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WEAK SEPARATION LATTICES OF GRAPHS

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1. Introduction. In an interesting, but apparently largely unknown, paper [**1**] Halin has introduced the concept of a *primitive* set of vertices of a graph. This concept, or rather a slight modification of it, seems to provide the key to a new approach to the well-known and important class of graph-theoretical problems centering on the notion of separation. As is well known, if A, B, C are sets of vertices of a (non-oriented) graph X, C is said to *separate* A and B if and only if every AB-path in X contains a vertex of C. Holding A fixed, let us write $C \leqq_A B$ for the fact that C separates A and B. This relation of separation is easily seen to be a quasi-order on the power set of the vertex-set of X.

In general, $\leqq_A$ is not a partial order. In order to force it to be one, there is, as always, the possibility of passing to the quotient structure associated with the canonical equivalence defined by $B\sigma_A C$ if and only if $B \leqq_A C$ and $C \leqq_A B$. In the context of separation theory it is, however, more natural to consider sets of vertices rather then equivalence classes of such sets. The customary procedure in such a situation is to search for a "canonical" representative of each equivalence class modulo σ_A and to work with these representatives. This approach was used by Pym and Perfect [**6**] in their study of independence structures associated with separation in (directed) graphs. They were able to show ([**6**, Lemma 7.3]) that the set of canonical representatives of σ_A endowed with the relation $\leqq_A$ is a complete lattice. This closely parallels Halin's result ([**1**, Satz 1]) that the primitive sets (relative to A) form a complete lattice with respect to $\leqq_A$. It should be noted, however, that Halin-lattices and Pym and Perfect-lattices are not the same thing.

The very fact that complete lattices make their appearance in connection with separation is sufficiently striking to merit a detailed investigation. Surprisingly, this was neither carried out by Pym and Perfect nor by Halin. Only recently, Pym and Perfect-lattices were systematically studied by Polat [**5**].

The present article is inspired by Halin's approach. We replace the equivalence σ_A by a larger one, τ_A, also defined in terms of separation and permitting the choice of a unique natural representative for each equivalence class in such a way that, once again, restriction of $\leqq_A$ to the set of these representatives produces a complete lattice. In order to see how this can be accomplished, let B and C be two sets of vertices with $B\sigma_A C$, and take any $x \in B \cup C$, say

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$x \in B$. Let $W = (x_0, \ldots, x_m)$ be an Ax-path, i.e., $x_0 \in A$, $x_m = x$. Since $C \leqq_A B$ there is an $m_1 \leqq m$ such that $x_{m_1} \in C$. Thus $(x_0, \ldots, x_{m_1})$ is an AC-path, hence by $B \leqq_A C$ there is an $m_2 \leqq m_1$ with $x_{m_2} \in B$. Continuing, one obtains a sequence of subscripts $m \geqq m_1 \geqq \ldots \geqq 0$ such that the corresponding vertices alternatingly belong to B and C. Since one must have $m_i = m_{i+1}$ after a finite number of steps, it follows that $x_k \in B \cap C$, for some $k \leqq m$, i.e., $(B \cap C) \leqq_A B$ and $(B \cap C) \leqq_A C$ (which, incidentally, is equivalent to $(B \cap C)\sigma_A B$ and $(B \cap C)\sigma_A C$). However, one cannot prove that $B = C$. Nevertheless, this simple argument shows how to remedy the situation. If x is the *first* vertex of B on the path W (counting from A), then all the subscripts $m, m_1, \ldots$ must be equal, i.e., $x \in C$. Thus, if one considers only sets all of whose members are "first" vertices on some AB-path, one will be able to arrange that $\leqq_A$ is also anti-symmetric. Halin's primitive sets are defined on the basis of this observation.

In spite of the remarkable fact that the Halin-primitive sets form a complete lattice, their definition appears to be too narrow. The reason is that Halin defines primitivity in terms of the vertices of the given graph without availing himself fully of the amount of structure provided by the edges. In Section 2 we redefine the notion of primitivity in a manner closely parallel to that of Halin, but based on the edges, and show that these new primitive sets also form a complete lattice relative to $\leqq_A$. This lattice will be called the *weak separation lattice* of (X, A), and will be denoted by $Q(X, A)$.

In Section 10 we compare the two notions of primitivity. We show that every Halin-primitive set is primitive in the new sense, and that for a rooted graph $(X, \{a\})$ the weak separation lattice $Q(X, \{a\})$ is isomorphic to the separation lattice in the sense of Halin of the line graph ∂X relative to a suitably chosen subset of $V(\partial X)$. Thus when considering *classes* of lattices the theory we are developing in this paper may be viewed as a special case of the theory of Halin. However, for a given *individual pair* (X, A) exactly the opposite is the case; the weak separation lattice $Q(X, A)$ is in general much richer than the corresponding Halin-lattice.

One aspect of the assignment of separation lattices to pairs (X, A) (in whatever sense: weak, Halin, or Pym and Perfect) is that it is *functorial*. In order to make this a meaningful statement suitable morphisms between pairs have to be defined. For weak separation lattices the appropriate mappings (called *rim stable maps*) are introduced in Section 4. We denote the resulting category by $\mathbf{G}_{\mathrm{rim}}$. In Section 5 we establish that Q is a contravariant functor from $\mathbf{G}_{\mathrm{rim}}$ to the category of complete lattices and complete meet-preserving maps. In Section 6 we characterize the lattice of primitive sets of a rooted tree and show that in this case the lattice is always distributive. Section 8 deals with the closure system associated with the complete lattice $Q(X, A)$, providing a geometric description of the corresponding closure operator. Finally, in Section 9 we show that for 3-connected graphs the lattice $Q(X, A)$ is always coatomic.

Further papers will deal with the relationships between $Q(X, A)$ and the weak separation lattices of the contractions of X, the question of the existence of adjoints of the functor Q, as well as with certain geometric lattices related to weak separation lattices (which in general are not semi-modular and hence not geometric), thus providing a framework in which the vast theory of geometric lattices can be brought to bear on problems in separation.

For a graph X we denote the vertex-set by $V(X)$, the edge-set by $E(X)$. Thus, $E(X)$ is an irreflexive, symmetric binary relation on $V(X)$. Y is a *subgraph* of X if $V(Y) \subset V(X)$ and $E(Y) \subset E(X)$; it is a *restriction* of X if it is a subgraph and $x, y \in V(Y)$, $[x, y] \in E(X)$ always implies $[x, y] \in E(Y)$ (i.e., Y is a *full* subgraph). If $Y \subset X$ we denote by $X \backslash Y$ the least subgraph of X whose edge-set is $E(X) - E(Y)$. $X - Y$ will denote the restriction of X to $V(X) - V(Y)$. For $x \in V(X)$ we put $V(x;X) = \{y \in V(X) : [x, y] \in E(X)\}$ and $E(x;X) = \{[x, y] : y \in V(x;X)\}$, i.e., $V(x;X)$ is the set of neighbors of x, and $E(x;X)$ the set of edges incident with x.

A *path* $W = (x_0, \ldots, x_n)$ is a graph with $V(W) = \{x_0, \ldots, x_n\}$ (all x_i distinct) and $E(W) = \{[x_{i-1}, x_i] : i = 1, \ldots, n\}$.

Throughout this paper, X will denote a connected graph (unless otherwise stated).

We conclude this section with an important result concerning the relation σ_A. For a graph X we shall denote by $\mathfrak{P}_X$ the complete lattice of all subsets of $V(X)$ with union and intersection as operations.

1.1. THEOREM. *Every σ_A-class is a complete sublattice of $\mathfrak{P}_X$.*

Because it will be useful later we state first the following obvious (and well-known) result.

1.2 LEMMA. *Let A, B_i, $i \in I$, be subsets of $V(X)$, and put $C = \bigcup_{i \in I} B_i$. Then $B \leqq_A C$ if and only if $B \leqq_A B_i$ for each $i \in I$.*

Proof (of Theorem 1.1). Let B, B_i, $i \in I$, be subsets of $V(X)$, and put $C = \bigcup_{i \in I} B_i$, $D = \bigcap_{i \in I} B_i$. We have to show that if $B\sigma_A B_i$ for each $i \in I$, then $B\sigma_A C$ and $B\sigma_A D$.

That σ_A is compatible with arbitrary unions can be seen as follows. By 1.2, $B \leqq_A C$. On the other hand, $C \leqq_A B$ is a consequence of $B_i \subset C$ since each AB-path contains a vertex of B_i.

Concerning intersections, let $W = (x_0, \ldots, x_n)$ be an AB-path, and $m \leqq n$ the least subscript with $x_m \in B$. Since $B_i \leqq_A B$ for each i, there is an $r_i \leqq m$ with $x_{r_i} \in B$. If $r_i < m$, application of $B \leqq_A B_i$ to the AB_i-path $(x_0, \ldots, x_{r_i})$ produces an $s_i \leqq r_i$ with $x_{s_i} \in B$, contradicting the minimality of m. Thus, $r_i = m$ for each $i \in I$, i.e., $x_m \in D$. Therefore $D \leqq_A B$. On the other hand, $B \leqq_A D$ is trivial, since every AD-path is also an AB_i-path for each $i \in I$. Hence $B\sigma_A D$, completing the proof. In fact, it is easy to show that every σ_A-class is an interval in $\mathfrak{P}_X$.

Theorem 1.1 indicates how a canonical representative of each σ_A-class can be obtained. In principle there are two choices, either to take the greatest element of $\sigma_A[B]$, i.e., $\bigcup\sigma_A[B]$, or the least element, $\bigcap\sigma_A[B]$. Since in separation theory the emphasis is on minimal separators, one adopts the second alternative. This leads to the sets first introduced by Pym and Perfect and studied in detail by Polat [**5**]. Polat calls them *primitive* but it should be observed that his notion of primitivity coincides neither with that of Halin nor with our own.

The author is indebted to the referee for having pointed out an error in the formulation of Theorem 1.1 and for having shortened its proof, as well as for numerous other remarks and suggestions.

2. Primitive sets.

2.1 *Definition.* Let $A, B \subset V(X)$. An *AB-accessibility path* in X is a path $W = (x_0, \ldots, x_n) \subset X$ such that

$$x_0 \in A \quad \text{and} \quad x_0, \ldots, x_{n-1} \notin B$$

(the vertex x_n may or may not belong to B). x_n will be called an *AB-accessible vertex*. By $\mathfrak{W}_{AB}$ we denote the set of all AB-accessibility paths in X, and we set

$$X_{AB} = \bigcup \mathfrak{W}_{AB}.$$

In the following we will keep A fixed whereas B is allowed to range over all subsets of $V(X)$. With this in mind we say that X_{AB} is the *accessibility graph* determined by B. These graphs correspond to (but are not identical with) what is called *Verbindungsgraph* in [**1**, p. 35]. In the terminology just introduced a Verbindungsgraph $X(A \to B)$ in the sense of Halin is the restriction of X to $V(X_{AB})$, and might thus be called a *full accessibility graph* (see also Section 10).

It is clear from Definition 2.1 that no AB-accessibility path can contain two vertices of B.

Figure 1 shows an example of an accessibility graph ($A = \{a\}$, $B = \{1, 2, 3', 4'\}$) on the left according to our definition, on the right the corresponding full accessibility graph.

In any graph X, if $x \in A$ then (x) is an AB-accessibility path for any $B \subset V(X)$. Hence $A \subset V(X_{AB})$.

If $A = \emptyset$ there are no AB-accessibility paths. This means that $X_{\emptyset B}$ is always empty. Since this is an uninteresting case *we will henceforth assume that* $A \neq \emptyset$.

If $B = \emptyset$, then clearly every path in X which has one endpoint in A is an AB-accessibility path. Consequently, $X_{A\emptyset} = X$.

2.2 Lemma. *$C \leqq_A B$ implies $\mathfrak{W}_{AC} \subset \mathfrak{W}_{AB}$, and hence $X_{AC} \subset X_{AB}$.*

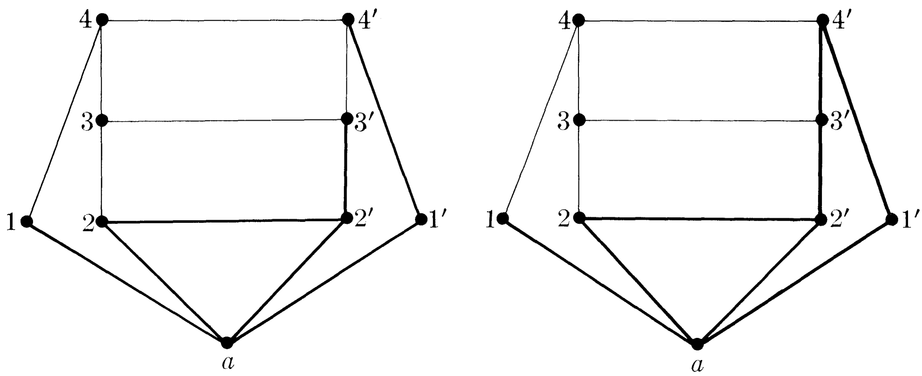

FIGURE 1

Proof. Let $W = (x_0, \ldots, x_n)$ be an AC-accessibility path. If $x_r \in B$ for some $r < n$, then $(x_0, \ldots, x_r)$ is an AB-path, whence by $C \leqq_A B$ there is an s, $0 \leqq s \leqq r$, with $x_s \in C$, a contradiction. Hence $x_0, \ldots, x_{n-1} \notin B$, i.e., W is an AB-accessibility path.

Under very slight hypotheses concerning X the converse of 2.2 is also true.

2.3 LEMMA. *If no vertex belonging to B is of degree* 1, *and if $\mathfrak{W}_{AC} \subset \mathfrak{W}_{AB}$, then $C \leqq_A B$.*

Proof. Suppose $\mathfrak{W}_{AC} \subset \mathfrak{W}_{AB}$ and let $W = (x_0, \ldots, x_n)$ be an AB-path, i.e., $x_0 \in A$, $x_n \in B$. Suppose, moreover, that $x_0, \ldots, x_n \notin C$, and consider an arbitrary vertex y adjacent to x_n. If $y \notin V(W)$, then $W' = (x_0, \ldots, x_n, y)$ is an AC-accessibility path, hence by hypothesis, $W' \in \mathfrak{W}_{AB}$. This means in particular that $x_n \notin B$, a contradiction. If, on the other hand, $y = x_m$ with $m \leqq n - 2$, then $W'' = (x_0, \ldots, x_m, x_n, x_{n-1})$ is an AC-accessibility path, and hence belongs to $\mathfrak{W}_{AB}$. Once again this yields $x_n \notin B$. It follows that x_{n-1} is the only vertex adjacent to x_n, i.e., x_n is of degree 1.

The following is now obvious:

2.4 PROPOSITION. *If X is a graph without vertices of degree* 1, *then $C \leqq_A B$ if and only if $\mathfrak{W}_{AC} \subset \mathfrak{W}_{AB}$.*

Extremely simple examples show that 2.3 is false in the presence of vertices of degree 1. One may even have $\mathfrak{W}_{AB} = \mathfrak{W}_{AC}$ for two sets B and C which are incomparable with respect to $\leqq_A$.

2.5 *Definition.* Let X be any graph (no connectedness assumed here), Y a subgraph of X. By the *rim* of Y relative to X is meant the set

$$\mathfrak{R}Y = \{y \in V(Y) : E(y;X) \not\subset E(Y)\} = V(Y \cap X\backslash Y).$$

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A vertex $y \in \mathfrak{R}Y$ is called a *rim vertex* of Y, int $Y = V(Y) - \mathfrak{R}Y$ is called the *interior* of Y, and a vertex $y \in$ int Y is an *interior vertex* of Y. By Int Y we denote the restriction of Y to int Y. If $A \subset V(X)$, the rim of the restriction of X to A is called the *boundary* of A and is denoted by $\mathfrak{B}A$. That is,

$$\mathfrak{B}A = \{a \in A : V(a;X) \not\subset A\}.$$

For $Y \subset X$ we define $\mathfrak{B}Y$ to be $\mathfrak{B}V(Y)$.

Clearly $\mathfrak{B}Y \subset \mathfrak{R}Y$. Equality holds if and only if $y \in \mathfrak{R}Y$ implies $E(y;X) \not\subset E(\bar{Y})$, where $\bar{Y}$ is the restriction of X to $V(Y)$. In particular, $\mathfrak{B}Y = \mathfrak{R}Y$ if Y is a restriction of X.

2.6 LEMMA. (i) $E(x;X_{AB}) = E(x;X)$ *for each* $x \in V(X_{AB}) - B$.

(ii) $\mathfrak{R}X_{AB} \subset B$ (*rim relative to* X).

(iii) *If* $x, y \in V(X_{AB})$ *but not both* $x, y \in \mathfrak{R}X_{AB}$, *and* $e = [x, y] \in E(X)$, *then* $e \in E(X_{AB})$. *In particular,* Int X_{AB} *is a restriction of* X.

(iv) *If* $x, y \in B \cap V(X_{AB})$, *then* $[x, y] \notin E(X_{AB})$. *Hence if two vertices* x, y *of* $B \cap V(X_{AB})$ *are adjacent in* X, *then* $x, y \in \mathfrak{R}X_{AB}$. *In other words, if two vertices in* B *are adjacent in* X, *then neither of them belongs to* int X_{AB}.

Proof. (i) $x \in V(X_{AB}) - B$ implies the existence of an AB-accessibility path $W = (x_0, \ldots, x_n)$ with $x_n = x$ and $V(W)$ disjoint from B. Take any $e = [x, y] \in E(x;X)$. If $y \notin V(W)$, then $W \cup (e)$ is an AB-accessibility path, whence $e \in E(X_{AB})$. If $y \in V(W)$, i.e., $y = x_m$, $m < n$, then $(x_0, \ldots, x_m) \cup (e)$ is an AB-accessibility path, so that again $e \in E(X_{AB})$.

(ii) is an immediate consequence of (i).

(iii) is trivial since e belongs to $E(x;X)$ as well as $E(y;X)$. Hence if $e \notin E(X_{AB})$, then $x, y \in \mathfrak{R}X_{AB}$.

(iv) is equally trivial, for if $e = [x, y] \in E(X_{AB})$, then e is an edge of some AB-accessibility path W. But no accessibility path contains two vertices of B.

2.7 LEMMA. *Let* $B_0 = \mathfrak{R}X_{AB}$. *Then* $X_{AB_0} = X_{AB}$.

Proof. $B_0 \subset B$ implies that every AB-accessibility path is also an AB_0-accessibility path, i.e., $\mathfrak{W}_{AB} \subset \mathfrak{W}_{AB_0}$, and hence $X_{AB} \subset X_{AB_0}$.

We show next that $V(X_{AB}) = V(X_{AB_0})$. Suppose $x \notin V(X_{AB})$. Then any Ax-path $(x_0, \ldots, x_n)$, where $x_n = x$, meets B. Let r be the least subscript with $x_r \in B$. Then $(x_0, \ldots, x_r)$ is an AB-accessibility path, hence $x_r \in V(X_{AB})$. Let s be the largest subscript such that $x_s \in V(X_{AB})$. Clearly $r \leqq s < n$ since $x_n = x \notin V(X_{AB})$. By maximality of s, $x_{s+1} \notin V(X_{AB})$, hence $x_s \in \mathfrak{R}X_{AB} = B_0$. It follows that no Ax-path is an AB_0-accessibility path, i.e., $x \notin V(X_{AB_0})$.

To complete the proof take any AB_0-accessibility path $W = (y_0, \ldots, y_p)$. By what has just been shown $V(W) \subset V(X_{AB})$. Hence if some edge $e = [y_i, y_{i+1}]$ does not belong to $E(X_{AB})$, then $y_i \in \mathfrak{R}X_{AB} = B_0$, a contradiction. Thus $W \subset X_{AB}$.

It should be observed that in general $\mathfrak{W}_{AB_0}$ and $\mathfrak{W}_{AB}$ are different. For example, if X is a (finite) circuit, and a, b are two distinct vertices of X, then the ab-accessibility paths are precisely those which start at a and do not contain b as an interior point. Thus $X_{ab} = X$, whence $B_0 = \mathfrak{R}X_{ab} = \emptyset$. But this means that $\mathfrak{W}_{aB_0}$ consists of *all* paths starting at a, i.e., $\mathfrak{W}_{aB_0}$ is properly larger than $\mathfrak{W}_{ab}$.

In the light of 2.6 (ii), which says that $\mathfrak{R}X_{AB} \subset B$ for any $B \subset V(X)$, we now define the basic concept of this paper.

2.8 *Definition.* A set $B \subset V(X)$ is *primitive* (or, more precisely, *A-primitive*) if and only if $\mathfrak{R}X_{AB} = B$.

The introduction of primitive sets is motivated by the observation that for them, separation is a partial order.

2.9 PROPOSITION. *Let B be an A-primitive set, C any subset of $V(X)$. Then $C \leqq_A B$ if and only if $X_{AC} \subset X_{AB}$.*

Proof. In view of 2.2 we only have to prove sufficiency. Assume $X_{AC} \subset X_{AB}$. Let $x \in B$, W an Ax-path, and assume that W does not meet C. Then W is an AC-accessibility path and $x \notin C$, i.e., $x \in V(X_{AC}) - C$. By 2.6 (i), $E(x;X_{AC}) = E(x;X)$. Since $X_{AC} \subset X_{AB}$ this gives $E(x;X_{AB}) = E(x;X)$, whence $x \notin \mathfrak{R}X_{AB} = B$, a contradiction.

To recapitulate: by 2.2 we have that $B\sigma_A C$ implies $X_{AB} = X_{AC}$ for arbitrary $B, C \subset V(X)$. In other words, if we define $B\tau_A C$ by $X_{AB} = X_{AC}$, then $\sigma_A \subset \tau_A$. By 2.6 (ii) and 2.7 we have that every set $B \subset V(X)$ contains an A-primitive subset which is equivalent to B modulo τ_A, viz. $\mathfrak{R}X_{AB}$. *Moreover, $\mathfrak{R}X_{AB}$ is the smallest set (relative to inclusion) which is τ_A-equivalent to B.* For if $X_{AB} = X_{AC}$, then $\mathfrak{R}X_{AB} = \mathfrak{R}X_{AC} \subset C$ by 2.6 (ii). Thus the A-primitive sets can be characterized as the (unique) smallest representatives of each equivalence class modulo τ_A. They may thus be considered as canonical representatives, and it is in this sense that they were alluded to in the introduction. Incidentally, it follows from 2.7 that any set C with $\mathfrak{R}X_{AB} \subset C \subset B$ belongs to the same equivalence class modulo τ_A as B. As for σ_A (see 1.1) this permits one to prove that every τ_A-class is closed under arbitrary intersections.

By $Q(X, A)$ we denote the set of all A-primitive subsets of $V(X)$. It follows from 2.9 together with the definition of A-primitive sets that *the restriction of the quasi-order $\leqq_A$ to $Q(X, A)$ is a partial order.*

As a consequence of Lemma 2.7 we have the existence of a mapping $\beta_A : \mathfrak{P}_X \to Q(X, A)$ ($\mathfrak{P}_X$ = power set of $V(X)$ partially ordered by inclusion) defined by

$$\beta_A B = \mathfrak{R}X_{AB}.$$

β_A is *order-inverting*. For if $B \subset B' \subset V(X)$, then clearly $X_{AB} \supset X_{AB'}$, but

$X_{AB} = X_{AC}, X_{AB'} = X_{AC'}$, where $C = \beta_A B$, $C' = \beta_A B'$. Hence $\beta_A B \geqq_A \beta_A B'$.
The following corresponds to Halin's Satz 1.

2.10 THEOREM. *$Q(X, A)$ is a complete lattice relative to the order $\leqq_A$. This lattice is called the weak separation lattice of (X, A).*

Proof. It suffices to show that every family $(B_i)_{i\in I}$ of A-primitive sets has an infimum. We will show that

$$\inf_{i\in I} B_i = \beta_A B, \quad \text{where} \quad B = \bigcup_{i\in I} B_i,$$

Abbreviate $\beta_A B$ by B_0. For $I = \emptyset$ we have $B = \emptyset$, whence $X_{AB} = X$, i.e., $\emptyset$ is the greatest element of $Q(X, A)$.

Since $\beta_A : \mathfrak{P}_X \rightarrow Q(X, A)$ inverts order, we have that for each $i \in I$,

$$B_i \subset B \Rightarrow B_i = \beta_A B_i \geqq_A \beta_A B = B_0.$$

Hence B_0 is a lower bound for the family $(B_i)_{i\in I}$. Now let C be an A-primitive set such that $B_i \geqq_A C$ for every $i \in I$. By Lemma 1.2, this says that $C \leqq_A B$. Since $B \supset B_0$, we have trivially $B \leqq_A B_0$, whence $C \leqq_A B_0$. Thus B_0 is indeed the greatest lower bound.

In the following it will also be convenient to have an explicit expression for suprema in the lattice $Q(X, A)$. For this we first prove a lemma which says that any union of accessibility graphs is again an accessibility graph. Here again we follow the ideas of Halin. The supremum of the empty family, i.e., the least element of $Q(X, A)$ is A since $A \subset V(X_{AB})$ for every $B \subset V(X)$ and X_{AA} is the discrete subgraph of X on A, the latter implying that A is A-primitive.

2.11 PROPOSITION. *For any non-empty family $(B_i)_{i\in I}$ of subsets of $V(X)$, $\bigcup_{i\in I} X_{AB_i} = X_{AB}$, where $B = \mathfrak{R} \bigcup_{i\in I} X_{AB_i}$.*

Proof. Abbreviate X_{AB_i} by X_i and put $Y = \bigcup_{i\in I} X_i$, $B = \mathfrak{R} Y$.

Take any $i \in I$ and an AB_i-accessibility path $W = (x_0, \ldots, x_n)$. Suppose that $x_m \in B$ for some $m < n$. Since $B = \mathfrak{R} Y$ there is an edge

$$e = [x_m, y] \in E(X \backslash Y).$$

But $(x_0, \ldots, x_m, y)$ is an AB_i-accessibility path, whence $e \in E(X_i) \subset E(Y)$, a contradiction. Hence W is an AB-accessibility path. This proves $Y \subset X_{AB}$.

For the reverse inclusion take any AB-accessibility path $W = (x_0, \ldots, x_n)$, and put $e_k = [x_k, x_{k+1}]$, $k = 0, \ldots, n - 1$. Without loss of generality we may assume that $n > 0$. It suffices to prove that for each $k = 0, \ldots, n - 1$ there is an $i \in I$ such that $e_k \in E(X_i)$. Suppose by way of contradiction that there is an m, $0 \leqq m \leqq n - 1$, such that $e_m \notin E(X_i)$ for every $i \in I$, i.e., $e_m \notin E(Y)$, and that m is the smallest such subscript. If $m = 0$, then $x_m = x_0 \in A \subset V(Y)$. If $m > 0$ then by minimality of m, $e_{m-1} \in E(X_j)$ for some $j \in I$, hence $e_{m-1} \in E(Y)$, hence again $x_m \in V(Y)$. But $e_m \notin E(Y)$ implies $x_m \in \mathfrak{R} Y = B$, contrary to W being an AB-accessibility path.

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2.12 COROLLARY. *If* $(B_i)_{i \in I}$ *is any family of* A*-primitive sets, then*

$$\sup_{i \in I} B_i = \Re \bigcup_{i \in I} X_{AB_i}.$$

This follows immediately from the preceding lemma.

3. One-sided sets. This section is primarily technical and provides the background for Section 4 as well as certain results in Section 8.

In view of 2.6 (ii), every set $B \subset V(X)$ can be divided into the following three parts:

$$B \cap \operatorname{int} X_{AB},\ \Re X_{AB}, \text{ and } B - V(X_{AB}).$$

In general, all three sets are non-empty. That is, vertices of B lie on either side of the rim of X_{AB}.

3.1 *Definition.* A set $B \subset V(X)$ is called *one-sided* if and only if

$$B \cap \operatorname{int} X_{AB} = \emptyset.$$

In particular, every primitive set is one-sided. For an example of a set which is not one-sided, see Figure 2a.

3.2 LEMMA. *Any union of one-sided sets is a one-sided set.*

Proof. Let $(B_i)_{i \in I}$ be a family of one-sided sets, $B = \bigcup_{i \in I} B_i$. Suppose there is a vertex $x \in B \cap \operatorname{int} X_{AB}$. Then for each $i \in I$,

$$E(x;X) \subset E(X_{AB}) \subset E(X_{AB_i}).$$

At the same time, $x \in B_j$ for some $j \in I$, whence $x \in B_j \cap \operatorname{int} X_{AB_j}$, a contradiction.

The intersection of two one-sided sets need not be one-sided. In general, even the intersection of two primitive sets is not one-sided.

3.3 LEMMA. *If* $(B_i)_{i \in I}$ *is a family of one-sided sets, then* $\mathfrak{W}_{AB} = \mathfrak{W}_{AC}$, *where* $B = \bigcup_{i \in I} B_i$ *and* $C = \bigcup_{i \in I} \beta_A B_i$.

Proof. $B \supset C$ implies $\mathfrak{W}_{AB} \subset \mathfrak{W}_{AC}$. For the reverse inclusion take any AC-accessibility path $W = (x_0, \ldots, x_n)$ and suppose that W is not an AB-accessibility path. Let m be the least subscript such that $x_m \in B$. Then $x_m \in B_i$ for some $i \in I$, and since $x_0, \ldots, x_{m-1} \notin B$, it follows that $x_m \in V(X_{AB_i})$. By one-sidedness of B_i, $x_m \in B_i \cap V(X_{AB_i})$ implies $x_m \in \Re X_{AB_i} = \beta_A B_i \subset C$, contrary to W being an AC-accessibility path.

By way of digression, we consider the question whether the relation τ_A defined in Section 2 is a congruence on the complete lattice $\mathfrak{P}_X$ (relative to union and intersection as operations). The answer is, in general, negative. In Figure 2a consider the two sets $B_1 = \{2, 4\}$, $B_2 = \{2', 4'\}$. If τ_a were a congruence, then in view of $B_i \tau_a \beta_a B_i = C_i$, $i = 1, 2$, we would have $(B_1 \cup B_2) \tau_a (C_1 \cup C_2)$, but this is

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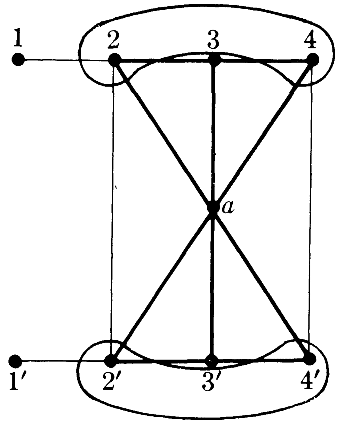

FIGURE 2a
Heavy lines = $X_{a, B_1 \cup B_2}$

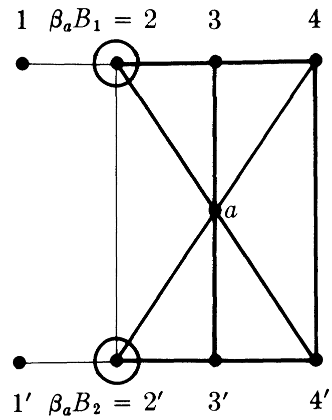

FIGURE 2b
Heavy lines = $X_{a, C_1 \cup C_2}$

obviously not the case, since $X_{a, B_1 \cup B_2}$ is a proper subgraph of $X_{a, C_1 \cup C_2}$.

On the positive side we have:

3.4 PROPOSITION. *τ_A is a semilattice congruence on the set of all one-sided sets.*

Proof. $B_i \tau_a C_i$ implies $\beta_A B_i = \beta_A C_i$, where $(B_i)_{i \in I}$ and $(C_i)_{i \in I}$ are families of one-sided sets. From 3.3 one then has immediately that $U \tau_a V$, where $U = \bigcup_{i \in I} B_i$, $V = \bigcup_{i \in I} C_i$.

Proposition 3.4 says in particular that every τ_A-class of one-sided sets contains a largest element. These sets will be described in Theorem 8.2.

Since the intersection of one-sided sets is not necessarily one-sided it is not surprising that the behavior of τ_A relative to intersections is even worse than for unions. In the graph of Figure 3 let $B_i = \{i, 3, 4, 5\}$, $i = 1, 2$. B_i is one-sided, and $\beta_a B_i = \{i, 3, 4\}$.

The heavy lines in Figure 3a show $X_{a, B_1 \cap B_2} = X_{a, \{3,4,5\}}$, Figure 3b shows $X_{a, \beta_a B_1 \cap \beta_a B_2} = X_{a, \{3,4\}}$. Since these two graphs are different, $\{3, 4, 5\}$ and $\{3, 4\}$ are not in relation τ_a. Thus, in general, intersections do not preserve τ_A even in the case of one-sided sets.

We conclude this section with a result which will be useful later (Section 8).

3.5 LEMMA. *If B is one-sided, then* Int X_{AB} *is the union of all AB-accessibility paths which are disjoint from B.*

Proof. Take any Ax-path W disjoint from B. Then $x \in \operatorname{int} X_{AB}$; for if $x \in \mathfrak{R} X_{AB}$, then $x \in B$ by 2.6 (ii), a contradiction. Conversely, given any $x \in \operatorname{int} X_{AB}$ there is an AB-accessibility path $Q = (x_0, \ldots, x_n)$ with $x_n = x$. By one-sidedness of B, x does not belong to B, hence W is disjoint from B.

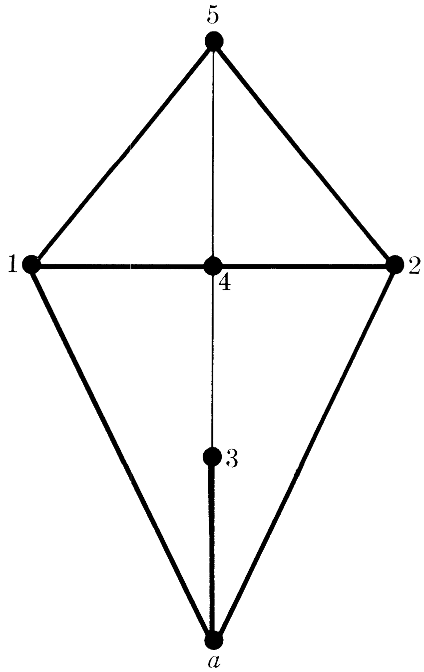

FIGURE 3a

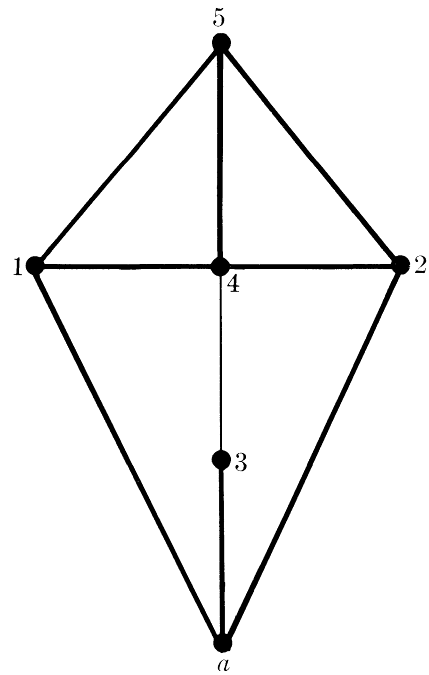

FIGURE 3b

Since Int X_{AB} is a section of X this proves equality of Int X_{AB} and the union of all AB-accessibility paths missing B.

We remark here that contrary to what one might be tempted to think *it is in general not the case* that for two one-sided sets B, C, int $X_{AB} \cup$ int X_{AC} is the interior of some accessibility graph.

4. Rim stable maps. In the next section it will be shown that the construction of the weak separation lattice of a pair (X, A) is "natural" in the sense that if pairs on the one hand and complete lattices on the other are regarded as suitable categories, then the correspondence Q is a functor. To set the stage we have to define an appropriate class of mappings between pairs.

In 4.1 and 4.2 *we do not assume that the graphs involved are connected.*

4.1 *Definition.* Let X, Y be graphs. A function $\varphi : V(X) \rightarrow V(Y)$ is a *weak contraction* if and only if $[x, y] \in E(X)$ implies $\varphi x = \varphi y$ or $[\varphi x, \varphi y] \in E(Y)$.

These are the maps which have been called *homomorphisms* by Ore [**4**].

In the sequel we shall use the following.

Notation. Let $\varphi : V(X) \to V(Y)$ be a weak contraction, $X' \subset X$. By $\varphi X'$ is meant the subgraph of Y with

$$V(\varphi X') = \varphi V(X'),$$
$$\text{and} \quad E(\varphi X') = \{[\varphi x, \varphi y] : [x, y] \in E(X') \quad \text{and} \quad \varphi x \neq \varphi y\}.$$

Similarly, if $Y' \subset Y$, we denote by $\varphi^{(-1)} Y'$ the restriction of X to the set $\varphi^{-1} V(Y')$.

It is an immediate consequence of 4.1 that if $\varphi : V(X) \to V(Y)$ is a weak contraction, and X' is a connected subgraph of X, then $\varphi X'$ is connected. In other words, *weak contractions preserve connectedness.*

4.2 *Definition.* Given two graphs X, Y a function $\varphi : V(X) \to V(Y)$ is a *contraction* if and only if (i) φ is a weak contraction, and (ii) for every $y \in \varphi V(X)$ the graph $\varphi^{(-1)}(y)$ is connected.

Let $\varphi : V(X) \to V(Y)$ be a contraction, $e = [y, y'] \in E(\varphi X)$. Then $V(\varphi^{(-1)}(e)) = V(\varphi^{(-1)}(y)) \cup V(\varphi^{(-1)}(y'))$, the two graphs $\varphi^{(-1)}(y)$ and $\varphi^{(-1)}(y')$ are connected, and there is an edge $[x, x'] \in E(X)$ with $\varphi x = y$, $\varphi x' = y'$. Thus $\varphi^{(-1)}(e)$ is connected. This says that if Y' is any connected subgraph of φX, then $\varphi^{(-1)} Y'$ is connected.

We now consider the class of all pairs (X, A), where X is a connected graph and $A \subset V(X)$ (if $A = \{a\}$ we write (X, a) instead of $(X, \{a\})$). Given two pairs (X, A) and (Y, B) we define a *map* $\varphi : (X, A) \to (Y, B)$ to be a function $\varphi : V(X) \to V(Y)$ such that $\varphi A \subset B$.

4.3 *Definition.* A map $\varphi : (X, A) \to (Y, B)$ is *rim stable* if and only if φ is a weak contraction and the inverse image of every one-sided subset of $V(Y)$ is again one-sided, i.e., for any $C \subset V(Y)$,

$$C \cap \operatorname{int} Y_{BC} = \emptyset \quad \text{implies} \quad \varphi^{-1} C \cap \operatorname{int} X_{A, \varphi^{-1} C} = \emptyset.$$

Obviously, *pairs and rim stable maps form a category.* This category, denoted by $\mathbf{G}_{\text{rim}}$, provides us with the natural setting for the study of weak separation lattices.

The following result shows that the rim stability of a weak contraction φ depends on the behaviour of φ on primitive sets.

4.4 PROPOSITION *A weak contraction $\varphi : (X, A) \to (Y, B)$ is rim stable if and only if the inverse image of every primitive set is one-sided.*

For the proof we need two auxiliary results.

4.5 LEMMA. $B \subset B' \subset V(X)$ *implies* $\operatorname{int} X_{AB'} \subset \operatorname{int} X_{AB}$.

Proof. $B \subset B'$ implies $X_{AB'} \subset X_{AB}$. Hence

$$x \in \operatorname{int} X_{AB'} \subset V(X_{AB'}) \subset V(X_{AB})$$

and $x \notin \operatorname{int} X_{AB}$ implies $x \in \Re X_{AB}$, i.e., $E(x;X) \not\subset E(X_{AB})$. But since $x \in \operatorname{int} X_{AB'}$, $E(x;X) \subset E(X_{AB'}) \subset E(X_{AB})$, a contradiction.

4.6 LEMMA. *If $\varphi : (X, A) \to (Y, B)$ is a weak contraction, then for any $C \subset V(Y)$, $\varphi X_{A,\varphi^{-1}C} \subset Y_{BC}$, and hence $X_{A,\varphi^{-1}C} \subset \varphi^{(-1)} Y_{BC}$.*

Proof. To prove the first part it suffices to show that $\varphi W \subset Y_{BC}$ for any $A, \varphi^{-1}C$-accessibility path $W = (x_0, \ldots, x_n)$. Let $e = [y, y'] \in E(\varphi W)$. There exists a subscript i, $1 \leqq i \leqq n$, such that $y = \varphi x_{i-1}$ and $y' = \varphi x_i$. Note that $\varphi x_0, \ldots, \varphi x_{n-1} \notin C$ since W is an $A, \varphi^{-1}C$-accessibility path. Let h be the least subscript such that e is incident with φx_h. Clearly $h \leqq i - 1 < n$. Consider $U = \varphi(x_0, \ldots, x_h)$. Since φ is a weak contraction, U is a connected subgraph of Y, hence U contains a $\varphi x_0, \varphi x_h$-path Q. $V(Q) \subset V(U) = \{\varphi x_j : 0 \leqq j \leqq h\}$ and since $h < n$, no vertex of Q belongs to C. Hence $Q \cup (e)$ is a BC-accessibility path which contains e, in other words, $e \in E(Y_{BC})$.

The second part is an immediate consequence of the first.

Proof (of Proposition 4.4). Necessity is obvious. Sufficiency: Take any $C \subset V(Y)$ and suppose there is an $x \in \varphi^{-1}C \cap \operatorname{int} X_{A,\varphi^{-1}C}$. $\beta_B C$ is a primitive subset of C; by hypothesis and 4.5,

$$\varphi^{-1}\beta_B C \cap \operatorname{int} X_{A,\varphi^{-1}C} \subset \varphi^{-1}\beta_B C \cap \operatorname{int} X_{A,\varphi^{-1}\beta_B C} = \emptyset,$$

from which it follows that $x \notin \varphi^{-1}\beta_B C$, i.e., $\varphi x \notin \Re Y_{BC}$. But since φ is a weak contraction, $x \in \operatorname{int} X_{A,\varphi^{-1}C} \subset V(X_{A,\varphi^{-1}C})$ implies $\varphi x \in V(Y_{BC})$ (4.6), whence $\varphi x \in \operatorname{int} Y_{BC}$. This says $C \cap \operatorname{int} Y_{BC} \neq \emptyset$, i.e., C is not one-sided.

The next two propositions provide examples of rim stable maps.

4.7 PROPOSITION. *If $\varphi : (X, A) \to (Y, B)$ is a contraction which is nowhere one-one (i.e., $|\varphi^{-1}y| \neq 1$ for every $y \in V(Y)$), then φ is rim stable. In fact, the inverse image under φ of every subset of $V(Y)$ is one-sided.*

Proof. Take any $C \subset V(Y)$ and suppose there is an $x \in \varphi^{-1}C \cap \operatorname{int} X_{A,\varphi^{-1}C}$. Put $\varphi x = y$. Since φ is nowhere one-one, $|\varphi^{-1}y| \geqq 2$. Since φ is a contraction, $\varphi^{(-1)}(y)$ is connected, hence there exists an edge $e = [x, x'] \in E(\varphi^{(-1)}(y))$. $x \in \operatorname{int} X_{A,\varphi^{-1}C}$ implies $e \in E(x;X) \subset E(X_{A,\varphi^{-1}C})$. At the same time, $x, x' \in \varphi^{-1}y \subset \varphi^{-1}C$, but this is a contradiction to 2.6 (iv).

4.8 PROPOSITION. *If φ is a contraction of (X, a) onto (Y, b) (i.e., $\varphi X = Y$ and $\varphi a = b$), then φ is rim stable.*

Proof. Take a b-primitive subset C of $V(Y)$ and suppose there is an $x \in \varphi^{-1}C \cap \operatorname{int} X_{a,\varphi^{-1}C}$. Put $y = \varphi x$. By the proof of 4.7, $\varphi^{-1}y = \{x\}$. Since $y \in C = \Re Y_{bC}$ there exists an edge $e_y = [y, y'] \in E(y;Y) - E(Y_{bC})$. By surjectivity of φ, $\varphi^{-1}y' \neq \emptyset$, and since φ is a contraction, $\varphi^{(-1)}(e_y)$ is a connected subgraph of X with $V(\varphi^{(-1)}(e_y)) = \{x\} \cup \varphi^{-1}y'$. Hence there is an $x' \in \varphi^{-1}y'$ such that $[x, x'] \in E(x;X) \subset E(X_{a,\varphi^{-1}C})$ (since $x \in \operatorname{int} X_{a,\varphi^{-1}C}$), which in turn implies $x' \in V(X_{a,\varphi^{-1}C})$. At the same time $x' \notin \varphi^{-1}C$, otherwise we have a contradiction to 2.6 (iv). Hence by 4.6, $y' = \varphi x' \in V(Y_{bC}) - C = \operatorname{int} Y_{bC}$, whence $e_y \in E(y';Y) \subset E(Y_{bC})$, a contradiction.

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The following example shows that in 4.8 the hypothesis that φ be a contraction (not just a *weak* contraction) is essential. Let X be the graph of Figure 2a, Y the restriction of X to $\{a, 1, 2, 3, 1', 2', 3'\}$, φ the weak contraction of (X, a) onto (Y, a) which maps $4 \mapsto 2$, $4' \mapsto 2'$ and which is the identity elsewhere. $C_1 = \{2\}$ is a-primitive in Y, but $\varphi^{-1}C_1 = B_1 = \{2, 4\}$ is not one-sided. Hence φ is not rim stable.

5. The functor Q**.** Any map $\varphi : (X, A) \to (Y, B)$ induces a function $Q(\varphi) : Q(Y, B) \to Q(X, A)$ by

$$Q(\varphi) = \beta_A \varphi^{-1}.$$

Since $Q(\varphi)$ is a mapping between lattices one might hope that with suitable conditions on φ, $Q(\varphi)$ will turn out to be a lattice homomorphism. Trivial cases aside, this is, however, not the case. We will show (Proposition 5.2) that if φ is rim stable, then $Q(\varphi)$ preserves arbitrary infima but fails to preserve suprema (see the counterexample given after 5.5). Nevertheless, rim stable maps appear to be the natural maps to work with in this context, since they arise from the closure systems associated with separation lattices (see Section 8). We have found no sufficiently wide subclass of the rim stable maps for which the induced maps are lattice homomorphisms. Possible candidates for such a class would be the contractions, but our counterexample shows that even restriction to surjective contractions will not force the induced maps $Q(\varphi)$ to preserve suprema. Thus there seems to be no good reason, at this point, for restricting the class of rim stable maps in any way. Nevertheless, one cannot help feeling that there is room for improvement.

5.1 Lemma. *If* $\varphi : (X, A) \to (Y, B)$ *is a weak contraction and* $\psi : (Y, B) \to (Z, C)$ *is rim stable, then* $Q(\psi \circ \varphi) = Q(\varphi) \circ Q(\psi)$. *In particular this holds when both* φ *and* ψ *are rim stable.*

Proof. Take any C-primitive set $K \subset V(Z)$ and put $H = \varphi^{-1}\beta_B\psi^{-1}K$. We shall show that $\mathfrak{W}_{A,\varphi^{-1}\psi^{-1}K} = \mathfrak{W}_{AH}$. Since $\varphi^{-1}\psi^{-1}K \supset H$ we have that $\mathfrak{W}_{A,\varphi^{-1}\psi^{-1}K} \subset \mathfrak{W}_{AH}$. Now let $W = (x_0, \ldots, x_n)$ be an AH-accessibility path in X. We have to show that W is also an $A, \varphi^{-1}\psi^{-1}K$-accessibility path. Suppose, by way of contradiction, that $x_m \in \varphi^{-1}\psi^{-1}K$ for some $m < n$, and that m is the least subscript with this property. Hence $\varphi x_0, \ldots, \varphi x_{m-1} \notin \psi^{-1}K$ so that $\varphi x_m \in V(Y_{B,\psi^{-1}K})$ (here one uses that φ is a weak contraction). Since $\varphi x_m \in \psi^{-1}K$, rim stability of ψ implies that $\varphi x_m \in \Re Y_{B,\psi^{-1}K} = \beta_B\psi^{-1}K$, i.e., $x_m \in H$, contrary to W being an AH-accessibility path.

5.2 Proposition. *If* $\varphi : (X, A) \to (Y, B)$ *is rim stable, then*

$$\beta_A\varphi^{-1} : Q(Y, B) \to Q(X, A)$$

preserves arbitrary infima.

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In particular, this says that $\beta_A\varphi^{-1}$ is an *order-preserving* map. For the proof of this proposition we need two lemmas.

5.3 LEMMA. *If* $\varphi : (X, A) \to (Y, B)$ *is a weak contraction, then* $\mathfrak{W}_{A,\varphi^{-1}C} = \mathfrak{W}_{A,\varphi^{-1}\beta_B C}$ *for every one-sided set* $C \subset V(Y)$.

Proof. Since $\varphi^{-1}C \supset \varphi^{-1}\beta_B C$ it suffices to prove that $\mathfrak{W}_{A,\varphi^{-1}\beta_B C} \subset \mathfrak{W}_{A,\varphi^{-1}C}$. Take any $A, \varphi^{-1}\beta_B C$-accessibility path $W = (x_0, \ldots, x_n)$, and suppose that there is an $r < n$ with $x_r \in \varphi^{-1}C$. r may be taken as the least subscript with this property. Then $x_0, \ldots, x_{r-1} \notin \varphi^{-1}C$, hence $x_r \in V(X_{A,\varphi^{-1}C})$. Since φ is a weak contraction, $\varphi x_r \in V(Y_{BC})$ by 4.6. This together with $\varphi x_r \in C$ and one-sidedness of C implies $\varphi x_r \in \Re Y_{BC} = \beta_B C$, so that $x_r \in \varphi^{-1}\beta_B C$, contrary to W being an $A, \varphi^{-1}\beta_B C$-accessibility path. Hence we have shown that W is an $A, \varphi^{-1}C$-accessibility path.

5.4 LEMMA. *If* $\varphi : (X, A) \to (Y, B)$ *is a weak contraction, then for any family* $(C_i)_{i\in I}$ *of primitive subsets of* $V(Y)$,

$$\mathfrak{W}_{A,\varphi^{-1}\,\mathrm{inf}\,C_i} = \mathfrak{W}_{A,\varphi^{-1}C},$$

where $C = \bigcup_{i\in I} C_i$.

Proof. By 3.2, C is a one-sided set. Hence $\mathfrak{W}_{A,\varphi^{-1}C} = \mathfrak{W}_{A,\varphi^{-1}\beta_B C}$ by 5.3, but $\beta_B C = \inf C_i$ by the proof of 2.10.

Proof (of Proposition 5.2). Let $(C_i)_{i\in I}$ be a family of primitive subsets of $V(Y)$. We wish to show that $\beta_A\varphi^{-1} \inf C_i = \inf \beta_A\varphi^{-1}C_i$. This is the same as $\beta_A\varphi^{-1}\beta_B C = \beta_A U$, where $C = \bigcup_{i\in I} C_i$ and $U = \bigcup_{i\in I} \beta_A\varphi^{-1}C_i$, and amounts to showing that $X_{A,\varphi^{-1}\,\mathrm{inf}\,C_i} = X_{AU}$. By Lemma 5.4 it suffices to show that $X_{A,\varphi^{-1}C} = X_{AU}$.

For each $i \in I$, $\beta_A\varphi^{-1}C_i \subset \varphi^{-1}C_i$; hence $U \subset \bigcup_{i\in I} \varphi^{-1}C_i = \varphi^{-1}C$, hence $X_{AU} \supset X_{A,\varphi^{-1}C}$.

For the reverse inclusion let $W = (x_0, \ldots, x_n)$ be an AU-accessibility path, i.e., $x_0, \ldots, x_{n-1} \notin U$. We will show that W is also an $A, \varphi^{-1}C$-accessibility path. Suppose, by way of contradiction, that there is an $r < n$ with $x_r \in \varphi^{-1}C = \bigcup_{i\in I} \varphi^{-1}C_i$, and let $s \leqq r$ be the least subscript such that $x_s \in \varphi^{-1}C_j$ for some $j \in I$. Then $x_0, \ldots, x_{s-1} \notin \varphi^{-1}C_j$, in other words, $x_s \in V(X_{A,\varphi^{-1}C_j})$. By rim stability of φ this together with $x_s \in \varphi^{-1}C_j$ implies $x_s \in \Re X_{A,\varphi^{-1}C_j} = \beta_A\varphi^{-1}C_j \subset U$, a contradiction.

We can sum up 5.1 and 5.2 as follows:

5.5 THEOREM. *The assignment* $(X, A) \mapsto Q(X, A)$, $\varphi \mapsto Q(\varphi)$ *is a contravariant functor from the category* $\mathbf{G}_{\mathrm{rim}}$ *into the category* $\mathbf{L}_{\mathrm{inf}}$ *of complete lattices and (complete)* inf-*preserving maps.*

Concerning suprema it is clear from 5.2 that for any rim stable $\varphi : (X, A) \to (Y, B)$ and any family $(C_i)_{i\in I}$ of primitive subsets of $V(Y)$

one has

$$\beta_A\varphi^{-1} \sup C_i \geqq_A \sup \beta_A\varphi^{-1}C_i,$$

but equality need not hold as is shown by the example of Figure 4. Note that the map used in this example is a *contraction* of (X, a) *onto* (Y, a). Figure 4 shows the two graphs and the mapping φ between them; Figure 5 shows the corresponding weak separation lattices. Sets are written without braces and commas; thus $131'$ means $\{1, 3, 1'\}$. Those elements of $Q(X, a)$ which are images under the induced mapping $\beta_a\varphi^{-1}$ are indicated by circles.

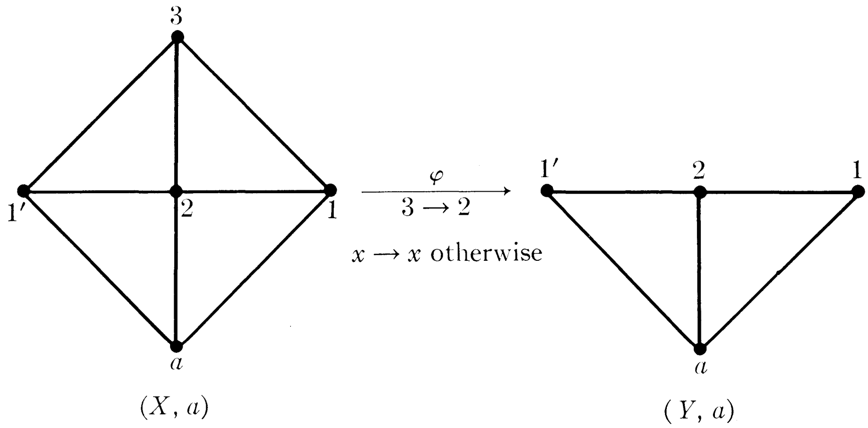

FIGURE 4

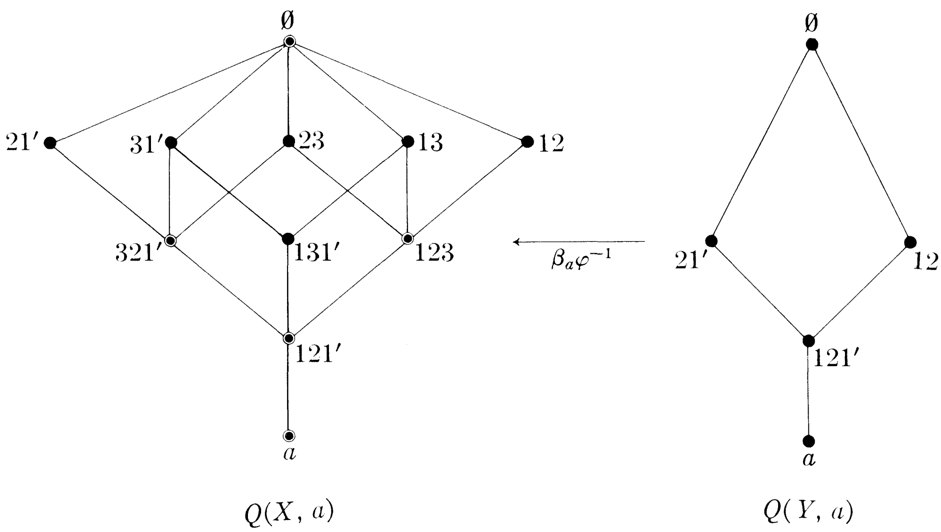

FIGURE 5

Take $C_1 = 21'$, $C_2 = 12$. Then in $Q(Y, a)$, $\sup_{i=1,2} C_i = \emptyset$, hence $\beta_a\varphi^{-1} \sup_{i=1,2} C_i = \emptyset$. On the other hand, $\beta_a\varphi^{-1}C_1 = 321'$, $\beta_a\varphi^{-1}C_2 = 123$, so that in $Q(X, a)$, $\sup_{i=1,2} \beta_a\varphi^{-1}C_i = \sup\{321', 123\} = 23$, which shows that $\beta_a\varphi^{-1}$ does not preserve suprema.

It will be noted that in this example the map $\beta_a\varphi^{-1}$ is one-one. The reason for this is that φ is a surjective contraction, as we shall now show.

5.6 PROPOSITION. *If φ is a contraction of (X, A) onto (Y, B) (i.e., $\varphi X = Y$ and $\varphi A = B$), then $Q(\varphi) : Q(Y, B) \to Q(X, A)$ is one-one.*

The proof depends on the following refinement of 4.6:

5.7 LEMMA. *If φ is a contraction of (X, A) onto (Y, B), then $\varphi X_{A,\varphi^{-1}C} = Y_{BC}$ for every $C \subset V(Y)$.*

Proof. Let $W = (y_0, \ldots, y_n)$ be a BC-accessibility path in Y, i.e., $y_0 \in B$, and $y_0, \ldots, y_{n-1} \notin C$. Put $e_i = [y_{i-1}, y_i]$, $i = 1, \ldots, n$. Since $V(\varphi^{(-1)}(e_i)) = \varphi^{-1}y_{i-1} \cup \varphi^{-1}y_i$ and $\varphi^{(-1)}(e_i)$ is connected, there exist vertices $z_{i-1} \in \varphi^{-1}y_{i-1}$ and $x_i \in \varphi^{-1}y_i$ such that $e_i' = [z_{i-1}, x_i] \in E(\varphi^{(-1)}(e_i))$, $i = 1, \ldots, n$. Choose an arbitrary $x_0 \in A \cap \varphi^{-1}y_0$ (this is possible since $y_0 \in B$ and $\varphi A = B$). Since $\varphi^{(-1)}(y_i)$ is connected, there is an x_iz_i-path P_i in $\varphi^{(-1)}(y_i)$, $i = 0, \ldots, n - 1$. Now put

$$S = P_0 \cup (e_1') \cup P_1 \cup \ldots \cup P_{n-1} \cup (e_n').$$

S is an Ax_n-path, and it is clear from the construction that no vertex of S (with the possible exception of x_n) belongs to $\varphi^{-1}C$. Hence S is an $A, \varphi^{-1}C$-accessibility path. Again from the construction of S it is clear that $\varphi S = W$. This means $W \subset \varphi X_{A,\varphi^{-1}C}$, and hence $Y_{BC} \subset \varphi X_{A,\varphi^{-1}C}$. Equality of the two graphs then follows from 4.6.

Proof (of Proposition 5.6). Let $C, C' \in Q(Y, B)$ and $\beta_A\varphi^{-1}C = \beta_A\varphi^{-1}C'$. Then $X_{A,\varphi^{-1}C} = X_{A,\varphi^{-1}C'}$, whence by 5.7, $Y_{BC} = \varphi X_{A,\varphi^{-1}C} = \varphi X_{A,\varphi^{-1}C'} = Y_{BC'}$. Finally, since C and C' are primitive, $C = C'$.

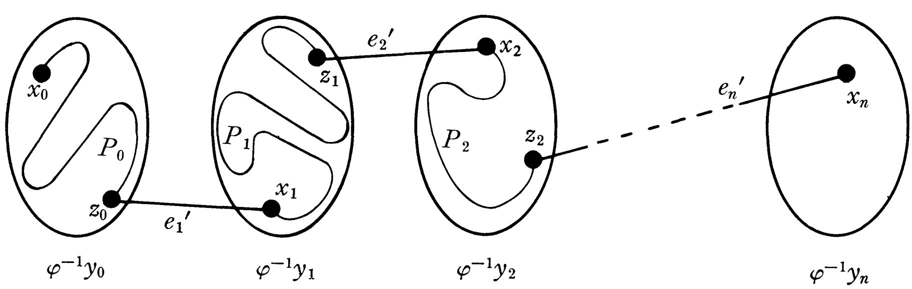

FIGURE 6
The path S

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6. The weak separation lattice of a rooted tree. A case where the structure of a weak separation lattice is particularly easy to describe is that of a rooted tree (T, a).

For $x, y \in V(T)$ define $y \leqq x$ if and only if y is a vertex of the (unique) path in T which joins a and x. We shall describe $Q(T, a)$ in terms of certain ideals of the partially ordered set $(V(T), \leqq)$.

Given an arbitrary partially ordered set $(A, \leqq)$ and an element $x \in A$ we shall mean by the *upper neighborhood* of x the set U_x of all elements $y \in A$ which cover x (i.e., which are $>x$ and for which there is no element z with $x < z < y$). Note that for a vertex x of a rooted tree (T, a) we have $U_x = \{y \in V(x;T) : y > x\}$.

6.1 *Definition.* An order ideal I of a partially ordered set $(A, \leqq)$ is a *neighborhood ideal* if and only if for any $x \in I$ either $U_x \cap I = \emptyset$ or $U_x \subset I$.

6.2 LEMMA. *For any partially ordered set the neighborhood ideals (if there are any) form a complete sublattice of the lattice of all order ideals.*

Proof. The lattice operations in the lattice of all order ideals are set-theoretic union and intersection. Let $(I_\alpha)_{\alpha \in A}$ be any family of neighborhood ideals, $I = \bigcup_{\alpha \in A} I_\alpha$. If $U_x \cap I \neq \emptyset$, then $U_x \cap I_\alpha \neq \emptyset$ for some α, hence $U_x \subset I_\alpha$ by definition of a neighborhood ideal. Hence $U_x \subset I$, i.e., I is a neighborhood ideal. The argument for intersections of neighborhood ideals is quite analogous.

We now characterize the primitive sets of a rooted tree in terms of its neighborhood ideals.

6.3 THEOREM. *For any rooted tree (T, a) the sets $V(T_{aB})$, $B \subset V(T)$, are precisely the neighborhood ideals of the partially ordered set $(V(T), \leqq)$.*

Proof. Take any $B \subset V(T)$ and let $x \in V(T_{aB})$. Then there is an aB-accessibility path $(x_0, \ldots, x_n)$ with $x_0 = a$, $x_n = x$. Hence if $y \leqq x$, then $y = x_i$ for some $i \leqq n$, hence $y \in V(T_{aB})$. Thus $V(T_{aB})$ is an order-ideal of $V(T)$.

To see that it is a neighborhood ideal take any vertex $y \in U_x \cap V(T_{aB})$ and an aB-accessibility path $(x_0, \ldots, x_n)$ for y, i.e., $x_0 = a$, $x_n = y$, and $x_0, \ldots, x_{n-1} \notin B$. Since T is a tree, $x_{n-1} = x$, and hence for each $z \in U_x$, $(x_0, \ldots, x_{n-1}, z)$ is an aB-accessibility path. Thus $U_x \subset V(T_{aB})$.

To prove that any neighborhood ideal I determines an accessibility graph let B be the set of all maximal elements of I (i.e., those $y \in I$ for which there is no $x \in I$ with $x > y$), and consider $V(T_{aB})$.

Suppose there is an $x \in I - V(T_{aB})$, and let $(x_0, \ldots, x_n)$ be the path in T joining a and x. Then $x_m \in B$ for some $m < n$. Since $x_n = x \in I$ this contradicts the fact that x_m is a maximal element of I. Hence $I \subset V(T_{aB})$.

If $x \in V(T_{aB})$ then the path $W = (x_0, \ldots, x_n)$ which joins a to x is an aB-accessibility path. If $x \notin I$ then there is a largest subscript $m < n$ for which $x_m \in I$, and hence $x_{m+1} \in U_{x_m}$. If $U_{x_m} \cap I = \emptyset$, then $x_m \in B$, but this is impossible since W is an aB-accessibility path. Hence $U_{x_m} \subset I$, whence $x_{m+1} \in I$, contrary to the maximality of m. Thus $V(T_{aB}) \subset I$, completing the proof.

It is worth noting that for rooted trees those subsets B which contain no end-vertex of T are one-sided. For if $x \in B \cap \operatorname{int} T_{aB}$, then $E(x;T) \subset E(T_{aB})$ and hence $U_x \subset V(T_{aB})$. $d(x;T) \geqq 2$ implies that $U_x \neq \emptyset$. Given $y \in U_x$ the path in T which joins a and y is an aB-accessibility path and contains x. Hence $x \notin B$, a contradiction. But this means that B is one-sided.

By Theorem 6.3 the lattice $Q(T, a)$ is isomorphic to the lattice of all neighborhood ideals of $(V(T), \leqq)$, the isomorphism being $B \mapsto V(T_{aB})$. It is well known that the lattice of order ideals of any p.o. set is distributive. Any sublattice of a distributive lattice is distributive; hence from 6.2 we have

6.4 THEOREM. *The weak separation lattice of a rooted tree is distributive.*

7. Chains. In this section we determine all rooted graphs whose weak separation lattice is a chain. It turns out that these graphs can be obtained by a simple construction from paths or rays (= one-way infinite paths). The restriction to rooted graphs entails no loss of generality (see beginning of Section 8).

Before stating the principal result of this section we introduce some terminology and notation.

7.1 *Definition.* Given a rooted graph (X, a) and a vertex $x \in V(X)$ let

$$U_x = \{y \in V(x;X) : \rho(a, y) > \rho(a, x)\},$$

where ρ denotes distance in X. The inequality $\rho(a, y) > \rho(a, x)$ is equivalent to $\rho(a, y) = \rho(a, x) + 1$. We shall say that x is *essential* if and only if $U_x \neq \emptyset$.

Observe that if X is a tree, then U_x is the upper neighborhood of x introduced in the preceding section.

7.2 THEOREM. *Any weak separation lattice $Q(X, a)$ which is a chain is isomorphic to a segment of the ordinal $\omega + 1$. Moreover, (X, a) is a cactus with the following properties:*

(i) *every block of X is either an edge or a triangle;*

(ii) *for every $x \in V(X)$, U_x contains at most one essential vertex; and*

(iii) *for every $x \in V(X)$ there is at most one edge e_x joining two vertices in U_x; if U_x contains an essential vertex u_x, then $e_x \in E(u_x;X)$.*

Conversely, if (X, a) satisfies (i), (ii) *and* (iii), *then $Q(X, a)$ is a chain.*

Conditions (i), (ii) and (iii) permit the following explicit description of (X, a) (actually, (i) is redundant; it is implied by (ii) and (iii)). For each $n \geqq 0$ let $A_n = \{x \in V(X) : \rho(a, x) = n\}$, and $d = \max\{n : A_n \neq \emptyset\}$. If X is non-trivial, then either d is a positive integer or $d = \infty$. Clearly, for every $n < d$, A_n contains an essential vertex, hence by condition (ii) exactly one

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such, say x_n, and $x_0 = a$. Then $W = (x_0, \ldots, x_{d-1})$ or $W = (x_0, x_1, \ldots)$ is a path or a ray (depending on whether d is finite or infinite) which comprises all essential vertices of X, and which is the skeleton of X in the sense that every other vertex of X is adjacent to some vertex in W. That is,

$$V(X) = \bigcup_{0 \leqslant n < d} V(x_n; X).$$

Moreover, for $0 < n < d$ we have

$$V(x_n;X) \cap A_{n-1} = \{x_{n-1}\}$$

by condition (ii), and

$$|V(x_n;X) \cap A_n| \leqq 1$$

by condition (iii) (if $V(x_n;X) \cap A_n = \{y_n\}$, say, then x_n, y_n and x_{n-1} form a triangle). $U_{x_n} = V(x_n;X) \cap A_{n+1}$ is non-empty and may be of any cardinality. By (iii) all but at most two vertices of U_{x_n} are of degree 1. If U_{x_n} contains two vertices of degree $\geqq 2$, then one of them is x_{n+1}, the other is of degree 2, and they are adjacent to each other.

All this says that (X, a) can be obtained by taking a path or a ray W starting at a and by attaching at each vertex of W (except at the other end-point, if W is a path) at most one triangle which must have an edge in common with W, as well as any number of vertices of degree 1 (see Figure 7).

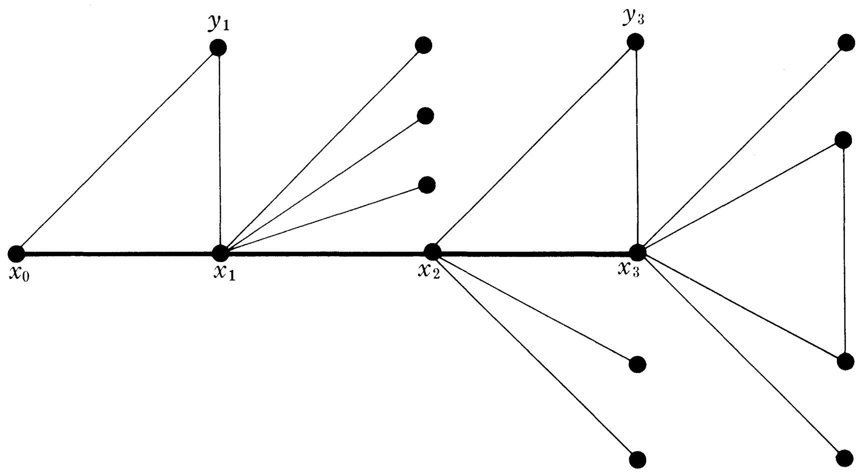

FIGURE 7

Proof (of Theorem 7.2). Sufficiency: Let $0 < n < d$. In terms of the description of (X, a) given above, every vertex in $B_n = U_{x_n} - \{x_{n+1}\}$ is of degree 1 (case 1_n) or B_n contains exactly one vertex of degree 2, say y_n (case 2_n). If

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$d < \infty$, then U_d contains no essential vertex, but it may nevertheless happen that U_d contains exactly two vertices of degree 2 (all others being of degree 1), x_d, y_d, say, and $[x_d, y_d] \in E(X)$ (case 2_d). The corresponding a-primitive sets are $\{x_n\}$ (case 1_n, $n < d$), or $\{x_n\}$ and $\{x_n, y_n\}$ (case 2_n, $n < d$), and $\{x_d, y_d\}$ (case 2_d), as well as $\emptyset$. It is straightforward to verify that the map $\eta : Q(X, a) \to \omega + 1$ defined by $\{x_n\} \mapsto 2n$, $\{x_n, y_n\} \mapsto 2n + 1$, $\emptyset \mapsto \omega + 1$, is a monomorphism. Clearly the image of η is isomorphic to a segment of $\omega + 1$.

Necessity: Suppose $Q(X, a)$ is a chain. Let Y be a block of X which is not an edge. Either $a \in V(Y)$ or $a \notin V(Y)$; in the latter case there is a unique cut-vertex c of X separating a from Y. Put $b = a$ or c as the case may be, and suppose there are two distinct edges $e_i = [x_i, y_i]$ of Y, $i = 0, 1$, not incident with b. Since Y is 2-connected there is a path in Y containing b and e_i but not e_{1-i}. Hence $B_i = \{x_i, y_i\}$, $i = 0, 1$, are two incomparable a-primitive sets, contrary to $Q(X, a)$ being a chain. Hence at most one edge of Y is not incident with b, i.e., Y is a triangle. This proves (i).

Since every block of X is either an edge or a triangle, it is clear that every essential vertex (with the possible exception of a) is a cut-vertex. In any graph, if x is a cut-vertex, then $x \in \Re X_{ax} \subset \{x\}$, i.e., $\{x\} \in Q(X, a)$.

Suppose $x \in V(X)$ is such that U_x contains two distinct essential vertices u_0, u_1 and let $e_i \in E(u_i; X)$, $e_i \neq [x, u_i]$, $i = 0, 1$. Since u_i is a cut-vertex one has $\{u_i\} \in Q(X, a)$ as well as $e_i \in E(X_{au_{1-i}}) - E(X_{au_i})$, $i = 0, 1$. Hence $\{u_0\}$ and $\{u_1\}$ are two incomparable elements of $Q(X, a)$, a contradiction.

Finally, suppose U_x contains an essential vertex u_x, and let $[y, z] \in E(X)$ with $y, z \in U_x$. If $y \neq u_x \neq z$, then $X_{a,\{y,z\}} = X \backslash e$. On the other hand, $e \in E(X_{ou_x})$, but since $X_{au_x} \neq X$, $\{u_x\}$ and $\{y, z\}$ are incomparable a-primitive sets. This completes the proof.

It should be noted that while $Q(X, a)$ is at most countable the order of X may be as high as one wishes. The reason is that at any vertex of the skeleton path (or ray) one may attach any number of vertices of degree 1.

8. The closure operator associated with the weak separation lattice of a rooted graph. It is clear from Proposition 2.11 that for any pair (X, A) the sets $E(X_{AB})$, $B \subset V(X)$, form the open sets (i.e., complements of closed sets) of a closure system. In this section we study briefly the closure operator arising from this system. We begin by showing that in so doing we may restrict ourselves to the case of a rooted graph, i.e., a pair where $A = \{a\}$. This is done by the standard procedure of adjoining a new vertex to the given graph.

Given an arbitrary pair (X, A) take a new vertex $a \notin V(X) \cup E(X)$ and consider the graph X' with

$$V(X') = V(X) \cup \{a\} \quad \text{and} \quad E(X') = E(X) \cup \{[a, x] : x \in A\}.$$

8.1 PROPOSITION. *The a-primitive sets of X' are $\{a\}$ and the A-primitive sets of X. In other words, the lattice $Q(X', a)$ consists of $Q(X, A)$ with the set $\{a\}$ adjoined as a new zero element.*

Proof. The only a-primitive set of X' which contains a is $\{a\}$. Any other a-primitive set B of X' is therefore a subset of $V(X)$. For any such B, if $(x_0, \ldots, x_n)$ is an AB-accessibility path in X, then $(a, x_0, \ldots, x_n)$ is an aB-accessibility path in X', and conversely. This says that

$$V(X'_{aB}) = V(X_{AB}) \cup \{a\} \quad \text{and}$$
$$E(X'_{aB}) = E(X_{AB}) \cup \{[a, x] : x \in A\}.$$

Moreover, $\mathfrak{R}X_{AB} = \mathfrak{R}'X'_{aB}$ (rims in X and X', respectively), since for $x \in V(X)$,

$$E(x; X') = \begin{cases} E(x;X), & \text{if } x \notin A \\ E(x;X) \cup \{[a, x]\}, & \text{if } x \in A. \end{cases}$$

Hence B is a-primitive if and only if it is A-primitive.

For the remainder of this section we shall deal with a given rooted graph (X, a). We first characterize the accessibility graphs in terms of their edge-complements, in other words, the closed sets of the underlying closure system.

8.2 THEOREM. *Given a one-sided set $B \subset V(X)$ with $a \notin B$ put $X \backslash X_{aB} = Y$. Then*

(i) *$a \notin V(Y)$;*
(ii) *$X - Y$ is connected;*
(iii) *Y is a restriction of X; and*
(iv) *Y has no isolated vertex.*

Conversely, if Y is a subgraph of X satisfying (i), (ii) *and* (iii), *then $X \backslash Y = X_{a,V(Y)}$. Moreover, if Y also satisfies* (iv), *then $V(Y)$ is a one-sided set, $\mathfrak{R}Y$ is a-primitive, and $X \backslash Y = X_{a,\mathfrak{R}Y}$. In other words, the non-zero elements of $Q(X, a)$ are precisely the rims of subgraphs satisfying* (i), . . . , (iv).

Note that since Y is a restriction of X the two sets $\mathfrak{R}Y$ and $\mathfrak{B}Y$ coincide.

Proof. We begin by showing that $X - Y = \operatorname{Int} X_{aB}$. Since by 2.6 (iii), $\operatorname{Int} X_{aB}$ is a restriction of X, it suffices to prove that $V(X) - V(Y) = \operatorname{int} X_{aB}$.

Since $a \notin B$, $X_{aB} \neq (a)$ and hence has no isolated vertex. This implies $V(X_{aB}) \cap V(Y) = \mathfrak{R}X_{aB} = \mathfrak{R}Y$, whence $\operatorname{int} X_{aB} \cap V(Y) = \emptyset$, i.e., $\operatorname{int} X_{aB} \subset V(X) - V(Y)$. For the reverse inclusion note that

$$X - X_{aB} \subset X \backslash X_{aB} = Y.$$

Hence $x \notin V(Y)$ implies $x \notin V(X - X_{aB})$, i.e., $x \in V(X_{aB})$. Also, $V(Y) \supset \mathfrak{R}Y = \mathfrak{R}X_{aB}$, whence $x \notin \mathfrak{R}X_{aB}$. Thus $V(X) - V(Y) \subset \operatorname{int} X_{aB}$.

Since B is one-sided any vertex in $\operatorname{int} X_{aB}$ can be joined with a by an aB-accessibility path W disjoint from B and hence disjoint from $\mathfrak{R}X_{aB} \subset B$. This means that $W \subset X - Y$, proving (i) and (ii).

To prove (iii) take $x, y \in V(Y)$ such that $e = [x, y] \in E(X) - E(Y)$. Then $e \in E(X_{aB})$ and hence $x, y \in V(X_{aB})$, i.e., $x, y \in V(X_{aB}) \cap V(Y) =$

$\Re X_{aB}$. But this is a contradiction, since no two vertices of $\Re X_{aB}$ are adjacent in X_{aB} (2.6 (iv)).

(iv) is clear from the definition of Y as an edge-complement.

Now suppose that Y satisfies (i), (ii), (iii) and put $V(Y) = B$. We have to show that $X\backslash Y = X_{aB}$. Let $W = (x_0, \ldots, x_n)$ be an aB-accessibility path. If $W \not\subset X\backslash Y$, then $[x_{i-1}, x_i] \in E(Y) \cap E(W)$ for some i, $1 \leqq i \leqq n$. But then $x_{i-1} \in B \cap V(W)$ and $i - 1 < n$, contrary to W being an aB-accessibility path. Hence $X_{aB} \subset X\backslash Y$. None of the conditions (i), (ii), (iii) is used for this inclusion.

For the reverse inclusion take any $e = [x, y] \in E(X\backslash Y)$. At least one of x, y belongs to $X - Y$, say x; for if $x, y \in V(Y)$ then by (iii), $e \in E(Y)$, a contradiction. By (i) and (ii) there is an ax-path W in $X - Y$. But then $W \cup (e)$ is an aB-accessibility path containing e, i.e., $e \in E(X_{aB})$.

Now assume (iv), and suppose there is an $x \in B \cap \operatorname{int} X_{aB}$. (iv) implies $\emptyset \neq E(x;Y) \subset E(x;X) = E(x;X_{aB}) = E(x;X\backslash Y)$, an obvious contradiction. Thus B is one-sided.

Since we already know that $X\backslash Y = X_{aB}$ we have from (iv) that $\Re Y = \Re X_{aB}$, hence by 2.7, $\Re Y$ is a-primitive and $X\backslash Y = X_{a,\Re Y}$. This completes the proof.

For future reference we state the equation $X\backslash Y = X_{a,V(Y)}$ as a separate corollary.

8.3 COROLLARY. *For any one-sided set B, $X_{aB} = X_{a,V(X\backslash X_{aB})}$.*

8.4 COROLLARY. *Let $B_i \in Q(X, a)$, $Y_i = X\backslash X_{aB_i}$, $i = 1, 2$. Then $B_1 \leqq_a B_2$ if and only if Y_2 is a restriction of Y_1.*

Proof. $B_1 \leqq_a B_2 \Leftrightarrow X_{aB_1} \subset X_{aB_2} \Leftrightarrow Y_1 \supset Y_2 \Leftrightarrow Y_2$ is a restriction of Y_1, the last equivalence following from the fact that Y_1, Y_2 are restrictions of X (8.2 (iii)).

For a given rooted graph (X, a) let $R(X, a)$ be the collection of all connected restrictions Z of X with $a \in V(Z)$. Partially ordered by inclusion, $R(X, a)$ forms a complete lattice in which the supremum and infimum of a family $(Z_i)_{i\in I}$ are given by

$$\sup_{i\in I} Z_i = \text{restriction of } X \text{ to } \bigcup_{i\in I} Z_i,$$

$$\inf_{i\in I} Z_i = Z,$$

where Z is the component of $\bigcap_{i\in I} Z_i$ which contains the vertex a. The graph (a) and X are the zero and one-element, respectively, of $R(X, a)$.

8.5 PROPOSITION. *For any rooted graph (X, a) the mapping $\zeta : B \mapsto \operatorname{Int} X_{aB}$ is an injective complete meet-preserving mapping of $Q(X, a)$ into $R(X, a)$, i.e., in the category $\mathbf{L}_{\mathrm{inf}}$, $Q(X, a)$ is a subobject of $R(X, a)$.*

Proof. By 8.2, ζ is indeed a mapping of $Q(X, a)$ into $R(X, a)$. Its injectivity can immediately be seen from the following factorization:

$$B \mapsto X_{aB} \mapsto X \backslash X_{aB} \mapsto X - (X \backslash X_{aB}) = \operatorname{Int} X_{aB}.$$

Let $B_0 = \inf B_i$ in $Q(X, a)$, $Z = \inf \operatorname{Int} X_{aB_i}$ in $R(X, a)$. Since ζ preserves order,

$$\operatorname{Int} X_{aB_0} = \zeta B_0 \subset \inf \zeta B_i = Z.$$

In order to prove the reverse inclusion note that Z is the union of all paths $W \subset Z$ which start at a. Since $Z \subset \bigcap_{i \in I} \operatorname{Int} X_{aB_i}$ we have that any such path W is disjoint from every B_i (3.5), hence W is disjoint from $B = \bigcup_{i \in I} B_i$. This means $W \subset \operatorname{Int} X_{aB}$ and hence $Z \subset \operatorname{Int} X_{aB} = \operatorname{Int} X_{aB_0}$, completing the proof.

Given any edge $e \in E(X)$ denote by V_e the set consisting of the two vertices incident with e.

For the rooted graph (X, a) let $\mathfrak{P}_a = \mathfrak{P}(E(X - a))$. Define a mapping $\xi : \mathfrak{P}_a \to Q(X, a)$ by

$$F \mapsto \beta_a \bigcup_{e \in F} V_e.$$

Note that $B = \bigcup_{e \in F} V_e$ is a one-sided set. This follows from 2.6 (iv) and the observation that any $x \in B$ is incident with some $e = [x, y] \in F$, so that $y \in B$.

8.6 LEMMA. *ξ maps unions to infima.*

Proof. Let $F_i \in \mathfrak{P}_a$ and put $B_i = \bigcup_{e \in F_i} V_e$, $i \in I$. Then

$$\xi \bigcup_{i \in I} F_i = \beta_a \bigcup_{e \in \bigcup_{i \in I} F_i} V_e = \beta_a \bigcup_{i \in I} B_i = \beta_a \bigcup_{i \in I} \beta_a B_i = \beta_a \bigcup_{i \in I} \xi F_i$$
$$= \inf_{i \in I} \xi F_i,$$

the essential equality (in the middle) being provided by 3.3.

In particular, Lemma 8.6 says that ξ is an order-inverting map.

The following is the main result of this section.

8.7 THEOREM. *The mapping $\alpha : \mathfrak{P}_a \to \mathfrak{P}_a$ defined by $F \mapsto E(X \backslash X_{a,\xi F})$ is a closure operator on $E(X - a)$, and the lattice $\mathfrak{F}_a$ of α-closed subsets of $E(X - a)$ is anti-isomorphic (as a complete lattice) to $Q_0(X, a)$, the sublattice consisting of all non-zero elements of $Q(X, a)$.*

Proof. Take any $F \subset E(X - a)$. Since $X_{a,\xi F} = X_{aB}$, where $B = \bigcup_{e \in F} V_e$, the existence of an edge $e = [x, y] \in F \cap E(X_{a,\xi F}) = F \cap E(X_{aB})$ implies that $x, y \in B \cap V(X_{aB})$, which is impossible by 2.6 (iv). Hence

$$F \subset E(X \backslash X_{a,\xi F}) = F^{\alpha}.$$

That α is isotone follows from the fact that ξ inverts order.

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Still using the above notation we have by 8.3,

$$\begin{aligned}\xi F^{\alpha} &= \xi E(X\backslash X_{a,\xi F}) = \beta_a V(X\backslash X_{a,\xi F})\\ &= \beta_a V(X\backslash X_{aB}) = \beta_a B = \xi F.\end{aligned}$$

This implies that the operator α is idempotent.

Consider the restriction of ξ to $\mathfrak{F}_a$. We shall denote the restriction also by ξ. Our claim is that $\xi : \mathfrak{F}_a \to Q_0(X, a)$ is a complete lattice anti-isomorphism.

ξ maps suprema in $\mathfrak{F}_a$ onto infima in $Q(X, a)$. Let $F_i \in \mathfrak{F}_a$, $i \in I$. Then

$$\xi \sup_{i\in I} F_i = \xi\Big(\bigcup_{i\in I} F_i\Big)^{\alpha} = \xi \bigcup_{i\in I} F_i = \inf_{i\in I} \xi F_i$$

the last equality by 8.6.

ξ maps infima in $\mathfrak{F}_a$ (= intersections) onto suprema in $Q(X, a)$. Given a family of closed sets F_i, $i \in I$, $F = \bigcap_{i\in I} F_i$ is again closed. Hence $F = E(X\backslash X_{a,\xi F})$. On the other hand, by 2.10,

$$\begin{aligned}F = \bigcap_{i\in I} E(X/X_{a,\xi F_i}) &= E(X) - \bigcup_{i\in I} E(X_{a,\xi F_i})\\ &= E(X) - E(X_{a,\sup \xi F_i}) = E(X\backslash X_{a,\sup \xi F_i}).\end{aligned}$$

Thus $E(X\backslash X_{a,\xi F}) = E(X\backslash X_{a,\sup \xi F_i})$, whence $X_{a,\xi F} = X_{a,\sup \xi F_i}$. Since both ξF and $\sup \xi F_i$ are primitive this implies $\xi F = \sup_{i\in I} \xi F_i$.

ξ is one-one. Let F_1, F_2 be two closed sets with $\xi F_1 = \xi F_2$. Then $F_1 = E(X\backslash X_{a,\xi F_1}) = E(X\backslash X_{a,\xi F_2}) = F_2$.

Finally, ξ is onto. Let $B \in Q_0(X, a)$ be given, and put $F = E(Y)$, where $Y = X\backslash X_{aB}$. Then clearly $V(Y) = \bigcup_{e\in F} V_e$. By the first part of 8.2, Y satisfies (i) . . . (iv). Hence by the second part of the same theorem, $X_{aB} = X\backslash Y = X_{a,V(Y)} = X_{a,\xi F}$. This says that F is closed and that $B = \xi F$.

9. Structural properties of $Q_0(X, a)$. As established in 8.7, $Q_0(X, a)$ is a complete lattice, and it is easily seen that its zero-element is $\beta_a V(a;X)$. In this section we shall deal with the atomicity and coatomicity of $Q_0(X, a)$ leaving other structural properties of this lattice for a later study. We continue to use the notation of Section 8. To avoid useless complications we shall assume that $V(a;X)$ is primitive.

Little that is of interest can be said about the atoms. We sum up the situation in the following proposition, leaving the proof to the reader.

9.1 PROPOSITION. *Any atom of $Q_0(X, a)$ is of the form*

$$\beta_a((V(a;X) - \{b\}) \cup (V(b;X) - \{a\})),$$

where $b \in V(a;X)$.

Thus the atoms depend exclusively on the local structure of X near the vertex a, which implies immediately that the lattice $Q_0(X, a)$ which depends on X globally, will not in general be atomic. For example, using the fact that there are at most $d(a;X)$ atoms, one can easily see that for any graph with more than $2^{d(a;X)}$ edges $Q_0(X, a)$ cannot be atomic.

In view of Theorem 8.7 the coatoms $Q_0(X, a)$ are of much greater interest.

If $e = [x, y] \in E(X - a)$ is not a bridge ([**2**, p. 26]), then both x and y are aV_e-accessible, and since $e \notin E(X_{aV_e})$ it follows that $V_e \subset \Re X_{aV_e}$, so that V_e is primitive. Hence if X is a bridgeless graph, then by the proof of 8.7 and by 2.10, any $B \in Q_0(X, a)$ can be written as

$$B = \xi E(X \backslash X_{aB}) = \inf\{V_e : e \in E(X \backslash X_{aB})\}.$$

$e \in E(X \backslash X_{aB})$ is equivalent to $V_e \geqq_a B$ in $Q_0(X, a)$. Thus

$$B = \inf\{V_e : V_e \geqq_a B\};$$

in other words, the sets V_e, $e \in E(X - a)$, generate $Q_0(X, a)$ as a complete meet-semilattice.

We now characterize the coatoms of $Q_0(X, a)$.

9.2 Lemma. *If X is bridgeless, then the coatoms of $Q_0(X, a)$ are precisely those sets V_e, $e \in E(X - a)$, which are not cut sets of X.*

An edge e of X for which V_e is not a cut set will be called *non-separating*; otherwise e is called a *separating* edge.

Proof (of Lemma 9.2). In a bridgeless graph, to say that V_e is not a cut set implies that neither endpoint of e is a cut vertex. Now let $B \in Q_0(X, a)$. If $X \backslash X_{aB}$ has two distinct edges e_1, e_2, then $V_{e_i} \geqq_a B$, $i = 1, 2$, and at least one of these inequalities is strict. Hence if B is a coatom, then $X \backslash X_{aB} = (e)$, i.e., $B = V_e$ for some e not incident with a. Moreover, V_e cannot cut X, for if it does, let Y be any component of $X - V_e$ which does not contain a, and e' any edge of X incident with some vertex of Y. Then e and e' are two distinct edges of $X \backslash X_{aV_e}$, contrary to V_e being a coatom.

Since always $|V_e| = 2$, Lemma 9.1 and the remarks preceding it yield immediately:

9.3 Proposition. *For a 3-connected graph X the lattice $Q_0(X, a)$ is coatomic, the coatoms being precisely the sets V_e, $e \in E(X - a)$.*

In the 2-connected case, if every edge $e \in E(X - a)$ is a separating edge of X, then clearly $Q_0(X, a)$ has no coatoms whatsoever. Such a graph has the property that every vertex, with the possible exception of a, has infinite degree. What happens to the coatomicity of $Q_0(X, a)$ when vertices of finite degree are present, is not known. However, the following result shows that every vertex of finite degree induces at least one coatom.

9.4 Theorem. *Every vertex of finite degree of a 2-connected graph is incident with a non-separating edge.*

In particular, this says that every vertex of a locally finite 2-connected graph is incident with a non-separating edge. Mercier [**3**] has pushed this result further by showing that for every locally rayless 2-connected graph X the lattice $Q_0(X, a)$ is coatomic. X is said to be *locally rayless* if and only if for every ray $R \subset X$ and every $x \in V(X)$ the set $V(x;X) \cap V(R)$ is finite. Trivially, every locally finite graph is locally rayless.

In the proof of Theorem 9.4 we make use of the well-known concept of the *block-cutpoint tree* of a graph ([**2**, p. 36]). We recall the definition.

Let X be a (connected) graph, A its set of cut vertices (not to be confused with the set A used earlier in this paper), $\mathbf{B}$ its set of blocks. One defines the *block-cutpoint graph* T of X by

$$V(T) = A \cup \mathbf{B} \quad \text{and} \quad E(T) = \{[a, B] : a \in A, B \in \mathbf{B}, a \in V(B)\}.$$

T is known to be a tree ([**2**, Theorem 4.4]), and for $a \in A$, $d(a;T)$ is the number of blocks which contain a. Hence

$$d(a;T) \geqq 2 \quad \text{for every } a \in A.$$

Similarly, for $B \in \mathbf{B}$, $d(B;T)$ is the number of cut vertices of X which belong to B.

In statement and proof of the following lemma the word "cut-vertex" means "cut-vertex of X".

9.5 LEMMA. *Let X be a connected graph in which every infinite circuit* (= *2-way infinite path*) *contains at most finitely many cut-vertices. Then X has a block containing at most one cut-vertex.*

Proof. Suppose every block of X contains two cut-vertices. Then in the block-cutpoint tree of X

$$d(B;T) \geqq 2 \quad \text{for every } B \in \mathbf{B}.$$

This means $\min_{y \in V(T)} d(y;T) \geqq 2$. Hence T contains a circuit C, and since T is a tree, C must be infinite. The vertices of C are alternatingly blocks and cut-vertices of X, i.e.,

$$C = (\ldots, a_{-1}, B_{-1}, a_0, B_0, a_1, B_1, \ldots).$$

For every integer i let W_i be an $a_i a_{i+1}$-path in B_i. Then $C' = \bigcup W_i$ is an infinite circuit in X, and C' contains infinitely many cut-vertices, a contradiction.

9.6 *Definition.* Let T be a tree. For $x, y \in V(T)$ let $T(xy)$ be the (unique) path in T from x to y, and for $A \subset V(X)$ put

$$T_A = \bigcup_{x,y \in A} T(xy).$$

A is called a set of *generators* of T if and only if $T_A = T$.

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9.7 PROPOSITION. *Let T be a tree, A a set of generators of T such that $d(a;T) \geqq 2$ for every $a \in A$. Then T contains an infinite circuit*

$$C = (\ldots, c_{-1}, c_0, c_1, \ldots)$$

such that $c_i \in A$ for infinitely many positive as well as infinitely many negative subscripts i. In particular, this says that both A and T are infinite.

Proof. Given any $x \in A$ and $e \in E(x;T)$ let

$$A(x, e) = \{a \in A : e \in E(T(ax))\}.$$

Since A generates T the set $A(x, e)$ is never empty. Now fix an $x_0 \in A$ and an $e_0 \in E(x_0;T)$, and let $x_1 \in A(x_0, e_0)$. Since x_1 is an end-vertex of the path $T(x_0x_1)$, and since $d(x_1;T) \geqq 2$ there is an $e_1 \in E(x_1;T)$ which does not belong to $T(x_0x_1)$. Now take an $x_2 \in A(x_1, e_1)$, etc. It is then clear that

$$R = T(x_0x_1) \cup T(x_1x_2) \cup \ldots$$

is a ray starting at x_0, and that $V(R)$ contains infinitely many members of A (viz. x_i, $i = 0, 1, \ldots$).

Since $d(x_0;T) \geqq 2$ there is an $e_0' \in E(x_0;T) - \{e_0\}$ and one can repeat the previous construction beginning with x_0 and e_0', obtaining a ray R', also starting at x_0 but otherwise disjoint from R, and containing infinitely many members of A. $C = R \cup R'$ is then the required infinite circuit.

Proof (of Theorem 9.4). What we shall show is the following stronger statement: *Given a vertex x_0 of a 2-connected graph X such that all edges in $E(x_0;X)$ separate X. Then $d(x_0;X)$ is infinite, and there is an infinite circuit $C \subset X - x_0$ which contains infinitely many vertices adjacent to x_0.*

Since X is 2-connected, $Y = X - x_0$ is connected. The assumption that every edge in $E(x_0;X)$ separates X means that every vertex in $A_0 = V(x_0;X)$ is a cut-vertex of Y, i.e., $A_0 \subset A$, where A is the set of cut-vertices of Y.

We now show that A_0 generates T, the block-cutpoint tree of Y. Take any edge $[a, B]$ of T, and let $e \in E(a;B)$. By 2-connectedness of X there is a finite circuit Q containing x_0 and e. $W = Q - x_0$ is a path in Y joining two vertices $a_0, a_1 \in A_0$, and $e \in E(W)$. Now consider W^*, the path in T induced by W. Since the endpoints a_0, a_1 of W are cut-vertices of Y it follows that they are also the endpoints of W^*; in other words, $W^* = T(a_0a_1)$. Further, $e \in E(W)$ implies $[a, B] \in E(W^*)$. Thus, $[a, B] \in E(T_{A_0})$, and since $[a, B]$ was an arbitrary edge of T it follows that $T = T_{A_0}$.

In any block-cutpoint tree one has $d(a;T) \geqq 2$ for every $a \in A$. In particular this holds for the vertices in A_0; hence by 9.6, T contains an infinite circuit C which contains infinitely members of A_0. As in the proof of 9.5, C determines a circuit $C' \subset Y$ with $V(C') \cap A_0 = V(C) \cap A_0$.

In general, Theorem 9.4 is false when $d(x_0;X)$ is infinite (see Figure 8). Every edge e_i is separating; nevertheless, the graph is 2-connected.

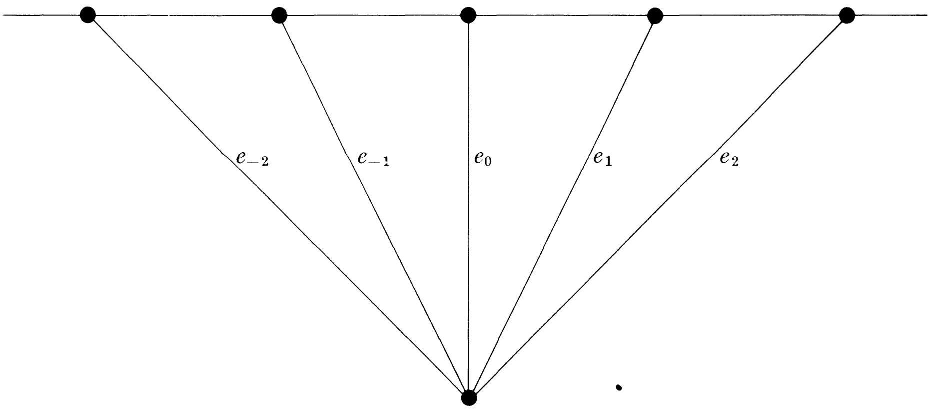

FIGURE 8

10. Comparison with primitivity in the sense of Halin. In this section we establish that the weak separation lattice of a rooted graph is isomorphic to the separation lattice in the sense of Halin of the *line graph* or *derivative* of the given graph. This does not mean, however, that the theory of primitive sets as developed in the earlier sections of this paper coincides with Halin's theory when applied to derivatives. Both Halin's primitive sets and our own are sets of *vertices*, not of edges as would be the case if Halin's definitions were simply transferred to derivatives.

For a subgraph Y of X let $\bar{Y}$ be the restriction of X to $V(Y)$. On the basis of the observation that for any $B \subset V(X)$, $\mathfrak{B}\bar{X}_{AB} \subset B$, Halin [**1**, (1.3)] defines B to be primitive if and only if $B = \mathfrak{B}\bar{X}_{AB}$ (in Halin's notation $\bar{X}_{AB}$ is $X(A \rightarrow B)$). To avoid confusion we shall call such a set (H)-*primitive*.

From the definitions of boundary and rim and 2.6 (ii) one has immediately that

$$\mathfrak{B}\bar{X}_{AB} \subset \mathfrak{R}X_{AB} \subset B.$$

Hence

10.1 PROPOSITION. *Every* (H)-*primitive set is primitive.*

Let us denote by $Q^{(H)}(X, A)$ the lattice of all (H)-primitive sets.

10.2 PROPOSITION. *For a rooted graph* (X, a) *in which every vertex* $\neq a$ *is either a cut-vertex or a vertex of degree* 1, $Q(X, a) = Q^{(H)}(X, a)$. *Conversely, if* $X - a$ *is locally finite, and* $Q(X, a) = Q^{(H)}(X, a)$, *then every vertex* $x \in V(X)$, $x \neq a$, *is either a cut-vertex or* $d(x;X) = 1$.

Proof. Let $x \in B \in Q(X, a)$, $x \neq a$. Since $x \in V(X_{aB})$, there is an aB-accessibility path $W = (x_0, \ldots, x_n)$, i.e., $x_0 = a$, $x_n = x$, and since $x \in \mathfrak{R}X_{aB}$, there is an $e \in E(x;X)$ different from $[x_{n-1}, x_n]$. Hence $d(x;X) \geqq 2$, whence x is a cut-vertex. Let Y be a block of X containing x and edge-disjoint from W. Then $V(x;Y)$ is non-empty and disjoint from $V(X_{aB})$, i.e., $x \in \mathfrak{B}X_{aB}$. Thus $B = \mathfrak{B}X_{aB}$; in other words, $Q(X, a) = Q^{(H)}(X, a)$.

For the converse take any vertex $x \in V(X)$, $x \neq a$, and suppose that x is not a cut-vertex of X and that $d(x;X) \geqq 2$. This implies that the unique block Z of X which contains the terminal edge of every ax-path is not a bridge, and hence is 2-connected. Since $X - a$ is locally finite we have from the proof of 9.4 that x is incident with a non-separating edge e of Z. e is also a non-separating edge of X, whence $X_{aV_e} = X\backslash e$, and consequently $V_e \in Q(X, a)$ but $V_e \notin Q^{(H)}(X, a)$. This completes the proof.

As a corollary we have that $Q(X, a) = Q^{(H)}(X, a)$ for any rooted tree.

In general, if $Q(X, A) \neq Q^{(H)}(X, A)$, the relationship between the two lattices is not very strong. In view of the similarity of 2.10, 2.11, 2.12, with Halin's Satz 1 (and its proof) it is tempting to think that $Q^{(H)}(X, A)$ is a sublattice of $Q(X, A)$. That this is, however, not the case can be seen as follows. In the lattice $Q^{(H)}(X, A)$ the supremum is given by

$$\sup_{i\in I}{}^{(H)}B_i = \mathfrak{B} \bigcup_{i\in I} \bar{X}_{AB_i} = \mathfrak{B} \bigcup_{i\in I} V(X_{AB_i});$$

hence in order to show that $\sup B_i$ and $\sup^{(H)} B_i$ are different it suffices to find an example where $\bigcup_{i\in I} \bar{X}_{AB_i}$ is a proper spanning subgraph of X. Such an example is provided by Figure 1. If one takes $B_1 = \{1, 2, 3', 4'\}$, $B_2 = \{1', 2', 3, 4\}$ (these are (H)-primitive sets), then $\sup^{(H)}\{B_1, B_2\} = \mathfrak{B}X = \emptyset$, whereas $\sup\{B_1, B_2\} = \{3, 4, 3', 4'\}$. On the positive side, *the inclusion* $Q^{(H)}(X, A) \rightarrow Q(X, A)$ *preserves arbitrary infima*. To establish this we show that for any family $B_i \in Q^{(H)}(X, A)$, $i \in I$,

$$\mathfrak{R}X_{AB} = \mathfrak{B}X_{AB},$$

where $B = \bigcup_{i\in I} B_i$. It suffices to prove $\mathfrak{R}X_{AB} \subset \mathfrak{B}X_{AB}$.

Let $x \in \mathfrak{R}X_{AB}$. By 2.6 (ii), $x \in B$, hence $x \in B_j$ for some $j \in I$. Since $B_j \in Q^{(H)}(X, A)$, there is a $y \in V(x;X) - V(X_{AB_j})$. If $x \notin \mathfrak{B}X_{AB}$, then $V(x;X) \subset V(X_{AB})$, hence y is AB-accessible, and therefore also AB_j-accessible. But this means $y \in V(X_{AB_j})$, a contradiction. It follows that

$$\inf{}^{(H)}B_i = \beta_A{}^{(H)}B = \mathfrak{B}X_{AB} = \mathfrak{R}X_{AB} = \beta_A B = \inf B_i.$$

Going the other way, there is a natural map $\beta_A{}^{(H)} : B \mapsto \mathfrak{B}\bar{X}_{AB}$ from $Q(X, A)$ to $Q^{(H)}(X, A)$. This map is clearly *onto* and *preserves arbitrary suprema*. For if $(B_i)_{i\in I}$ is any family of primitive sets, then

$$\beta_A{}^{(H)} \sup B_i = \mathfrak{B}X_{A,\sup B_i} = \mathfrak{B} \bigcup_{i\in I} X_{AB_i} = \mathfrak{B}V\Big(\bigcup_{i\in I} X_{AB_i}\Big)$$

$$= \mathfrak{B} \bigcup_{i\in I} V(X_{AB_i}) = \mathfrak{B} \bigcup_{i\in I} V(X_{A,\beta_A{}^{(H)}B_i}) = \sup{}^{(H)}\beta_A{}^{(H)}B_i.$$

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On the other hand, very simple examples can be given which show that $\beta_A^{(H)}$ does not in general preserve infima; thus $\beta_A^{(H)}$ is not a lattice homomorphism.

In order to state the principal result of this section we recall that for a given graph X the *derivative* ∂X of X is the graph with $V(\partial X) = E(X)$ and $E(\partial X)$ consisting of all (unordered) pairs of adjacent edges of X. Frequently, edge-separation properties of X can be translated into vertex-separation properties of ∂X. The following theorem is an example of this situation. It says essentially that the weak separation lattice of a rooted graph (X, a) is isomorphic to the Halin-lattice of the pair $(\partial X, E(a;X))$. Note in this connection that Halin-lattices are defined in terms of vertex-separation.

10.3 THEOREM. *The mapping* $\partial_a : B \mapsto B' = \bigcup_{b\in B} E(b;X_{aB})$ *is an isomorphism between* $Q_0(X, a)$ *and* $Q^{(H)}(\partial X, E(a;X))$.

Proof. For abbreviation we shall put $\partial X = Y$ and $E(a;X) = A$.

Given $Z \subset Y$ we denote by $\partial^{-1}Z$ the smallest subgraph of X with $E(\partial^{-1}Z) = V(Z)$.

If $W = (x_0, \ldots, x_n) \in \mathfrak{W}_{aB}$, then $\partial W = (e_1, \ldots, e_n)$ is an AB'-accessibility path in Y, where $e_i = [x_{i-1}, x_i]$, $i = 1, \ldots, n$. For if $e_i \in E(b;X_{aB})$ for some $i < n$, then $x_{i-1} = b \in B$ or $x_i = b \in B$, contrary to $W \in \mathfrak{W}_{aB}$. Thus $V(\partial X_{aB}) \subset V(Y_{AB'})$.

Conversely, let $e \in V(Y_{AB'})$ and $S = (e_1, \ldots, e_n)$ an AB'-accessibility path in Y with $e_n = e$. Then $\partial^{-1}S$ is a connected subgraph of X containing a and both ends of e. Let $P = (x_0, \ldots, x_p)$ be a path in $\partial^{-1}S$ with $x_0 = a$ and x_p an end of e. None of the edges $e_1, \ldots, e_{n-1}$ belongs to B', i.e., is incident with a vertex of B, hence $x_0, \ldots, x_{p-1} \notin B$, i.e., $P \in \mathfrak{W}_{aB}$, so that $e \in E(X_{aB})$. This proves

$$(10.3.1) \quad V(Y_{AB'}) = V(\partial X_{aB}) = E(X_{aB}).$$

Take $e \in B'$. Then $e \in E(b;X_{aB})$ for some $b \in \Re X_{aB}$, hence there is an $e' = [b, c] \in E(X \backslash X_{aB})$. From (10.3.1) this gives $e' \notin V(Y_{AB'})$. But $[e, e'] \in E(Y)$, whence $e' \in \mathfrak{B}_Y Y_{AB'}$, i.e., $B' \in Q^{(H)}(Y, A)$. Thus ∂_a is a map $Q_0(X, a) \to Q^{(H)}(Y, A)$.

By 2.6 (iv) any edge of X_{aB} is incident with at most one vertex of B. Hence for $e \in B'$ exactly one end of e, say b_e, belongs to B. This means that $B = \{b_e : e \in B'\}$ which immediately implies that the mapping ∂_a is one-one.

Given $C \in Q^{(H)}(Y, A)$ put $B = \Re\partial^{-1} Y_{AC}$. We claim that

$$(10.3.2) \quad \partial^{-1} Y_{AC} = X_{aB}.$$

Let $e \in E(X_{aB})$ and $W = (x_0, \ldots, x_n) \in \mathfrak{W}_{aB}$ with $x_0 = a$, $[x_{n-1}, x_n] = e$. Then ∂W is an AC-accessibility path in Y. For if $[x_{m-1}, x_m] \in C = \mathfrak{B}_Y Y_{AC}$, where $0 < m < n$, then x_{m-1} or x_m is incident with some edge $e' \notin V(Y_{AC}) = E(\partial^{-1} Y_{AC})$. Accordingly, x_{m-1} or $x_m \in \Re\partial^{-1} Y_{AC} = B$, a contradiction. Since $e \in V(\partial W) \subset V(Y_{AC}) = E(\partial^{-1} Y_{AC})$, this shows that $X_{aB} \subset \partial^{-1} Y_{AC}$. For the reverse inclusion let $e \in E(\partial^{-1}Y_{AC}) = V(Y_{AC})$, and take any AC-accessibility path S in Y of which e is a vertex. Let $P = (x_0, \ldots, x_p)$ be a path in $\partial^{-1}S$ with $x_0 = a$ and $e = [x_{p-1}, x_p]$. $V(\partial P)$ being contained in $V(S)$, ∂P is an AC-accessibility path in Y. We claim that $P \in \mathfrak{W}_{aB}$. For if not, take any subscript $m < n$ such that $e_m = [x_{m-1}, x_m] \in B$. This means x_m is incident with an edge e' not in $E(\partial^{-1}Y_{AC})$. Having x_m in common, e_m and e' are adjacent in Y. Since $e_m \in V(Y_{AC})$ this implies that $e_m \in \mathfrak{B}_Y Y_{AC} = C$, a contradiction to ∂P being an AC-accessibility path.

From (10.3.2) we have immediately that $B = \mathfrak{N}\partial^{-1}Y_{AC} = \mathfrak{N}X_{aB}$, which means $B \in Q(X, a)$. Since B is always $\neq \{a\}$, except in the case where C is empty, we have that $B \in Q_0(X, a)$.

To show now that ∂_a is onto, take C and B as in the preceding paragraph. By (10.3.2) and (10.3.1),

$$V(Y_{AC}) = E(\partial^{-1}Y_{AC}) = E(X_{aB}) = V(Y_{AB'}),$$

hence $C = \mathfrak{B}_Y Y_{AC} = \mathfrak{B}_Y Y_{AB'} = B'$ since we have already established that $B \in Q(X, a)$.

For a family $B_i \in Q(X, a)$, $i \in I$, we have from (10.3.1) that

$$V(Y_{A, \sup^{(H)} B_i'}) = V\Big(\bigcup_{i \in I} Y_{AB_i'}\Big) = E\Big(\bigcup_{i \in I} X_{aB_i}\Big)$$
$$= E(X_{a, \sup B_i}) = V(Y_{A, (\sup B_i)'});$$

in other words, $\partial_a \sup B_i = \sup^{(H)} \partial_a B_i$.

Since ∂_a is a bijection and preserves suprema, it also preserves infima.

Theorem 10.3 can be strengthened in the following sense: not only is every weak separation lattice *isomorphic* to a Halin-lattice but actually *coincides* with one such.

10.4 THEOREM. *Given a rooted graph (X, a) there exists a rooted graph (X', a) such that X is a restriction of X' and $Q(X, a) = Q^{(H)}(X', a)$.*

Proof. We construct X' as follows. Let Q be a set disjoint from $V(X) \cup E(X)$ which is in one-one correspondence with $E(X)$. For $e \in E(X)$ let q_e be the element of Q corresponding to e. Put

$$V(X') = V(X) \cup Q,$$
$$E(X') = E(X) \cup \{[q_e, x] : e \in E(x;X), x \in V(X)\}.$$

In other words, every edge $e = [x, y]$ of X is being replaced by a triangle T_e whose vertices are x, y, q_e. Note that X is a restriction of X', and that each q_e is a vertex of degree 2. An example of this construction is given in Figure 9.

For any $Y \subset X$ we shall denote by Y' the graph $\bigcup \{T_e : e \in E(Y)\}$. Our claim is that $Q(X, a) = Q^{(H)}(X', a)$.

To begin with we show that for every $B \subset V(X)$,

(10.4.1) $\quad X'_{aB} = (X_{aB})' \supset X_{aB}$.

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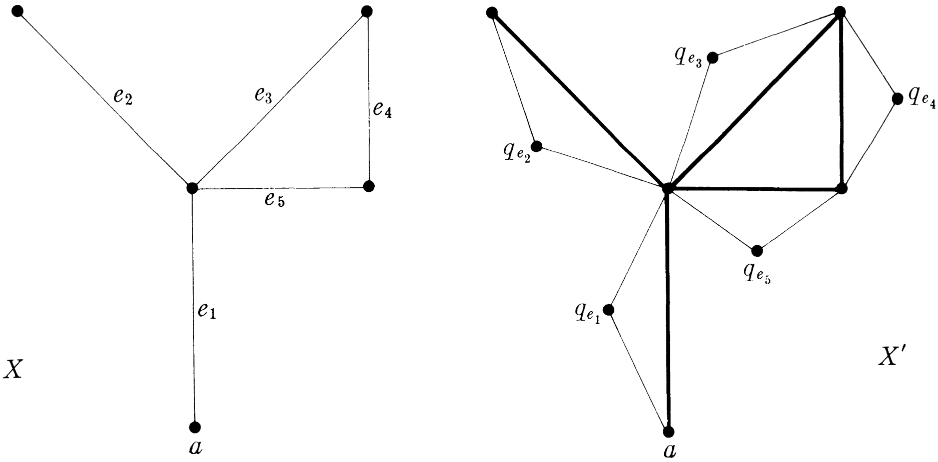

FIGURE 9
(The heavy lines are the edges of X)

Trivially, any $W = (x_0, \ldots, x_n) \in \mathfrak{W}_{aB}$ also belongs to $\mathfrak{W}'_{aB}$. Let $e_i = [x_{i-1}, x_i]$, $0 < i < n$. Then $W_i = (x_0, \ldots, x_{i-1}, q_{e_i}, x_i, \ldots, x_n)$ likewise belongs to $\mathfrak{W}'_{aB}$. Thus $T_e \subset X'_{aB}$ for every $e \in E(X_{aB})$, i.e., $(X_{aB})' \subset X'_{aB}$.

Let $S = (s_0, \ldots, s_m)$ be a path in X' starting at a. Since $s_0 = a$, we have $s_0 \in V(X)$. If for some i, $0 < i < m$, the vertex s_i is of the form q_e for some e, then $s_{i-1}, s_{i+1} \in V(X)$, and $e = [s_{i-1}, s_{i+1}]$. Similarly, if $s_m = q_e$, then $s_{m-1} \in V(X)$, and $e = [s_{m-1}, s']$ for some $s' \in V(X)$. Thus we can reduce S to a path $S_{\text{red}} = (x_0, \ldots, x_n)$, where $x_0 = a$, $x_n = s_m$, and $x_0, \ldots, x_{n-1} \in V(X)$ as follows. Let $0 = i_0 < i_1 < \ldots < i_n \leqq m$ be those subscripts with $s_{i_j} \in V(X)$ and $s_k \notin V(X)$ for every k, $i_j < k < i_{j+1}$. It is clear that $i_{j+1} = i_j + 1$ or $i_j + 2$. Put $s_{i_j} = x_j$, $j = 0, \ldots, n$. If S is an aB-accessibility path in X' so also is S_{red}. Moreover, S_{red} is an aB-accessibility path in X, i.e., $S_{\text{red}} \subset X_{aB}$. Thus, whenever a vertex of S is of the form q_e, then $e \in E(X_{aB})$, and since the two edges of X' incident with q_e belong to T_e this means $X'_{aB} \subset (X_{aB})'$.

For any $B \subset V(X)$,

$$(10.4.2) \quad \mathfrak{B}'X_{aB}' = \mathfrak{R}X_{aB}.$$

Let $x \in \beta_a B$. Then there is an $e \in E(x;X\backslash X_{aB})$; $e = [x, y]$, say. Case (i): $y \notin V(X_{aB})$. Then q_e is not aB-accessible in X', since every aq_e-path in X' contains either x or y, and hence a vertex of B. Case (ii): $y \in V(X_{aB})$. Then $x, y \in B$, and since every aq_e-path in X' must contain either x or y, we have again $q_e \notin V(X_{aB}')$. In either case $x \in \mathfrak{B}'X_{aB}'$. Conversely, if $x \in \beta_a^{(H)}B$, then $x \in B$, and there is an edge $[x, w] \in E(X_{aB}')$, where either $w \in V(X)$ or $w = q_e$ with $e = [x, y] \in E(X)$. In the first case, w is either aB-accessible in X', and then $w \in B$, so that $[x, w] \notin E(X_{aB})$; $w \notin V(X_{aB})$ by (10.4.1). Again $[x, w] \notin E(X_{aB})$. Similarly, if $w = q_e$, then we have the same two alternatives for y which we just had for w, so that $[x, y] \notin E(X_{aB})$. Thus $x \in \mathfrak{R}X_{aB}$.

(10.4.2) implies in particular that $Q(X, a) \subset Q^{(H)}(X', a)$.

Next we show that the inclusion map $Q(X, a) \rightarrow Q^{(H)}(X', a)$ is a complete homomorphism. Let $B_i \in Q(X, a)$, $i \in I$. Then

$$\sup{}^{(H)}B_i = \mathfrak{B}' \bigcup_{i \in I} X_{aB_i}{}' = \mathfrak{B}'\Big(\bigcup_{i \in I} X_{aB_i}\Big)'$$

$$= \mathfrak{B}'X_{a,\sup B_i}{}' = \mathfrak{R}X_{a,\sup B_i} = \sup B_i,$$

the second equality following from (10.4.1), the next to last one from (10 4.2). Also, from (10.4.2),

$$\inf{}^{(H)}B_i = \beta_a{}^{(H)}B = \mathfrak{B}'X_{aB}{}' = \mathfrak{R}X_{aB} = \beta_a B = \inf B_i,$$

where $B = \bigcup_{i \in I} B_i$.

Every $C \in Q^{(H)}(X', a)$ is contained in $V(X)$. Suppose $q_e \in C$ for some $e = [x, y] \in E(X)$. Then q_e is aC-accessible in X' and adjacent to an aC-inaccessible vertex. This means that exactly one of x, y belongs to $X_{aC}{}'$, say $x \in V(X_{aC}{}')$, $y \notin V(X_{aC}{}')$. This means that every aC-accessibility path in X' joining a and q_e has the form $S = (a, \ldots, x, q_e)$. Hence $x \notin C$. Thus $S \cup (e)$ is an aC-accessibility path in X' joining a and y, i.e., $y \in V(X_{aC}{}')$, a contradiction.

It remains to show that $C \in Q^{(H)}(X', a)$ implies $C \in Q(X, a)$. But this is now obvious, since $C = \mathfrak{B}'X_{aC}{}' = \mathfrak{R}X_{aC} \subset C$ by (10.4.2).

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