The Minkowski metric tensor η_{ab} (η^{ab}) can be used to lower (raise) Latin indices, just as $g_{\mu\nu}$ ($g^{\mu\nu}$) can be used to lower (raise) Greek indices. Thus, we also have

$$g^{\mu\nu} = e^\mu_a \, e^\nu_b \, \eta^{ab}. \tag{10.28}$$

Taking the determinant of Eq. (10.27), we are able to replace the Jacobian factor $\sqrt{-g}$ by $\mathbf{e} \equiv det(e^a_\mu)$.

Spinors transform under LLTs as

$$\psi_m(x) \to \psi'_m(x') = \Lambda_{\frac{1}{2}mn}\psi_n(x) \tag{10.29}$$

where $\Lambda_{\frac{1}{2}mn} = \left[\mathrm{e}^{-\mathrm{i}\epsilon_{rs}(x)\sigma_{rs}}\right]_{mn}$, and the spinor index $m = 1$–4, and $\sigma_{rs} = \frac{\mathrm{i}}{2}[\gamma_r, \gamma_s]$.[3] The Dirac matrices satisfy $\{\gamma_r, \gamma_s\} = 2\eta_{rs}$ in local Lorentz space. They are related to the curved space gamma matrices via $\gamma^\mu = e^\mu_r \gamma^r$, and where

$$\{\gamma^\mu(x), \gamma^\nu(x)\} = 2g^{\mu\nu}(x). \tag{10.30}$$

The transformation parameter ϵ_{rs} is antisymmetric on rs and includes six parameters: three rotations and three boosts.

In order to define a covariant derivative for spinor fields $D_\mu\psi$ such that

$$D_\mu\psi \to D'_\mu\psi' = \Lambda_{\frac{1}{2}}\left(D_\mu\psi\right), \tag{10.31}$$

[3] We are economizing notation here by not writing the transformation matrix as $\mathcal{D}(\Lambda_{1/2})_{mn}$, as is the practice by many authors.

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we introduce *spin connection* fields ω_μ^{rs} such that

$$D_\mu\psi = \partial_\mu\psi - \frac{\mathrm{i}}{4}\omega_\mu^{rs}\sigma_{rs}\psi, \qquad (10.32)$$

and, as usual, require these to transform so that (10.31) is satisfied. The covariant derivative of the vierbein will involve both connection and spin connection fields:

$$D_\mu e_\nu^a = \partial_\mu e_\nu^a - \Gamma_{\mu\nu}^\lambda e_\lambda^a + \omega_{\mu b}^a e_\nu^b. \qquad (10.33)$$

A field strength tensor can be computed from the spinor field covariant derivative, just as from a vector field covariant derivative.

Exercise *Evaluate the commutator of spinor covariant derivatives and show that it can be written as*

$$[D_\mu, D_\nu]\psi = -\frac{\mathrm{i}}{4}\sigma_{uv}R_{\mu\nu}^{uv}\psi \qquad (10.34\text{a})$$

where

$$R_{\mu\nu}^{uv} = \partial_\mu\omega_\nu^{uv} - \partial_\nu\omega_\mu^{uv} + \omega_{\mu r}^u\omega_\nu^{rv} - \omega_{\mu r}^v\omega_\nu^{ur}. \qquad (10.34\text{b})$$

This quantity is related to the Riemann curvature tensor via

$$R_{\mu\nu}^{uv} = e_\rho^u\, e_\sigma^v\, R_{\mu\nu}^{\rho\sigma}. \qquad (10.34\text{c})$$

Hint: Recall the generators $M_{ab} = \sigma_{ab}/2$ of the Lorentz group obey the algebra (4.6).

We can again apply the principle of equivalence as we did to obtain $D_\mu g_{\nu\lambda} = 0$, but this time for the vierbein: $D_\mu e_\nu^a = 0$. This gives $4 \times 6 = 24$ constraints, the number of independent components of the spin connection, which can be eliminated as an independent field. Indeed, the spin connection fields ω_μ^{ab} can be constructed from knowledge of the vierbein via,

$$\omega_\mu^{ab} = \frac{1}{2}e^{a\nu}(\partial_\mu e_\nu^b - \partial_\nu e_\mu^b) + \frac{1}{4}e^{a\rho}e^{b\sigma}(\partial_\sigma e_\rho^c - \partial_\rho e_\sigma^c)e_{c\mu} - (a \leftrightarrow b)\,. \qquad (10.35)$$

10.2 Local supersymmetry implies (super)gravity

Our next goal is to examine what happens when we allow the parameters α that characterize SUSY transformations to be spacetime dependent; i.e. when we allow SUSY to be a local symmetry. Such local SUSY transformations are known as *supergravity* transformations since, as we will see presently, a consistent implementation of local SUSY transformations necessarily brings a massless spin 2 field

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into the theory. Moreover, this spin 2 field couples to the energy-momentum tensor for matter, just as in general relativity, and its quanta are identified with gravitons. The spin $\frac{3}{2}$ Rarita–Schwinger field is needed since SUSY requires that the gravitons must have fermionic partners with spin differing by $1/2$. Its quanta are referred to as gravitinos.

An aside on the spin $\frac{3}{2}$ Rarita–Schwinger field *We briefly discuss the basics of massive spin $\frac{3}{2}$ fields, since after supersymmetry breaking the gravitinos acquire a mass. A free massive gravitino may be described by a "vector-spinor" field $\psi_\lambda(x)$, each of whose Majorana spinor components (the spinor index is suppressed) satisfies the Dirac equation,*

$$(\mathrm{i}\not{\partial} - m)\psi_\lambda = 0, \tag{10.36a}$$

and is subject to the subsidiary condition,

$$\gamma^\lambda \psi_\lambda = 0. \tag{10.36b}$$

Contracting (10.36a) with γ^λ, it is easy to see that,

$$\partial^\lambda \psi_\lambda = 0. \tag{10.36c}$$

To understand why ψ_λ describes a spin $\frac{3}{2}$ particle, let us examine the plane wave solutions $\psi_\lambda(x) = u_\lambda(k)\mathrm{e}^{-\mathrm{i}kx}$ of (10.36a) in the rest frame of the particle. Eq. (10.36a) then implies

$$\gamma^0 u_\lambda = u_\lambda. \tag{10.37a}$$

It is most convenient to do the analysis using the standard representation for the gamma matrices. Exactly as for the case of a massive spin $\frac{1}{2}$ particle in its rest frame, we find that the lower two components of all four u_λ must vanish. The subsidiary condition (10.36b) implies that

$$u_0 = \vec{\gamma} \cdot \vec{u}, \tag{10.37b}$$

where $\vec{u}$ has as its components the three four-spinors u_1, u_2, and u_3, all of whose lower components vanish, and whose three upper components are the three two-spinors χ_1, χ_2, and χ_3. Using the explicit form of the $\vec{\gamma}$ matrices, we see from (10.37b) that,

$$\begin{aligned} u_0 &= 0 \\ \vec{\sigma} \cdot \vec{\chi} &= 0. \end{aligned} \tag{10.37c}$$

The two constraints (10.37c) imply that just four of the six components of $\vec{\chi}$ are truly independent. Since the spinors ψ_λ are completely fixed by $\vec{\chi}$, we see that these

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are specified by four independent components, just the right number to describe a massive spin $\frac{3}{2}$ particle in its rest frame.

Exercise *Show that the Lagrangian density*

$$\mathcal{L} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma^5\gamma_\nu\partial_\rho\psi_\sigma - \frac{1}{4}m\bar{\psi}_\mu[\gamma^\mu, \gamma^\nu]\psi_\nu \tag{10.38}$$

yields the Dirac equation (10.36a) as well as the constraint conditions (10.36b) and (10.36c), assuming $m \neq 0$. You may find the identity

$$\gamma^5\gamma^\nu = \frac{\mathrm{i}}{3!}\epsilon^{\nu\rho\sigma\tau}\gamma_\rho\gamma_\sigma\gamma_\tau$$

useful.

Notice that the Lagrangian for the massless case is invariant under the transformation $\psi_\mu \to \psi_\mu + \partial_\mu\alpha$. For this case, the constraints do not follow from the Lagrangian, but have to be imposed as gauge fixing conditions.

To obtain a locally supersymmetric theory, we will adopt the Noether procedure, which was used to derive the simplest supergravity Lagrangians. The Noether procedure is a systematic technique for obtaining a theory invariant under a local symmetry transformation, starting from a theory that is invariant under the corresponding global transformation.

QED serves as an illustrative example. We may start with the simple Dirac Lagrangian for an electron $\mathcal{L} = \mathrm{i}\bar{\psi}\not{\partial}\psi$ which is invariant under a global phase transformation $\psi \to \mathrm{e}^{\mathrm{i}\alpha}\psi$, where α is a constant. If we make the transformation local, so that $\alpha \to \alpha(x)$, then this Lagrangian is no longer invariant, changing by an amount $\delta\mathcal{L} = -\bar{\psi}\gamma^\mu\psi\partial_\mu\alpha$. Invariance can be restored by adding a gauge field term to $\mathcal{L}$ given by $\mathcal{L}' = -e\bar{\psi}\gamma^\mu A_\mu\psi$, where the gauge field transforms as $A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha$, and e in this case is the magnitude of the electric charge. The final QED Lagrangian is obtained by adding the gauge field kinetic term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and an electron mass term $-m\bar{\psi}\psi$, which are separately locally gauge invariant.

To illustrate why local supersymmetry necessarily implies gravity, we apply the Noether procedure to the Wess–Zumino model introduced in Chapter 3. To simplify our analysis, we will examine only the free, massless case with the fields "on shell", meaning that these satisfy their equations of motion. Then we do not have to worry about the auxiliary fields which can be set to zero. Furthermore, from (3.7d) and (3.7e), we see that the SUSY transforms of the auxiliary fields also vanish as long as the fermion field satisfies its equation of motion, $\not{\partial}\psi = 0$.

The Lagrangian for this very simplified model takes the form,

$$\mathcal{L} = \mathcal{L}_{\mathrm{kin}} = \frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}(\partial_\mu B)^2 + \frac{\mathrm{i}}{2}\bar{\psi}\not{\partial}\psi, \tag{10.39}$$

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and is invariant under

$$\delta A = \mathrm{i}\bar{\alpha}\gamma_5\psi, \tag{10.40a}$$

$$\delta B = -\bar{\alpha}\psi, \tag{10.40b}$$

$$\delta\psi = -\mathrm{i}\partial\!\!\!/(-B + \mathrm{i}\gamma_5 A)\alpha. \tag{10.40c}$$

If we now let $\alpha \to \alpha(x)$, and define the local transformation so that the derivative in (10.40c) acts only on the fields, a straightforward calculation shows that the Lagrangian no longer transforms as a total derivative, but instead as,

$$\delta\mathcal{L}_{\text{kin}} = \partial^\mu\left(\frac{1}{2}\bar{\alpha}\gamma_\mu\partial\!\!\!/(-B + \mathrm{i}\gamma_5 A)\psi\right) + (\partial^\mu\bar{\alpha})\left(\partial\!\!\!/\gamma_\mu(-B + \mathrm{i}\gamma_5 A)\right)\psi. \tag{10.41}$$

The additional term can be cancelled by adding to the Lagrangian a term given by

$$\mathcal{L}_1 = -\kappa\bar{\psi}_\mu\partial_\nu(-B + \mathrm{i}\gamma_5 A)\gamma^\nu\gamma^\mu\psi, \tag{10.42}$$

where ψ_μ is a spin $\frac{3}{2}$ field. It has mass dimensionality $[\psi_\mu] = \frac{3}{2}$, so that a dimensional constant κ with $[\kappa] = -1$ must be included to give a dimension four Lagrangian term. The field ψ_μ is effectively a gauge field for the local supersymmetry transformation, just as A_μ was the gauge field for a local phase transformation in the QED example. If ψ_μ transforms under local SUSY as $\bar{\psi}_\mu \to \bar{\psi}_\mu + \frac{1}{\kappa}\partial_\mu\bar{\alpha}$, then the transformation term involving $\partial^\mu\bar{\alpha}$ will cancel!

This of course does not mean that the action corresponding to the Lagrangian density $\mathcal{L}_{\text{kin}} + \mathcal{L}_1$ is supersymmetric because we must now apply the local SUSY transformation laws to the additional Lagrangian term $\mathcal{L}_1$ as well. Clearly, the terms resulting from this transformation are $\mathcal{O}(\kappa)$. Indeed a somewhat lengthy calculation shows that,

$$\delta(\mathcal{L}_{\text{kin}} + \mathcal{L}_1) = -2\mathrm{i}\kappa\bar{\psi}_\mu\gamma_\nu T^{\mu\nu}\alpha + \cdots, \tag{10.43a}$$

where the ellipsis denotes terms involving derivatives of α or total derivatives, and

$$T^{\mu\nu} = (\partial^\mu A)(\partial^\nu A) + (\partial^\mu B)(\partial^\nu B) - \frac{1}{2}\eta^{\mu\nu}\left[(\partial_\rho A)^2 + (\partial_\rho B)^2\right] + \frac{\mathrm{i}}{2}\bar{\psi}\gamma^\mu\partial^\nu\psi, \tag{10.43b}$$

is the canonical energy–momentum tensor for the WZ model.

Exercise *Verify the transformation (10.43a).*

To obtain the $T^{\mu\nu}$ term on the right-hand side of (10.43a) which is written up to derivatives of the parameter α, we need to consider only "global" SUSY transformations when performing the variation. We need to Fierz transform the fermion quartic term so that the transformation parameter α is contracted with the gravitino spinor: only the vector and axial-vector combinations survive. Moreover, since we write this variation only up to a total derivative, and for "on-shell fields",

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many terms vanish due to (10.36a)–(10.36c). Finally, we note that the $\eta^{\mu\nu}$ terms for the scalar field contributions to $T^{\mu\nu}$ vanish when the fields are on-shell.

This term can now be cancelled by adding another term,

$$\mathcal{L}_2 = -g_{\mu\nu}T^{\mu\nu}, \tag{10.44}$$

to the Lagrangian density. We see that the Noether procedure forces us to introduce a massless spin 2 field $g_{\mu\nu}$ that couples to the energy momentum tensor as in General Relativity. The quanta of this field are the gravitons. We require this spin 2 field $g_{\mu\nu}$ to transform as

$$\delta g_{\mu\nu} = -\mathrm{i}\kappa\bar{\alpha}(\gamma_\nu\psi_\mu + \gamma_\mu\psi_\nu). \tag{10.45}$$

We see that local supersymmetry implies gravity. The dimensionful coupling constant κ that we have been forced to introduce can be related to Newton's gravitational constant.

The procedure we have outlined was for the simple case of the massless, non-interacting on-shell WZ model. The locally supersymmetric couplings of the (on-shell) scalar supermultiplet of the Wess–Zumino model can be found in Ferrara *et al.*, and includes many more terms.[4] One must, of course, also include kinetic terms for both the graviton and gravitino fields, and derivatives must be made covariant with respect to general co-ordinate and local Lorentz transformations. A complete derivation is beyond the scope of this text, and we will simply present the answer. The relevant Lagrangian is given by a sum of a pure (supersymmetrized) gravity piece together with a second piece that describes the supersymmetrized gravitational couplings of matter:

$$\mathcal{L} = \mathcal{L}_\mathrm{G} + \mathcal{L}_\mathrm{M}. \tag{10.46}$$

Here, $\mathcal{L}_\mathrm{G}$ is given by a sum of the Einstein Lagrangian and the kinetic term (10.38) for the massless Rarita–Schwinger field:

$$\mathcal{L}_\mathrm{G} = -\frac{\mathbf{e}}{2\kappa^2}R - \frac{1}{2}\epsilon^{\lambda\rho\mu\nu}\bar{\psi}_\lambda\gamma_5\gamma_\mu D_\nu\psi_\rho, \tag{10.47}$$

where $\mathbf{e}$, the determinant of the vierbein, is the Jacobian factor $\sqrt{-g}$ that appears in the Einstein Lagrangian. A comparison with Eq. (10.19) shows that the constant κ introduced in our discussion of local supersymmetry transformations indeed coincides with the same constant that appears in our discussion of general relativity.

[4] S. Ferrara, D. Freedman, P. van Nieuwenhuizen, P. Breitenlohner, F. Gliozzi and J. Scherk, *Phys. Rev.* **D15**, 1013 (1977).

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The (super)gravitational interactions of the matter supermultiplet take the form,

$$
\begin{aligned}
\mathcal{L}_{\mathrm{M}} = \mathbf{e} g_{\mu\nu}\left(\frac{1}{2}\partial^{\mu}A\partial^{\nu}A + \frac{1}{2}\partial^{\mu}B\partial^{\nu}B\right) + \mathbf{e}\,\frac{\mathrm{i}}{2}\bar{\psi}\,\not{D}\psi \\
-\frac{\kappa}{2}\mathbf{e}\,\bar{\psi}_{\mu}\partial_{\nu}(-B+\mathrm{i}\gamma_5 A)\gamma^{\nu}\gamma^{\mu}\psi - \frac{\kappa^2}{16}\mathbf{e}\,(\bar{\psi}\gamma_5\gamma_{\mu}\psi)\,(\bar{\psi}\gamma_5\gamma^{\mu}\psi) \\
-\mathrm{i}\frac{\kappa^2}{8}(B\ \overset{\leftrightarrow}{\partial}_{\sigma}\ A)\left[\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_{\mu}\gamma_{\nu}\psi_{\rho} - \mathrm{i}\mathbf{e}\,\bar{\psi}\gamma_5\gamma^{\sigma}\psi\right] \\
+\frac{\kappa^2}{16}\bar{\psi}\gamma_5\gamma_{\sigma}\psi\left[\mathrm{i}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_{\mu}\gamma_{\nu}\psi_{\rho} + \mathbf{e}\,\bar{\psi}_{\mu}\gamma_5\gamma^{\sigma}\psi_{\mu}\right],
\end{aligned}
\tag{10.48}
$$

which includes relativistically covariant kinetic energy terms for scalar and spinor fields together with interaction terms involving the gravitational coupling constant κ. At low energies, these terms are suppressed by inverse powers of M_{P}. The covariant derivatives that appear in (10.47) and (10.48) are given by,

$$\not{D}\psi = \gamma^{\mu}(\partial_{\mu} - \frac{\mathrm{i}}{4}\omega_{\mu}^{rs}\sigma_{rs})\psi \quad \text{and} \tag{10.49a}$$

$$D_{\nu}\psi_{\rho} = \partial_{\nu}\psi_{\rho} - \frac{\mathrm{i}}{4}\omega_{\nu}^{rs}\sigma_{rs}\psi_{\rho} - \Gamma_{\rho\nu}^{\sigma}\psi_{\sigma}. \tag{10.49b}$$

Of course, the last term of $D_{\nu}\psi_{\rho}$ makes no contribution to the kinetic energy of the gravitino once the connection is written as a function of the metric (using the equations of motion) so that it is symmetric in its lower indices.

The local SUSY transformation laws are given by

$$\delta A = \mathrm{i}\bar{\alpha}\gamma_5\psi, \tag{10.50a}$$

$$\delta B = -\bar{\alpha}\psi, \tag{10.50b}$$

$$
\begin{aligned}
\delta\psi = -\mathrm{i}\not{\partial}(-B+\mathrm{i}\gamma_5 A)\alpha + \mathrm{i}\frac{\kappa}{2}(\bar{\psi}_{\mu}\psi)\gamma^{\mu}\alpha + \mathrm{i}\frac{\kappa}{2}(\bar{\psi}_{\mu}\gamma_5\psi)\gamma^{\mu}\gamma_5\alpha \\
+\frac{1}{4}\kappa^2\left[\bar{\alpha}(-B+\mathrm{i}\gamma_5 A)\gamma_5\psi\right]\gamma_5\psi,
\end{aligned}
\tag{10.50c}
$$

$$\delta e_{\mu}^{a} = -\mathrm{i}\kappa\bar{\alpha}\gamma^{a}\psi_{\mu}, \quad \text{and} \tag{10.50d}$$

$$\delta\psi_{\mu} = \frac{2}{\kappa}D_{\mu}\alpha + \frac{\mathrm{i}}{2}\kappa(B\ \overset{\leftrightarrow}{\partial}_{\mu}\ A)\gamma_5\alpha - \frac{\kappa^2}{4}[\bar{\alpha}(-B+\mathrm{i}\gamma_5 A)\gamma_5\psi]\gamma_5\psi_{\mu}. \tag{10.50e}$$

Notice that the transformation law for the vierbein reproduces the SUSY transformation (10.45) for the metric that we had obtained above. We do not write the transformation law for the connection fields as these are complicated, and are not needed for our discussion. The gravitino and vierbein fields can be combined into what is called the metric superfield. This is the gravitational analogue of the gauge superfield and, hence, is a *real* superfield. Since the gravitino carries a vector index, the metric superfield is a real vector superfield.

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The reader may have noticed that the supergravity Lagrangian contains non-renormalizable terms. This was also true of the Lagrangian for Einsteinian gravity. Such non-renormalizable terms enter because the gravitational coupling constant κ has dimensions of inverse mass. The situation is analogous to Fermi's theory of β-decay which though non-renormalizable was practically useful, and which has since been understood as the low energy limit of a more fundamental theory (the Standard Model). In the same vein, we will regard the non-renormalizable supergravity Lagrangian as the low energy limit of an as yet unformulated locally supersymmetric fundamental theory (perhaps, superstring theory) to be discovered in the future.

10.3 The supergravity Lagrangian

We have seen that the construction of locally supersymmetric field theories forces us to consider non-renormalizable interactions. If we give up the restriction of renormalizability the globally supersymmetric Lagrangian for gauge theories in (6.47) can be generalized to,

$$\mathcal{L} = -\frac{1}{4}\int \mathrm{d}^4\theta\, K\left(\hat{\mathcal{S}}^\dagger \mathrm{e}^{-2gt_A\hat{\Phi}_A}, \hat{\mathcal{S}}\right) - \frac{1}{2}\left[\int \mathrm{d}^4x\,\mathrm{d}^2\theta_{\mathrm{L}}\,\hat{f}(\hat{\mathcal{S}}) + \quad \text{h.c.}\right]$$
$$-\frac{1}{4}\int \mathrm{d}^2\theta_{\mathrm{L}}\, f_{AB}(\hat{\mathcal{S}})\overline{\hat{W}^c_A}\hat{W}_B. \tag{10.51}$$

In particular, the Kähler potential and the superpotential functions are no longer restricted to be quadratic and cubic polynomials, though the latter is still required to be an analytic function of the fields. Moreover, we have introduced the *gauge kinetic function* $f_{AB}(\hat{\mathcal{S}})$ which, like the superpotential $\hat{f}(\hat{\mathcal{S}})$, is an analytic function of the chiral superfields $\hat{\mathcal{S}}_i$ so that, like the superpotential term, the last term is also an F-term of a chiral superfield (and hence supersymmetric). Renormalizability (and gauge invariance) restricted $f_{AB} = \delta_{AB}$ in (6.47), but now the more general form is possible. To preserve gauge invariance, f_{AB} must transform as the symmetric product of two adjoints of the gauge group. As before, choosing the Kähler potential $K(\hat{\mathcal{S}}^\dagger, \hat{\mathcal{S}})$ and the superpotential $\hat{f}(\hat{\mathcal{S}})$ to be invariant under *global* gauge transformations guarantees local gauge invariance of (10.51). Except for these restrictions from gauge invariance, the Kähler potential, the superpotential and the gauge kinetic function are arbitrary functions of all chiral superfields.

Although it is possible in principle to obtain the complete Lagrangian for locally supersymmetric gauge theories by applying the Noether procedure to the globally supersymmetric Lagrangian (10.51), in practice, more efficient techniques involving tensor calculus of local supersymmetry have been developed to obtain the complete result including all auxiliary fields. A discussion of these techniques is

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beyond the scope of this text. The final result, analogous to our master formula, but for local supersymmetry, was first obtained in 1982 by Cremmer *et al.*[5] We simply present it here, in terms of component fields, after all auxiliary fields have been eliminated. It is customary to factor out the Jacobian term **e**, and to write the result in units with $M_P = 1$. The reduced Planck mass can be re-inserted term-by-term by requiring the dimensionality of each term be equal to four.

Although the Lagrangian for a general non-renormalizable supersymmetric theory depends on three independent functions, K, $\hat{f}$, and f_{AB}, the remarkable feature of the supergravity Lagrangian is that it depends on the gauge kinetic function and just one combination,

$$G(\hat{\mathcal{S}}^\dagger, \hat{\mathcal{S}}) = K(\hat{\mathcal{S}}^\dagger, \hat{\mathcal{S}}) + \log|\hat{f}(\hat{\mathcal{S}})|^2, \tag{10.52}$$

of the Kähler potential and superpotential. We will refer to G as the Kähler function, not to be confused with the Kähler potential K.[6] In what follows, derivatives of the Kähler function with respect to chiral superfields are denoted by,

$$G^i = \left.\frac{\partial G}{\partial \hat{\mathcal{S}}_i}\right|_{\hat{\mathcal{S}}=\mathcal{S}} \quad \text{and} \quad G_j = \left.\frac{\partial G}{\partial \hat{\mathcal{S}}^{j\dagger}}\right|_{\hat{\mathcal{S}}=\mathcal{S}}. \tag{10.53a}$$

Also,

$$G^i_j = \left.\frac{\partial^2 G}{\partial \hat{\mathcal{S}}_i \partial \hat{\mathcal{S}}^{j\dagger}}\right|_{\hat{\mathcal{S}}=\mathcal{S}} \tag{10.53b}$$

defines the Kähler metric.[7] Higher derivatives of G are analogously defined. Finally, we define the inverse of the metric by,

$$(G^{-1})^i_j G^j_k = \delta^i_k. \tag{10.53c}$$

Exercise *If the Lagrangian depends only on the combination G rather than separately on K and $\hat{f}$, the choice of the superpotential is not unique. Show that (classically) the transformations,*

$$K(\hat{\mathcal{S}}^\dagger, \hat{\mathcal{S}}) \to K - [h(\hat{\mathcal{S}})]^\dagger - h(\hat{\mathcal{S}})$$

$$\hat{f}(\hat{\mathcal{S}}) \to \exp\big(h(\hat{\mathcal{S}})\big)$$

[5] E. Cremmer, S. Ferrara, L. Girardello and A. van Proeyen, *Nucl. Phys.* **B212**, 413 (1983).

[6] Some authors refer to G as the Kähler potential. Moreover, what we call K is sometimes denoted by d and, to make matters worse, a different K is defined by $d = -3\log(-K/3)$.

[7] The use of j, the index labeling the adjoint of the jth field, as a superscript is merely conventional and should not cause confusion. It allows for contraction of upper and lower indices according to "usual rules" of tensor calculus. For notational clarity, we write the gauge generator matrix with only lower indices.

leave G (and hence the Lagrangian) invariant. This means that we can move all the analytic terms in the Kähler potential to the superpotential if we wish or, alternatively, that we may choose the superpotential to be a positive constant.

We are now in a position to write down the locally supersymmetric Lagrangian for a Yang–Mills gauge theory coupled to gravity. We break up this Lagrangian into purely bosonic terms $\mathcal{L}_B$, and terms with fermions $\mathcal{L}_F$. We further divide each of these terms into two parts: one part $(\mathcal{L}_B^C)$ independent of the gauge kinetic function, and the other $(\mathcal{L}_B^G)$ containing all the dependence on f_{AB}. The latter piece is, of course, absent in a theory without gauge fields. The purely bosonic Lagrangian can be written as,

$$\mathcal{L}_B = \mathcal{L}_B^C + \mathcal{L}_B^G \tag{10.54}$$

with (in units where the coupling $\kappa = 1$)

$$\mathbf{e}^{-1}\mathcal{L}_B^C = -\frac{R}{2} + G^i_j D_\mu \mathcal{S}_i D^\mu \mathcal{S}^{j*} - \mathrm{e}^G \left(G_i (G^{-1})^i_j G^j - 3 \right) \tag{10.55a}$$

and

$$\begin{aligned}\mathbf{e}^{-1}\mathcal{L}_B^G = &-\frac{1}{4}(\mathrm{Re}\, f_{AB}) F_{A\mu\nu} F_B^{\mu\nu} - \frac{1}{4}(\mathrm{Im}\, f_{AB}) F_{A\mu\nu} \tilde{F}_B^{\mu\nu} \\ &- \frac{g^2}{2}(\mathrm{Re}\, f_{AB}^{-1}) G^i t_{Aij} \mathcal{S}_j G^k t_{Bk\ell} \mathcal{S}_\ell,\end{aligned} \tag{10.55b}$$

where $\tilde{F}_B^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{B\rho\sigma}$. Here, $\mathcal{L}_B^G$ has been written as though the gauge group is simple: if the gauge group has several factors, a sum over each of these factors is implied. The first term in $\mathcal{L}_B^C$ is the Einstein Lagrangian (10.19). The second term contains the kinetic energy terms for the scalar components of the chiral superfields (hence the superscript C on this Lagrangian) while the last term in $\mathcal{L}_B^C$ is the part of the scalar potential that originates in the superpotential. Notice that, unlike the scalar potential for globally supersymmetric theories, this term may be negative. The kinetic terms for the gauge fields are contained in $\mathcal{L}_B^G$.

The part of the Lagrangian involving fermions is more complicated, and for convenience of writing we further split it into terms which give the kinetic energy terms $\mathcal{L}_{F,\mathrm{kin}}$ and other terms that only contain interactions, i.e.

$$\mathcal{L}_F = \mathcal{L}_{F,\mathrm{kin}} + \mathcal{L}_{F,\mathrm{Int}}. \tag{10.56a}$$

As before, each of these is further split into pieces, depending on whether or not there is dependence on the gauge kinetic function. We then have,

$$\mathcal{L}_{F,\mathrm{kin}} = \mathcal{L}_{F,\mathrm{kin}}^C + \mathcal{L}_{F,\mathrm{kin}}^G, \tag{10.56b}$$

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and

$$\mathcal{L}_{F,\mathrm{Int}} = \mathcal{L}^C_{F,\mathrm{Int}} + \mathcal{L}^G_{F,\mathrm{Int}}. \tag{10.56c}$$

The terms that appear in $\mathcal{L}_{F,\mathrm{kin}}$ are given by,

$$\begin{aligned}
\mathbf{e}^{-1}\mathcal{L}^C_{F,\mathrm{kin}} = & -\frac{\mathbf{e}^{-1}}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu D_\rho\psi_\sigma + \left(\frac{\mathrm{i}}{2}G^i_j\bar{\psi}_{i\mathrm{R}}\gamma^\mu D_\mu\psi^j_\mathrm{R} + \quad \mathrm{h.c.}\right)\\
& + \left(\frac{\mathbf{e}^{-1}}{8}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_\nu\psi_\rho G^i D_\sigma\mathcal{S}_i + \quad \mathrm{h.c.}\right)\\
& + \left(\frac{\mathrm{i}}{2}\bar{\psi}_{i\mathrm{R}}\, \not{D}\mathcal{S}_j\psi^k_\mathrm{R}(-G^{ij}_k + \frac{1}{2}G^i_k G^j)\right.\\
& \left. + \frac{\mathrm{i}}{\sqrt{2}}G^j_i\bar{\psi}_{\mu\mathrm{R}}\,\not{D}\mathcal{S}^{i\dagger}\gamma^\mu\psi_{j\mathrm{L}} + \quad \mathrm{h.c.}\right)
\end{aligned} \tag{10.57a}$$

and

$$\begin{aligned}
\mathbf{e}^{-1}\mathcal{L}^G_{F,\mathrm{kin}} = & \left[\frac{1}{2}\mathrm{Re}(f_{AB})\left(\frac{\mathrm{i}}{2}\bar{\lambda}_A\,\not{D}\lambda_B + \frac{1}{4}\bar{\lambda}_A\gamma^\mu\sigma^{\nu\rho}\psi_\mu F_{B\nu\rho}\right.\right.\\
& \left. - \frac{\mathrm{i}}{2}G^i D^\mu\mathcal{S}_i\bar{\lambda}_{A\mathrm{L}}\gamma_\mu\lambda_{B\mathrm{L}}\right) + \frac{1}{8}\mathrm{Im}\,(f_{AB})\mathbf{e}^{-1}D_\mu(\mathbf{e}\bar{\lambda}_A\gamma_5\gamma^\mu\lambda_B)\\
& \left. - \frac{1}{4\sqrt{2}}\frac{\partial f_{AB}}{\partial\mathcal{S}_i}\bar{\psi}_{i\mathrm{R}}\sigma^{\mu\nu}F_{A\mu\nu}\lambda_{B\mathrm{L}}\right] + \quad \mathrm{h.c.}
\end{aligned} \tag{10.57b}$$

The first two terms in (10.57a) contain the kinetic energies of the gravitino and the chiral fermions, while the first term of (10.57b) contains the kinetic energy of the gauginos. Finally, the pieces of $\mathcal{L}_{F,\mathrm{Int}}$ are given by

$$\begin{aligned}
\mathbf{e}^{-1}\mathcal{L}^C_{F,\mathrm{Int}} = & \left[\frac{\mathrm{i}}{2}\mathrm{e}^{G/2}\bar{\psi}_{\mu\mathrm{L}}\sigma^{\mu\nu}\psi_{\nu\mathrm{R}} + \frac{1}{2}gG^i t_{Aij}\mathcal{S}_j\bar{\psi}_{\mu\mathrm{R}}\gamma^\mu\lambda_{A\mathrm{R}}\right.\\
& - g\sqrt{2}G^j_i t_{Ajk}\mathcal{S}_k\bar{\lambda}_{A\mathrm{L}}\psi^i_\mathrm{R}\\
& - \frac{1}{2}\mathrm{e}^{G/2}(-G^{ij} - G^i G^j + G^{ij}_k(G^{-1})^k_\ell G^\ell)\bar{\psi}_{i\mathrm{R}}\psi_{j\mathrm{L}}\\
& - \frac{1}{\sqrt{2}}\mathrm{e}^{G/2}G^i\bar{\psi}_{\mu L}\gamma^\mu\psi_{i\mathrm{L}}\\
& + \frac{\mathrm{i}}{16}G^j_i\bar{\psi}_{i\mathrm{L}}\gamma_d\psi_{j\mathrm{L}}\left(\epsilon^{abcd}\bar{\psi}_a\gamma_b\psi_c - \mathrm{i}\bar{\psi}^a\gamma^5\gamma^d\psi_a\right)\\
& \left. + \left(\frac{1}{8}G^{ij}_{kl} - \frac{1}{8}G^{ij}_m(G^{-1})^m_n G^n_{kl} - \frac{1}{16}G^i_k G^j_l\right)\bar{\psi}_{i\mathrm{R}}\psi_{j\mathrm{L}}\bar{\psi}^k_\mathrm{L}\psi^l_\mathrm{R}\right]\\
& + \quad \mathrm{h.c.}
\end{aligned} \tag{10.58a}$$

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and

$$
\begin{aligned}
\mathbf{e}^{-1}\mathcal{L}^{G}_{F,\mathrm{Int}} = \Bigg[& \frac{1}{4}\mathrm{e}^{G/2}\frac{\partial f^*_{AB}}{\partial \mathcal{S}^{j*}}(G^{-1})^j_k G^k \bar{\lambda}_{A\mathrm{L}}\lambda_{B\mathrm{R}} \\
& + \frac{g}{2\sqrt{2}}(\mathrm{Re}\, f_{AB})^{-1}\frac{\partial f_{BC}}{\partial \mathcal{S}_k} G^i t_{Aij}\mathcal{S}_j \bar{\psi}_{k\mathrm{R}}\lambda_{C\mathrm{L}} \\
& - \frac{1}{32}(G^{-1})^k_l \frac{\partial f_{AB}}{\partial \mathcal{S}_l}\frac{\partial f^*_{CD}}{\partial \mathcal{S}^{k*}}\bar{\lambda}_{A\mathrm{R}}\lambda_{B\mathrm{L}}\bar{\lambda}_{C\mathrm{L}}\lambda_{D\mathrm{R}} \\
& + \frac{3}{32}\left[\mathrm{Re}\,(f_{AB})\bar{\lambda}_{A\mathrm{R}}\gamma_\mu\lambda_{B\mathrm{R}}\right]^2 + \frac{\mathrm{i}}{16}\mathrm{Re}\,(f_{AB})\bar{\lambda}_A\gamma^\mu\sigma^{\rho\sigma}\psi_\mu\bar{\psi}_\rho\gamma_\sigma\lambda_B \\
& + \frac{\mathrm{i}}{4\sqrt{2}}\frac{\partial f_{AB}}{\partial \mathcal{S}_i}\left(\bar{\psi}_{i\mathrm{R}}\sigma^{\mu\nu}\lambda_{A\mathrm{L}}\bar{\psi}_{\nu\mathrm{R}}\gamma_\mu\lambda_{B\mathrm{R}} + \frac{\mathrm{i}}{2}\bar{\psi}_{\mu\mathrm{L}}\gamma^\mu\psi_{i\mathrm{L}}\bar{\lambda}_{A\mathrm{R}}\lambda_{B\mathrm{L}}\right) \\
& + \frac{1}{16}\bar{\psi}_{i\mathrm{R}}\gamma^\mu\psi^j_{\mathrm{R}}\bar{\lambda}_{D\mathrm{L}}\gamma_\mu\lambda_{C\mathrm{L}}\left[G^i_j\mathrm{Re}\,(f_{CD}) + \frac{1}{2}\mathrm{Re}\left(f^{-1}_{AB}\frac{\partial f_{AC}}{\partial \mathcal{S}_i}\frac{\partial f^*_{BD}}{\partial \mathcal{S}^{j*}}\right)\right] \\
& - \frac{1}{16}\bar{\psi}_{i\mathrm{R}}\psi_{j\mathrm{L}}\bar{\lambda}_{C\mathrm{R}}\lambda_{D\mathrm{L}} \\
& \times\left(-2G^{ij}_k(G^{-1})^k_l\frac{\partial f_{CD}}{\partial \mathcal{S}_l} + 2\frac{\partial^2 f_{CD}}{\partial \mathcal{S}_i\partial \mathcal{S}_j} - \frac{1}{2}\mathrm{Re}\, f^{-1}_{AB}\frac{\partial f_{AC}}{\partial \mathcal{S}_i}\frac{\partial f_{BD}}{\partial \mathcal{S}_j}\right) \\
& - \frac{1}{128}\bar{\psi}_{i\mathrm{R}}\sigma_{\mu\nu}\psi_{j\mathrm{L}}\bar{\lambda}_{C\mathrm{R}}\sigma^{\mu\nu}\lambda_{D\mathrm{L}}\mathrm{Re}\left(f^{-1}_{AB}\frac{\partial f_{AC}}{\partial \mathcal{S}_i}\frac{\partial f_{BD}}{\partial \mathcal{S}_j}\right)\Bigg] + \quad \mathrm{h.c.}
\end{aligned}
\tag{10.58b}
$$

The transformation laws of local supersymmetry are given by,

$$\delta\mathcal{S}_i = -\mathrm{i}\sqrt{2}\bar{\alpha}\psi_{i\mathrm{L}}, \tag{10.59a}$$

$$
\begin{aligned}
\delta\psi_{i\mathrm{L}} &= \sqrt{2}\,\not{D}\mathcal{S}_i\alpha_\mathrm{R} + \mathrm{i}\sqrt{2}\mathrm{e}^{G/2}(G^{-1})^j_i G_j\alpha_\mathrm{L} \\
&\quad -\frac{\mathrm{i}}{2\sqrt{2}}\alpha_\mathrm{L}\overline{\lambda_A}\lambda_{B\mathrm{R}}(G^{-1})^j_i\frac{\partial f^*_{AB}}{\partial \mathcal{S}^{*j}} + \cdots,
\end{aligned}
\tag{10.59b}
$$

$$\delta e^a_\mu = -\mathrm{i}\bar{\alpha}\gamma^a\psi_\mu, \tag{10.59c}$$

$$\delta\psi_\mu = 2D_\mu\alpha + \mathrm{i}\mathrm{e}^{G/2}\gamma_\mu\alpha + \cdots, \tag{10.59d}$$

$$\delta V^\mu_A = -\mathrm{i}\bar{\alpha}\gamma^\mu\lambda_A, \tag{10.59e}$$

$$\delta\lambda_{A\mathrm{R}} = \frac{\mathrm{i}}{2}\sigma^{\mu\nu}F_{A\mu\nu}\alpha_\mathrm{R} - \mathrm{i}g\mathrm{Re}\,(f^{-1}_{AB})G^i(t_B)_{ij}\mathcal{S}_j\alpha_\mathrm{R} + \cdots \tag{10.59f}$$

The ellipses represent additional terms that we will not need for our subsequent discussion. These terms contain products of fermion fields, or, as in (10.59d), contain derivatives of scalar fields.

Except for the restrictions from gauge invariance and analyticity already mentioned, there is no known principle for the choice of the Kähler potential, the superpotential, and the gauge kinetic function in a general non-renormalizable theory.

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Supergravity couplings, however, depend only on the gauge kinetic function and the Kähler function G. Choosing the Kähler potential and the gauge kinetic function to be what they are in renormalizable theories,

$$K = \sum_i \hat{\mathcal{S}}^{i\dagger}\hat{\mathcal{S}}_i \tag{10.60a}$$

and

$$f_{AB}(\hat{\mathcal{S}}) = \delta_{AB}, \tag{10.60b}$$

leads to canonical kinetic energy terms for "matter" (scalar and fermion) fields and for gauginos, respectively. The theory that is obtained from the general supergravity Lagrangian (10.54)–(10.58b) for this choice of the Kähler potential and the gauge kinetic function is sometimes referred to as "minimal supergravity".[8]

Exercise *Verify that for any gauge-invariant superpotential $\hat{f}(\hat{\mathcal{S}})$,*

$$\frac{\partial \hat{f}}{\partial \hat{\mathcal{S}}_i} t_{Aij}\hat{\mathcal{S}}_j = 0.$$

Exercise (Recovering the Lagrangian for global SUSY) *The locally supersymmetric Lagrangian must reduce to the globally supersymmetric Lagrangian in our master formula (6.44) if we take the limit $M_P \to \infty$.*

(a) *Identify the kinetic energy terms for all the fields.*

(b) *Convince yourself that the coupling of matter and gauge fields with gravitons and gravitinos, as well as all contributions from non-minimal terms in K and f_{AB}, all result in non-renormalizable interactions suppressed by powers of the reduced Planck mass. We can thus confine ourselves to the minimal supergravity choice,*

$$K = \sum_i \frac{\hat{\mathcal{S}}^{i\dagger}\hat{\mathcal{S}}_i}{M_P^2},$$

and

$$f_{AB} = \delta_{AB}$$

for these functions in the remainder of this exercise. Notice that we have inserted appropriate powers of the reduced Planck mass required to make K dimensionless.

[8] This should be distinguished from the minimal supergravity model discussed in the next chapter, where the same Kähler potential and a related gauge kinetic function are used. The reader should also note that the field-independent choice of the gauge kinetic function leaves gauginos massless at the tree level.

(**c**) *Verify that the last term of (10.55a) reduces to* $-\sum_i \left|\frac{\partial \hat{f}}{\partial \mathcal{S}_i}\right|^2$*, the part of the scalar potential that originates in the superpotential on the third line of the master formula. Using the result of the previous exercise, show that the last term of (10.55b) reduces to the remainder of the scalar potential in our master formula. Remember that we have written the supergravity Lagrangian in units where* $\kappa = 1/M_{\mathrm{P}} = 1$*. You will have to reinsert this factor on the various terms using dimensional analysis.*

(**d**) *Finally, convince yourself that the terms on the second and third lines of (10.58a) reduce to the couplings of gauginos to the sfermion–fermion pair and to chiral fermion superpotential Yukawa couplings, respectively.*

10.4 Local supersymmetry breaking

In Chapter 7, we showed that in order for global SUSY to be broken, the variation of a spinorial operator had to be non-zero. The same holds for models with local supersymmetry, i.e. we may have either $\langle 0|\delta\psi_i|0\rangle \neq 0$, or $\langle 0|\delta\lambda_A|0\rangle \neq 0$.

When we considered the spontaneous breaking of global supersymmetry without also breaking the Poincaré symmetry, we were led to just two possibilities: F-type SUSY breaking with $\langle 0|\mathcal{F}_i|0\rangle \neq 0$, or D-type breaking with $\langle 0|\mathcal{D}_A|0\rangle \neq 0$. For both cases, some auxiliary fields acquired a vacuum expectation value. For the case of local supersymmetry, the same is true although we cannot see this because we have written these supersymmetry transformations, (10.59a)–(10.59f), with the auxiliary fields already eliminated. If we assume that fermion fields cannot acquire vacuum expectation values, the condition for local SUSY breaking from (10.59b) is

$$\langle 0|\mathrm{e}^{G/2}(G^{-1})_i^j G_j|0\rangle \neq 0 \tag{10.61a}$$

or from (10.59f),

$$\langle 0|\mathrm{Re}\,(f_{AB})^{-1} G^i t_{Bij} S_j|0\rangle \neq 0. \tag{10.61b}$$

The terms denoted by ellipses in the supergravity transformations (10.59a)–(10.59f) cannot acquire VEVs, and so are not relevant for this discussion. These conditions are the generalization of the conditions for global supersymmetry breaking that we found in Chapter 7.

These conditions take a simpler form for the minimal supergravity case introduced earlier. Then $(G^{-1})_i^j = \delta_i^j$, and $G_i = \mathcal{S}_i + \frac{1}{\hat{f}^*}\partial \hat{f}^*/\partial \mathcal{S}^{i*}$, and the F-type breaking condition reduces to,

$$\frac{\partial \hat{f}}{\partial \mathcal{S}_i} + \frac{\mathcal{S}^{i*}\hat{f}}{M_{\mathrm{P}}^2} \neq 0. \tag{10.62}$$

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Clearly, this condition reduces to Eq. (7.3a) in the limit $M_{\rm P} \to \infty$. It is easy to see that (10.61b) is, similarly, a generalization of the D-term SUSY breaking condition Eq. (7.3b).

Our discussion of local supersymmetry breaking up to now has omitted one important possibility that was actually mentioned in Section 7.5. Supersymmetry may also be broken if the last term in Eq. (10.59b) acquires a VEV. This is not possible if gauge couplings remain perturbative. There may, however, be gauge interactions (not contained in the MSSM) that become strong at a high scale, and cause the associated gauginos to condense.[9]

It is also instructive to calculate the form of the scalar potential for minimal supergravity. In this case, from the $\mathcal{L}_B$ terms in the supergravity Lagrangian, we obtain

$$V = \mathrm{e}^{\frac{\mathcal{S}^{i\dagger}\mathcal{S}_i}{M_{\rm P}^2}} \left(-\frac{3}{M_{\rm P}^2}|\hat{f}|^2 + \left| \frac{\partial \hat{f}}{\partial \mathcal{S}_i} + \frac{\mathcal{S}^{i\dagger}\hat{f}}{M_{\rm P}^2} \right|^2 \right) + \frac{g^2}{2}\mathcal{S}^{i\dagger} t_{Aij}\mathcal{S}_j \mathcal{S}^{k\dagger} t_{Ak\ell}\mathcal{S}_\ell. \quad (10.63)$$

The negative term above offers the possibility of a small or even zero cosmological constant in supergravity theories (even if supersymmetry is broken), whereas in global SUSY the scalar potential was always positive semi-definite. There is no known reason though why the negative and positive terms should (almost) cancel, and a small cosmological constant is only possible by severe fine-tuning.

10.4.1 Super-Higgs mechanism

Recall that Goldstone bosons are the relics of spontaneous breaking of global symmetries: corresponding to every symmetry generator that does not annihilate the ground state, there is a massless boson (with derivative couplings) in the physical spectrum. If instead the spontaneously broken symmetry is local, the Goldstone boson is "eaten by the gauge fields", in that it becomes the longitudinal component of a gauge field which then acquires a mass. This is the well-known Higgs mechanism.

The situation for supersymmetry is quite similar. We have already seen that when global SUSY is spontaneously broken we obtain a massless Goldstone fermion, the goldstino, in the spectrum. In supergravity theories, where we have invariance under local SUSY transformations, the gravitino plays the same role that gauge fields play in local gauge theories. If SUSY is spontaneously broken, it is then natural to examine whether the goldstino degrees of freedom become the longitudinal degrees of freedom of the gravitino, the gauge field of supergravity, thereby endowing it

[9] In this context, we should mention that condensation of chiral fermions associated with new gauge interactions is also a possibility. Indeed, if there are chiral fermions in the adjoint representation of the gauge group, hybrid $\bar{\psi}\lambda$ condensates may also be possible. In these cases, the terms denoted by the ellipses in the supergravity transformations may be relevant.

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with a mass. Although we do not analyze the details of the "supersymmetric Higgs mechanism" here, we see from the first term of Eq. (10.58a) that the gravitino becomes massive if the Kähler function G acquires a VEV:

$$\frac{i}{2}e^{\frac{G}{2}}\bar{\psi}_\mu\sigma^{\mu\nu}\psi_\nu \to \frac{i}{2}e^{\frac{G_0}{2}}\bar{\psi}_\mu\sigma^{\mu\nu}\psi_\nu, \tag{10.64}$$

where G_0 is the VEV of G. Thus the gravitino mass can be identified as

$$m_{3/2}^2 = e^{G_0}M_P^2. \tag{10.65}$$

The goldstino associated with either D- or F-type SUSY breaking is absorbed by the gravitino, and does not appear in the physical spectrum, while the gravitino becomes massive. Indeed, with an appropriate (field-dependent) choice of the local SUSY transformation parameter (unitarity gauge choice), the goldstino field can be completely eliminated from the Lagrangian.

We should also mention that the supertrace formula (7.35) that we obtained in Chapter 7 is also modified if the supersymmetry is local. For the case of minimal supergravity with N chiral supermultiplets, from Cremmer *et al.* we have

$$STr\mathcal{M}^2 = 2\sum_A \mathcal{D}_A Tr(gt_A) + (N-1)(2m_{3/2}^2 - \frac{\mathcal{D}_A\mathcal{D}_A}{M_P^2}). \tag{10.66}$$

The first term is the same as the case for global SUSY but the last term is new. This term will play an important role in the next chapter where we consider realistic supergravity models of particle physics.

Exercise *For the flat Kähler metric show that the gravitino mass, assuming that the cosmological constant vanishes, can be written as*

$$m_{3/2}^2 = \frac{\langle F_i F^{i*}\rangle}{3M_P^2}, \tag{10.67a}$$

where

$$F_i = e^{\frac{G}{2}}\left(G^{-1}\right)_i^j G_j \tag{10.67b}$$

is the auxiliary field whose VEV (10.61a) breaks supersymmetry.

An illustrative example: the Polonyi superpotential

A particularly simple illustration of the ideas that we have just discussed is obtained for the minimal supergravity model with a single chiral scalar superfield coupled via the Polonyi superpotential $\hat{f}$ given by,

$$\hat{f} = m^2\left(\hat{\mathcal{S}} + \beta\right), \tag{10.68}$$

where m^2 and β are real constants.

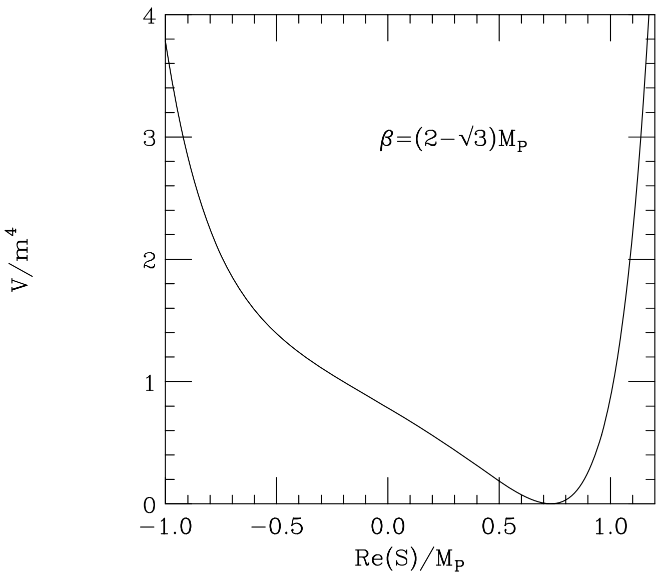

Figure 10.1 The scalar potential of the Polonyi model (in units of m^4) versus $\mathrm{Re}\,(\mathcal{S})/M_\mathrm{P}$ for the choice $\beta = (2-\sqrt{3})M_\mathrm{P}$ with $\mathrm{Im}\,\mathcal{S}$ set to zero. Supersymmetry is necessarily broken, and the cosmological constant is zero for this choice of parameters.

It is straightforward to obtain the scalar potential which is given by,

$$V = m^4 \mathrm{e}^{\mathcal{S}^*\mathcal{S}} \left(\left|1 + \mathcal{S}^*(\mathcal{S}+\beta)\right|^2 - 3\,|\mathcal{S}+\beta|^2 \right), \tag{10.69a}$$

while the condition (10.62) for SUSY to remain unbroken becomes,

$$1 + \mathcal{S}^*(\mathcal{S}+\beta) = 0. \tag{10.69b}$$

It is easy to see that SUSY is broken if $\beta^2 < 4$ (in Planck units).

The scalar potential has several extrema. In the following, we confine ourselves to those minima with $V = 0$, which implies

$$\left|1 + \mathcal{S}^*(\mathcal{S}+\beta)\right|^2 = 3\,|\mathcal{S}+\beta|^2 .$$

We can see that $\mathcal{S} = (\sqrt{3}-1)M_\mathrm{P}$ is one such minimum if $\beta = (2-\sqrt{3})M_\mathrm{P}$.[10] The shape of the scalar potential is shown in Fig. 10.1 for $\mathrm{Im}\,\mathcal{S} = 0$. In this minimum, the gravitino mass is given by,

$$m_{3/2} = \mathrm{e}^{G_0/2} M_\mathrm{P} = \mathrm{e}^{(2-\sqrt{3})} \frac{m^2}{M_\mathrm{P}^2} M_\mathrm{P}. \tag{10.70}$$

Thus, if the parameter $m \sim 10^{10}$ GeV, then $m_{3/2} \sim 100$ GeV.

[10] This is an incredible fine-tuning. For any other value of β the cosmological constant would be very large.

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