# Constructions connected with Euclid VI., 3 and A, and the Circle of Apollonius.

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1. Let ABC be a triangle and AD the bisector of the angle A meeting BC in D. (Fig. 8.)

For simplicity let us suppose $AB > AC$.

From AB cut off $AE = AC$, and join DE. Then by Euclid I. 4 it follows that the triangles AED, ACD are equal.

$$\begin{aligned} \text{Thus } BD : DC &= \triangle BAD : \triangle DAC \quad - \quad - \qquad \text{(Euclid VI. 1)} \\ &= \triangle BAD : \triangle EAD \\ &= BA : EA \quad - \quad - \quad - \quad (\;\text{,,} \quad \text{,,} \quad \text{,,}) \\ &= BA : AC. \end{aligned}$$

This proof of VI. 3 has the merit of depending only on I. 4 and VI. 1. A similar proof for VI. A can be got by cutting off $AE' = AC$ from BA produced.

2. Next let AO be drawn parallel to DE to meet BC in O. (Fig. 8.)

It is clear from the construction that OAD is an isosceles triangle with $OA = OD$.

If we suppose B, D and C fixed, but the vertex A veriable, it is obvious that the ratio $BA : AC$ will be constant, being $= BD : DC$.

And $$\frac{OD}{CD} = \frac{OA}{DE} = \frac{AB}{EB} = \frac{AB}{AB - AC} = \text{constant}.$$

Thus O is a fixed point. Hence OA which is $= OD$ is a fixed length, and the locus of A is a circle with centre O and radius OD.

3. The following simple linkage is closely connected with the construction we are considering.

Let AQPQ′ in Fig. 9 be a rhombus and AB any line drawn from A to meet PQ′ in B.

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Draw

| | | | | |
|---|---|---|---|---|
| BDO | parallel to PQ, meeting AP in D and AQ in O, | | | |
| B′DC′O′ | ,, | ,, PQ′, | ,, | PQ in B′, AB in C′, and AQ′ in O′, |
| C′R | ,, | ,, PQ, | ,, | AP in R. |

Complete the parallelogram DC′RC and produce its sides to meet those of the rhombus AQPQ′.

The parallelograms having diagonals collinear with that of AQPQ′ are evidently rhombuses, and the figure is symmetrical about the diagonal AP, so that A, C′, B being collinear, A, C, B′ are so also.

Now take eight equal rods and place them to form the figure just described so far as the lines parallel to the sides of AQPQ′ are concerned. Let the rods be freely jointed together at their intersections. Then the whole figure will have one internal freedom of movement, of the nature of a shearing strain.

Obviously A, C′, B and A, C, B′ will be collinear in all positions, and will be in lines of equal length, equally inclined to AP, and divided in the constant ratio of Q′B to BP.

Thus BA : AC is constant, and A traces out a circle whose centre is O and radius OA.

4. We might extend the figure so as to bring in the exterior bisector of the angle BAC, by taking 14 rods of double the length of those used before, and placing them to form a figure of which AQPQ′ would be one fourth part, or by taking four figures identical with AQPQ′ and placing them so as to form a larger rhombus of four times the area of AQPQ′.

We might also cut down the linkage to some extent without losing its essential properties.

5. Another construction is the following, suggested to me by one of C. Taylor's constructions in connection with conjugate diameters of an ellipse.

In Figure 10, ABC is a triangle, and AD the bisector of the angle A, meeting BC in D.

Draw DH parallel to CA to meet BA in H;

„ DK „ „ BA „ „ CA „ K.

Let HK meet BC in O.

AHDK is a rhombus, and since its diagonals bisect one another at right angles, O is equally distant from A and D.

O is also the centre of the circle of Apollonius.

To prove BA : AC = BD : DC we have

$$\frac{BA}{AC} = \frac{DK}{KC} = \frac{AK}{KC} = \frac{BD}{DC}.$$

Thus, B, D, C being fixed, the ratio BA : AC is fixed.

Again
$$\frac{OC}{OD} = \frac{CK}{DH} = \frac{CK}{KA} = \frac{CD}{DB};$$

$$\therefore \quad \frac{OC}{CD} = \frac{CD}{DB - CD} = \text{constant}.$$

Hence OC is constant, and O is a fixed point. Hence OA which is = OD, is a fixed length and the locus of A is the circle whose centre is O and radius OD.

Note also that OC : OD = OK : OH = OD : OB;

$$\therefore \quad OC \, . \, OB = OD^2.$$

If we make a corresponding construction, starting with the *external* bisector AD′, we get a diagonal H′K′ of the rhombus A′H′D′K′ which, like HK, bisects DD′ in O. See Fig. 10.

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