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UNIFORM KADEC-KLEE LORENTZ SPACES $L_{w,1}$ AND UNIFORMLY CONCAVE FUNCTIONS

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ABSTRACT. We consider the notion of a uniformly concave function, using it to characterize those Lorentz spaces $L_{w,1}$ that have the weak-star uniform Kadec-Klee property as precisely those for which the antiderivative ϕ of w is uniformly concave; building on recent work of Dilworth and Hsu. We also derive a quite general sufficient condition for a twice-differentiable ϕ to be uniformly concave; and explore the extent to which this condition is necessary.

1. **Introduction.** A dual Banach space $(X, \|\cdot\|_X)$, with predual $(Y, \|\cdot\|_Y)$, is said to have the weak-star uniform Kadec-Klee property (with respect to Y) if for every $\varepsilon > 0$, there exists an $\eta = \eta(\varepsilon) \in (0,1)$ such that whenever $(x_n)_{n=1}^{\infty}$ is a sequence in the closed unit ball $\mathbf{B}_X$ of X, converging weak-star to $x \in X$, and $\inf_{n \neq m} \|x_n - x_m\|_X \geq \varepsilon$, it follows that $\|x\|_X \leq 1 - \eta$.

The uniform Kadec-Klee property for the weak topology on a Banach space was introduced by Huff [Hu] as a useful substitute for uniform convexity, especially in many nonreflexive spaces. Van Dulst and Sims [DS], building on work of Brodskiĭ and Mil'man [BM] and Kirk [Ki1], showed that the uniform Kadec-Klee property for weak or weak-star topologies implied weak (resp. weak-star) normal structure. This geometric property implied in turn, via Kirk [Ki1], that every norm nonexpansive mapping on a weakly (resp. weak-star) compact, convex set must have a fixed point. (The weak-star result is due to [DS] and is also implicit in [Ki2]).

Many spaces have been found to have uniform Kadec-Klee (UKK) properties. Recent related papers concerning UKK properties in Lorentz spaces are [CDL], [DDDLS] and [HK]. We will continue the study of weak-star UKK in Lorentz spaces initiated in [CDLT] and [DH].

An admissible weight function w is a function from $(0,\infty)$ into $(0,\infty)$ that is decreasing (*i.e.*, non-increasing), satisfies $\lim_{t\to 0+} w(t) = \infty$ and $\lim_{t\to\infty} w(t) = 0$, while $\int_0^1 w(t)\,dt = 1$ and $\int_0^\infty w(t)\,dt = \infty$. We will denote the class of all such weights by W. If $w \in W$, we define $\phi(t) = \phi_w(t) := \int_0^t w(s)\,ds$, for all $t > 0$. The function ϕ is strictly increasing and concave, mapping $(0,\infty)$ onto $(0,\infty)$. We define the functions k_1,

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$k_2 : (0,1) \longrightarrow (0,\infty)$ by setting, for all $\alpha \in (0,1)$,

$$k_1(\alpha) := \sup_{t>0} \frac{\phi(\alpha t)}{\phi(t)} \quad \text{and} \quad k_2(\alpha) := \inf_{t>0} \frac{w(\alpha t)}{w(t)}.$$

We let C_1 be the subset of W consisting of all w for which ϕ_w is uniformly increasing; *i.e.*, $k_1(\alpha) < 1$ for all $\alpha \in (0,1)$. Similarly, we define C_2 to be the set of all those $w \in W$ for which w is uniformly decreasing; *i.e.*, $k_2(\alpha) > 1$ for all $\alpha \in (0,1)$. It is not hard to see that if $w \in C_1$, then k_1 is a strictly increasing function on $(0,1)$, with $\lim_{\alpha \to 0+} k_1(\alpha) = 0$ and $\lim_{\alpha \to 1-} k_1(\alpha) = 1$. Also, if $w \in C_2$, then k_2 is a strictly decreasing function on $(0,1)$, with $\lim_{\alpha \to 0+} k_2(\alpha) = \infty$ and $\lim_{\alpha \to 1-} k_2(\alpha) = 1$.

For all $w \in W$ the function space $L_{w,1} = L_{w,1}(0,\infty)$ is the set of all (equivalence classes of) scalar-valued, Lebesgue-measurable functions f on $(0,\infty)$ for which $\|f\|_{w,1} := \int_0^\infty f^*(t)w(t)\,dt$ is finite. Here, f^* is the decreasing rearrangement of f. Under the usual duality in function spaces, the Banach dual of $L_{w,1}$ is isometric to $L_{w,\infty}$; which is the set of all (equivalence classes of) scalar-valued, Lebesgue-measurable functions f on $(0,\infty)$ for which $\|f\|_{w,\infty} := \sup_{t>0} \frac{\int_0^t f^*(s)\,ds}{\phi(t)}$ is finite. Under the same duality, the subspace $L^o_{w,\infty}$, consisting of all $f \in L_{w,\infty}$ for which $\lim_{t\to 0+} \frac{\int_0^t f^*(s)\,ds}{\phi(t)} = 0$ and $\lim_{t\to\infty} \frac{\int_0^t f^*(s)\,ds}{\phi(t)} = 0$, is an isometric predual of $L_{w,1}$.

We will use the following inequality later. A qualitative consequence of it is the well-known fact that for an admissible weight w, $w \in C_1$ if and only if $k_1(\alpha_0) < 1$ for some $\alpha_0 \in (0,1)$.

LEMMA 1.1. *Let $w \in W$. Then for all $0 < \alpha < \beta < 1$,*

$$\frac{\alpha\big(1-k_1(\alpha)\big)}{1-\alpha} \le \frac{\beta\big(1-k_1(\beta)\big)}{1-\beta}.$$

PROOF. Fix $0 < \alpha < \beta < 1$. Let $\gamma := \beta/\alpha$. Then $\beta < 1 < \gamma$, and so there is a $\lambda \in (0,1)$ for which $1 = (1-\lambda)\beta + \lambda\gamma$. Indeed, $\lambda = \frac{\alpha(1-\beta)}{\beta(1-\alpha)}$. Thus, for all $t > 0$, since ϕ is concave and increasing, we see that

$$\begin{aligned}\frac{\phi(t)-\phi(\beta t)}{\phi(t)} &\ge \frac{(1-\lambda)\phi(\beta t)+\lambda\phi(\gamma t)-\phi(\beta t)}{\phi(t)} = \frac{\lambda\big(\phi(\gamma t)-\phi(\beta t)\big)}{\phi(t)} \\ &= \frac{\alpha(1-\beta)}{\beta(1-\alpha)}\Big(\frac{\phi(\gamma t)-\phi(\beta t)}{\phi(t)}\Big) \ge \frac{\alpha(1-\beta)}{\beta(1-\alpha)}\Big(\frac{\phi(\gamma t)-\phi(\beta t)}{\phi(\gamma t)}\Big).\end{aligned}$$

Consequently, $1 - k_1(\beta) \ge \frac{\alpha(1-\beta)}{\beta(1-\alpha)}\big(1-k_1(\alpha)\big)$. ∎

We remark that weights of class C_1 are known in the literature as regular weights. They have been studied for a long time. See, for example, [Ha], [Al] and [C].

2. **Uniformly concavity of ϕ and the weak-star UKK property.**

Recently Dilworth and Hsu [DH] proved the following theorem.

THEOREM 2.1. *Let $w \in W$. Then the Lorentz space $L_{w,1}$ has the weak-star uniform Kadec Klee property with respect to its predual $L^{\circ}_{w,\infty}$ if and only if $w \in C_1 \cap C_2$.*

This theorem makes uniform a result of Sedaev [S], who showed that $L_{w,1}$ has the weak-star Kadec-Klee property (w.r.t. $L^{\circ}_{w,\infty}$) if and only if ϕ is strictly concave; which is equivalent to the condition that w is strictly decreasing. However, for the uniform case, the condition that w is uniformly decreasing ($w \in C_2$) is not equivalent to the weak-star UKK property; as was shown in [DH]. Indeed, $w \in C_2$ does not imply $w \in C_1$, and $w \in C_1$ does not imply $w \in C_2$.

It is one of the purposes of this note to show that a single condition on ϕ, in the spirit of Sedaev's result, is equivalent to the weak-star UKK property in $L_{w,1}$: which is that ϕ is uniformly concave. The following definition gives this notion. Luxemburg [Lu] introduced the analogous notion of a uniformly convex function. Moreover, Kamińska [Ka1] extended a result of [Lu] to show that an Orlicz function space with the Luxemburg norm is uniformly convex if and only if the defining convex Orlicz function is a uniformly convex function and satisfies the Δ_2 condition (at 0 and ∞). (Also see [Ak],[Ka2]). We thank Anna Kamińska for showing us that a function ψ is uniformly concave in the sense of Definition 2.2 if and only if ψ^{-1} is uniformly convex and satisfies the Δ_2 condition.

DEFINITION 2.2. (a) Consider a mapping $\psi : (0,\infty) \longrightarrow (0,\infty)$. Then ψ is said to be *uniformly concave* if for every $\varepsilon \in (0,1)$, there exists a $\delta > 0$ such that for all $0 < a < b$ with $b - a \geq \varepsilon b$, it follows that

$$\psi\Big(\frac{a+b}{2}\Big) - \frac{1}{2}\big(\psi(a) + \psi(b)\big) > \delta\psi(b).$$

(b) We define, for any function $\psi : (0,\infty) \longrightarrow (0,\infty)$, the *modulus of uniform concavity*, $\delta = \delta_\psi : (0,1) \longrightarrow [0,\infty)$ by setting, for all $\varepsilon \in (0,1)$,

$$\delta(\varepsilon) := \inf\Big\{\frac{1}{\psi(b)}\Big[\psi\Big(\frac{a+b}{2}\Big) - \frac{1}{2}\big(\psi(a) + \psi(b)\big)\Big] : \ 0 < a < b \text{ and } 1 - \frac{a}{b} \geq \varepsilon\Big\}.$$

We note that $\psi : (0,\infty) \longrightarrow (0,\infty)$ is uniformly concave if and only if $\delta_\psi(\varepsilon) > 0$ for all $\varepsilon \in (0,1)$. We also note that since a uniformly concave function with range in $(0,\infty)$ is concave and increasing, we may replace "$> \delta\psi(b)$" by "$> \delta\psi(u)$" in Definition 2.2(a), where $u := (a+b)/2$, to get an equivalent definition of uniform concavity; and we may replace "$\frac{1}{\psi(b)}$" by "$\frac{1}{\psi(u)}$" in Definition 2.2(b) to get an equivalent modulus of uniform concavity.

THEOREM 2.3. *Let $w \in W$. The following are equivalent.*

(i) The Lorentz space $L_{w,1}$ has the weak-star uniform Kadec Klee property with respect to its predual $L^{\circ}_{w,\infty}$.

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(ii) $\phi = \phi_w$ is uniformly concave.

PROOF. We will use the fact, from Theorem 2.1, that (i) holds if and only if $w \in C_1 \cap C_2$.

(i) $\Longrightarrow$ (ii) Suppose $w \in C_1 \cap C_2$. Fix $0 < a < b$. Let $e := 1 - a/b \in (0,1)$, $u := (a+b)/2$ and $v := (b-a)/2$. Then

$$\begin{aligned}\Gamma &:= \frac{1}{\phi(b)}\Big[\phi(u) - \frac{1}{2}\big(\phi(a)+\phi(b)\big)\Big] \\ &= \frac{1}{2\phi(b)}\Big[\phi(u) - \phi(a) - \big(\phi(b)-\phi(u)\big)\Big] \\ &= \frac{1}{2\phi(b)}\Big[\int_{t=a}^{t=u} w(t)\,dt - \big(\phi(b)-\phi(u)\big)\Big] \\ &= \frac{1}{2\phi(b)}\Big[\int_{s=u}^{s=b} w(s-v)\,ds - \big(\phi(b)-\phi(u)\big)\Big].\end{aligned}$$

Now $s \longmapsto (s-v)/s$ is an increasing function on $[u,b]$ and k_2 decreases on $[0,1]$. Thus, for all $s \in [u,b]$,

$$\frac{w(s-v)}{w(s)} \geq k_2\Big(\frac{s-v}{s}\Big) \geq k_2\Big(\frac{b-v}{b}\Big) = k_2\Big(\frac{u}{b}\Big) = k_2\Big(1-\frac{e}{2}\Big);$$

and consequently,

$$\begin{aligned}\Gamma &\geq \frac{1}{2\phi(b)}\Big[k_2\Big(1-\frac{e}{2}\Big)\int_{s=u}^{s=b} w(s)\,ds - \big(\phi(b)-\phi(u)\big)\Big] \\ &= \frac{1}{2}\Big[k_2\Big(1-\frac{e}{2}\Big)-1\Big]\Big[1-\frac{\phi(u)}{\phi(b)}\Big] \geq \frac{1}{2}\Big[k_2\Big(1-\frac{e}{2}\Big)-1\Big]\Big[1-k_1\Big(\frac{u}{b}\Big)\Big] \\ &= \frac{1}{2}\Big[k_2\Big(1-\frac{e}{2}\Big)-1\Big]\Big[1-k_1\Big(1-\frac{e}{2}\Big)\Big] =: \xi(e).\end{aligned}$$

Note that ξ is a strictly increasing function on $(0,1)$ and $\lim_{\varepsilon\to 0+}\xi(\varepsilon) = 0$. From the above reasoning, we see that ϕ is uniformly concave and $\delta_\phi(\varepsilon) \geq \xi(\varepsilon) > 0$, for all $\varepsilon \in (0,1)$.

(ii) $\Longrightarrow$ (i) Suppose that $\phi = \phi_w$ is uniformly concave. Fix $\alpha \in (0,1)$. Fix $t > 0$. Let $\delta = \delta_\phi$ be the modulus of uniform concavity of ϕ. In Definition 2.2(b) substitute $a = \alpha t$, $b = t$ and let $u = \frac{a+b}{2} = \big(\frac{1+\alpha}{2}\big)t$. Note that $\frac{b-a}{b} = 1-\alpha$. Then

$$\begin{aligned}0 < \delta(1-\alpha) &\leq \frac{1}{2\phi(t)}\Big[\phi(u) - \phi(\alpha t) - \big(\phi(t)-\phi(u)\big)\Big] \\ &= \frac{1}{2\phi(t)}\Big[\int_{\alpha t}^{u} w(s)\,ds - \int_{u}^{t} w(s)\,ds\Big] \leq \frac{1}{2tw(t)}[w(\alpha t)(u-\alpha t) - w(t)(t-u)] \\ &= \frac{1}{2tw(t)}\Big(\frac{1-\alpha}{2}\Big)t[w(\alpha t) - w(t)] = \Big(\frac{1-\alpha}{4}\Big)\Big[\frac{w(\alpha t)}{w(t)} - 1\Big].\end{aligned}$$

So, $\frac{w(\alpha t)}{w(t)} \geq \frac{4\delta(1-\alpha)}{(1-\alpha)} + 1$, for all $t > 0$. Hence, $k_2(\alpha) \geq \frac{4\delta(1-\alpha)}{(1-\alpha)} + 1$, for all $\alpha \in (0,1)$; and so $w \in C_2$.

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Again fix $\alpha \in (0,1)$; and then $t > 0$. Making the same substitutions into Definition 2.2(b) as in the first part of the proof of (ii) $\Longrightarrow$ (i) gives us that

$$\begin{aligned} 0 < \delta(1-\alpha) &\le \frac{1}{\phi(t)}\Big[\phi(u) - \frac{1}{2}\big(\phi(\alpha t) + \phi(t)\big)\Big] \\ &= \frac{1}{2\phi(t)}\Big[\phi(u) - \phi(\alpha t) - \big(\phi(t) - \phi(u)\big)\Big] \\ &\le \frac{1}{2\phi(t)}[\phi(t) - \phi(\alpha t)] = \frac{1}{2}\Big[1 - \frac{\phi(\alpha t)}{\phi(t)}\Big]. \end{aligned}$$

So, $\frac{\phi(\alpha t)}{\phi(t)} \le 1 - 2\delta(1-\alpha)$, for all $t > 0$. Consequently, $k_1(\alpha) \le 1 - 2\delta(1-\alpha)$, for all $\alpha \in (0,1)$; and so $w \in C_1$. ■

We remark that the estimates for $k_1(\alpha)$ and $k_2(\alpha)$ in (ii) $\Longrightarrow$ (i) of the above proof are best for α close to 1. Moreover, $\lim_{\varepsilon \to 0+} \delta(\varepsilon)/\varepsilon = 0$, and we have the following estimates.

COROLLARY 2.4. *Let $w \in W$. Then, for all $\varepsilon \in (0,1)$,*

$$\frac{1}{4}[k_2(1-\varepsilon) - 1] \ge \frac{\delta(\varepsilon)}{\varepsilon} \ge \frac{1 - k_1(1/2)}{4}\Big[k_2\Big(1 - \frac{\varepsilon}{2}\Big) - 1\Big].$$

PROOF. The left hand inequality is simply the inequality in the first part of the proof of (ii) $\Rightarrow$ (i) in Theorem 2.3 above, with α replaced by $1-\varepsilon$. The right hand estimate is derived from the proof of (i) $\Rightarrow$ (ii) above and Lemma 1.1. Indeed, for each $\varepsilon \in (0,1)$, substitute $\alpha := 1/2 < 1 - \varepsilon/2 =: \beta$ in Lemma 1.1. Then the result readily follows. ■

3. **A second derivative characterization of the uniform concavity of ϕ.** Let $w \in W$ and $G(t) = G_w(t) := \frac{\phi(t)}{tw(t)}$, for all $t > 0$. Note that $\phi(t) \ge tw(t)$ implies that $G(t) \ge 1$, for all $t > 0$. The following characterization of the class C_1 is straightforward and well-known. We therefore omit the proof.

LEMMA 3.1. *Let $w \in W$. Then $w \in C_1 \Longleftrightarrow G$ is bounded on* $(0,\infty)$.

Next, let us assume that $w \in W$ is such that $w'(s)$ exists for all $s \in (0,\infty)$. Let W^0 be the collection of all such weights w. We consider now the function $\Gamma = \Gamma_w$, defined for all $w \in W^0$ by $\Gamma(t) := \frac{-t^2\phi''(t)}{\phi(t)} = \frac{-t^2 w'(t)}{\phi(t)}$, for all $t > 0$. Our next result gives a useful sufficient condition for the uniform concavity of ϕ.

THEOREM 3.2. *Let $w \in W^0$. If $\gamma := \inf_{t>0}\Gamma(t) > 0$, then $\phi = \phi_w$ is uniformly concave and $\delta_\phi(\varepsilon) \ge \frac{\gamma\varepsilon^2}{8}$, for all $\varepsilon \in (0,1)$.*

PROOF. Fix $0 < a < b$. Let $u = (a+b)/2$. Then, by Taylor's theorem,

$$\begin{aligned} \phi(a) &= \phi(u) + (a-u)\phi'(u) + \tfrac{(a-u)^2}{2}\phi''(\zeta), \quad \text{for some } \zeta \in (a,u), \text{ and} \\ \phi(b) &= \phi(u) + (b-u)\phi'(u) + \tfrac{(b-u)^2}{2}\phi''(\eta), \quad \text{for some } \eta \in (u,b). \end{aligned}$$

Hence, using our hypothesis on Γ,

$$\frac{1}{2}[\phi(a)+\phi(b)] = \phi(u) + \frac{(b-a)^2}{16}[\phi''(\zeta)+\phi''(\eta)] \leq \phi(u) + \frac{(b-a)^2}{16}\left[\frac{-\gamma\phi(\zeta)}{\zeta^2} + \frac{-\gamma\phi(\eta)}{\eta^2}\right].$$

But $s \longmapsto \phi(s)/s$ is decreasing on $(0,\infty)$ because $sw(s) \leq \phi(s)$ for all s; and consequently,

$$\frac{1}{2}[\phi(a)+\phi(b)] \leq \phi(u) - \gamma\frac{(b-a)^2}{16}\left[\frac{\phi(b)}{b^2}+\frac{\phi(b)}{b^2}\right] = \phi(u) - \frac{\gamma}{8}\left(\frac{b-a}{b}\right)^2\phi(b).$$

It follows that ϕ is uniformly concave with $\delta_\phi(\varepsilon) \geq \frac{\gamma\varepsilon^2}{8}$, for all $\varepsilon \in (0,1)$. ■

EXAMPLE 3.3. The converse to Theorem 3.2 generally fails. For example, define

$$w(t) := \frac{1}{t^{1/2}\left(\sqrt{2}+\sin(\frac{1}{2}\ln t)\right)}, \quad \text{for all } t > 0.$$

Then

$$w'(t) = -\frac{\left(w(t)\right)^2}{\sqrt{2}t^{1/2}}\left[1+\sin\left(\frac{1}{2}\ln t + \frac{\pi}{4}\right)\right] \leq 0, \quad \text{for all } t > 0.$$

So $w \in W^0$, and since $w'(t)$ vanishes at at least one t, $\gamma := \inf_{t>0}\frac{-t^2w'(t)}{\phi(t)} = 0$. Next, fix $\alpha \in (0,1)$. For all $t > 0$, let

$$F(t) := \frac{w(\alpha t)}{w(t)} = \frac{\sqrt{2}+\sin(\frac{1}{2}\ln t)}{\alpha^{1/2}\left(\sqrt{2}+\sin\left(\frac{1}{2}\ln(t\alpha)\right)\right)}.$$

It's not hard to see that $\alpha^{1/2}F$ has the same local extreme values as $H\colon \mathbf{R} \longrightarrow \mathbf{R}$ given by

$$H(s) := \frac{\sqrt{2}+\sin(s-B)}{\sqrt{2}+\sin(s+B)}, \quad s \in \mathbf{R},$$

where $B := \frac{1}{4}\ln\alpha$. (Indeed, $s = (1/2)\ln t + B$). H is 2π-periodic, and for $e^{-4\pi} < \alpha < 1$, $\sin B < 0$; so that elementary calculations give us that $H(s)$ is minimal when $H(s) = V(\alpha)$, where

$$V(\alpha) := \frac{\sqrt{2}-\frac{1}{\sqrt{2}}\cos^2 B + \sqrt{1-\frac{1}{2}\cos^2 B}\sin B}{\sqrt{2}-\frac{1}{\sqrt{2}}\cos^2 B - \sqrt{1-\frac{1}{2}\cos^2 B}\sin B}.$$

It follows that $k_2(\alpha) := \inf_{t>0}F(t) = \frac{V(\alpha)}{\alpha^{1/2}}$, for all $\alpha \in (e^{-4\pi},1)$. Basic calculations with power series give us that as $\alpha \longrightarrow 1-$, $k_2(\alpha) - 1 = \kappa(1-\alpha)^3 + O\left((1-\alpha)^4\right)$, where κ is a positive constant. In particular, for α close enough to 1 we have that $k_2(\alpha) > 1$; and so $w \in C_2$. Moreover, it is easy to check that $\phi(t) \leq Mtw(t)$ for all $t > 0$, for some $M \in (0,\infty)$; so that $w \in C_1$ by Lemma 3.1. It is now clear that $\phi = \phi_w$ is uniformly concave, and yet the hypothesis of Theorem 3.2 that $\gamma > 0$ fails. (We remark that $\kappa = 1/96$ and we may take $M = 2(\sqrt{2}+1)^2)$. This completes the discussion of our example.

We show below that by assuming more about the weight function w, we can prove a converse to Theorem 3.2. We do this by eliminating the possibility of an oscillatory w', such as the one described in 3.3 above. We will assume henceforth that $w \in W$ is such that $w'(s)$ exists for all $s \in (0,\infty)$ and w' is an increasing function. Let W^1 be the collection of all such weights w. Note that for all $w \in W^1$, $-w'$ is a non-negative, decreasing function on $(0,\infty)$, and w itself is both absolutely continuous and convex.

To give some idea of the regularizing effect on the function $\Gamma = \Gamma_w$ of the hypothesis that $w \in W^1$, we state the following proposition. We omit the proof, since we do not use this result later.

PROPOSITION 3.4. *Let $w \in W^1$. Then* $\sup_{t>0} \Gamma(t) < \infty$.

We come now to a converse of Theorem 3.2.

THEOREM 3.5. *Let $w \in W^1$ and suppose that $\phi = \phi_w$ is uniformly concave. Then* $\gamma := \inf_{t>0} \Gamma(t) > 0$.

PROOF. From Section 2 we know that $w \in C_1 \cap C_2$. Fix $\alpha \in (0,1)$. Then for all $t \in (0,\infty)$, $\Gamma(t) = R(t)S(t)$, where $R, S : (0,\infty) \longrightarrow (0,\infty)$ are given by

$$R(t) := \frac{t\big(w(t) - w(t/\alpha)\big)}{\phi(t)} \quad \text{and} \quad S(t) := \frac{-tw'(t)}{\big(w(t) - w(t/\alpha)\big)}.$$

Fix $t \in (0,\infty)$. Then, via Lemma 3.1, $M := \sup_{t>0} G(t) \in [1,\infty)$; and so

$$\begin{aligned} R(t) &= \Big(\frac{w(t)}{w(t/\alpha)} - 1\Big)\frac{tw(t/\alpha)}{\phi(t)} \geq \big(k_2(\alpha) - 1\big)\alpha\Big(\frac{(t/\alpha)w(t/\alpha)}{\phi(t/\alpha)}\Big) \\ &= \big(k_2(\alpha) - 1\big)\alpha\frac{1}{G(t/\alpha)} \geq \frac{1}{M}\big(k_2(\alpha) - 1\big)\alpha. \end{aligned}$$

Also, because $-w'$ is a decreasing, non-negative function on $(0,\infty)$,

$$\frac{1}{S(t)} = \frac{w(t) - w(t/\alpha)}{-tw'(t)} = \frac{\int_t^{t/\alpha} -w'(s)\,ds}{-t\,w'(t)} \leq \frac{-w'(t)\big((t/\alpha) - t\big)}{-tw'(t)} = \frac{1-\alpha}{\alpha}.$$

Thus for all $t > 0$, $\Gamma(t) = R(t)S(t) \geq \frac{1}{M}\frac{\big(k_2(\alpha)-1\big)\alpha^2}{1-\alpha}$. ∎

By Corollary 2.4 and the last line of the previous theorem, together with Theorem 3.2, we have the following estimates.

COROLLARY 3.6. *Let $w \in W^1 \cap C_1$, and γ and M be as defined in the statement and proof of Theorem 3.5. Then for the function $\phi = \phi_w$ we have that, for all $\varepsilon \in (0, 1)$,*

$$\frac{M\gamma\varepsilon^2}{4(1-\varepsilon)^2} \geq \delta_\phi(\varepsilon) \geq \frac{\gamma\varepsilon^2}{8}.$$

We remark that the right-most inequality in the above inequalities is still true if we just assume $w \in W^0$, by the proof of Theorem 3.2. Also, the coefficient of ε^2 in the right-most inequality is the largest possible. Indeed, for $1 < p < \infty$, let $w(t) := \frac{1}{p}t^{1/p-1}$, for all $t > 0$. Then $\phi(t) = t^{1/p}$; while the corresponding function Γ is a constant function, with value $\gamma = \frac{1}{pp'}$. Here, $p' := p/(p-1)$. Moreover, it is easy to calculate that

$$\delta_\phi(\varepsilon) = \Big(\frac{2-\varepsilon}{2}\Big)^{1/p} - \frac{1}{2}(1-\varepsilon)^{1/p} - \frac{1}{2}.$$

Consequently, $\delta(\varepsilon) = \frac{1}{8pp'}\varepsilon^2 + O(\varepsilon^3)$.

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