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Torsion properties of modified diagonal classes on triple products of modular curves

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Abstract. Consider three normalized cuspidal eigenforms of weight 2 and prime level p. Under the assumption that the global root number of the associated triple product L-function is +1, we prove that the complex Abel–Jacobi image of the modified diagonal cycle of Gross–Kudla–Schoen on the triple product of the modular curve $X_0(p)$ is torsion in the corresponding Hecke isotypic component of the Griffiths intermediate Jacobian. The same result holds with the complex Abel–Jacobi map replaced by its étale counterpart. As an application, we deduce torsion properties of Chow–Heegner points associated with modified diagonal cycles on elliptic curves of prime conductor with split multiplicative reduction. The approach also works in the case of composite square-free level.

1 Introduction

The study of diagonal cycles on triple products of Shimura curves has its origins in the work of Gross, Kudla, and Schoen [11, 12]. They introduced a null-homologous modification of the diagonal embedding of the curve in its triple product, referred to as the modified diagonal cycle, or more commonly today as the Gross–Kudla–Schoen cycle. Given three cuspidal newforms of weight 2 and square-free level N such that the sign of the functional equation of the associated triple product L-function is −1, Gross and Kudla [11] conjectured that the central value at $s = 2$ of the derivative of this L-function is given by a complex period times the Beilinson–Bloch height of the corresponding Hecke isotypic component of the modified diagonal cycle on the triple product of an indefinite Shimura curve determined by the local triple product root numbers. A proof of this conjecture was announced in work of Yuan et al. [28], but has yet to be published. The Shimura curve in question is the modular curve $X_0(N)$ precisely when the local triple product root numbers are +1 at all finite places.

1.1 Main results

In this article, we exhibit certain torsion properties of modified diagonal classes[1] on the triple product of the modular curve $X := X_0(p)$ defined over $\mathbb{Q}$ and of prime

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[1]This terminology refers to images of modified diagonal cycles under complex (or étale) Abel–Jacobi maps.

level p. The results hold more generally for composite square-free level N (see Section 1.4). Since the prime level case already contains all relevant ingredients of the proof, we have chosen to focus on this case.

The modified diagonal cycle depends on a base point e in $X(\bar{\mathbb{Q}})$. It will be denoted by $\Delta_{\mathrm{GKS}}(e)$ and viewed as an element of the Chow group $\mathrm{CH}^2(X^3)_0(\bar{\mathbb{Q}})$ of null-homologous codimension 2 algebraic cycles on X^3 over $\bar{\mathbb{Q}}$ modulo rational equivalence (see Section 1.7 for our slightly unconventional definition of Chow groups as functors). Let f_1, f_2, and f_3 be three normalized cuspidal eigenforms of weight 2 and level $\Gamma_0(p)$, and denote by $F := f_1 \otimes f_2 \otimes f_3$ their triple product. We place ourselves in the setting where the global root number $W(F)$ of the triple product L-function $L(F, s)$ associated with F is +1. This assumption forces $L(F, s)$ to vanish to even order at its centre $s = 2$. Comparing with the more classical situation of Heegner points studied in the seminal work of Gross and Zagier [13], it seems reasonable to expect that the "F-isotypic Hecke component" $\Delta^F_{\mathrm{GKS}}(e)$ (see Remark 3.2) of the modified diagonal cycle, with $e \in X(\mathbb{Q})$, is trivial in the Chow group $\mathrm{CH}^2(X^3)_0(\mathbb{Q}) \otimes_{\mathbb{Z}} K_F$ of cycles defined over $\mathbb{Q}$ with coefficients in the Hecke field of F, in line with the predictions of the Beilinson–Bloch conjectures [4]. While it appears difficult to prove a torsion statement directly in the Chow group, we can prove the corresponding result for the image of $\Delta^F_{\mathrm{GKS}}(e)$ under the complex Abel–Jacobi map

$$\mathrm{AJ}_{X^3} : \mathrm{CH}^2(X^3_{\mathbb{C}})_0(\mathbb{C}) \longrightarrow J^2(X^3_{\mathbb{C}}), \tag{1.1}$$

whose target is the Griffiths intermediate Jacobian of $X^3_{\mathbb{C}}$ viewed as a complex manifold.

Theorem 1.1 *Let f_1, f_2, and f_3 be three normalized eigenforms of weight 2 and level $\Gamma_0(p)$, denote by $F = f_1 \otimes f_2 \otimes f_3$ their triple product, and suppose that the global root number of $L(F, s)$ is +1. Then $\mathrm{AJ}_{X^3}(\Delta^F_{\mathrm{GKS}}(e))$ is trivial in $J^2(X^3_{\mathbb{C}}) \otimes_{\mathbb{Z}} K_F$, for all e in $X(\mathbb{Q})$.*

The kernel of the complex Abel–Jacobi map (restricted to cycles defined over $\bar{\mathbb{Q}}$) is conjectured to be torsion [16, Conjecture 9.12]. Conditional on this conjecture, Theorem 1.1 implies that $\Delta^F_{\mathrm{GKS}}(e)$ is trivial in $\mathrm{CH}^2(X^3)_0(\mathbb{Q}) \otimes_{\mathbb{Z}} K_F$. The same statement as in Theorem 1.1 holds with the complex Abel–Jacobi map replaced by its ℓ-adic étale counterpart [4]

$$\mathrm{AJ}^{\mathrm{et}}_{X^3} : \mathrm{CH}^2(X^3)_0(\mathbb{Q}) \longrightarrow H^1(\mathbb{Q}, H^3_{\mathrm{et}}(X^3_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(2))),$$

with ℓ a rational prime (see Remark 4.4).

In the special case $p = 37$, using numerical results due to Stein [5, Appendix], we deduce the following, where ξ_∞ denotes the cusp of X at infinity.

Theorem 1.2 *Let f and g be the normalized eigenforms of weight 2 and level $\Gamma_0(37)$ corresponding to the elliptic curves with Cremona labels 37b and 37a, and let $F := g \otimes g \otimes f$. Then $\mathrm{AJ}_{X_0(37)^3}(8\Delta^F_{\mathrm{GKS}}(\xi_\infty))$ is a nontrivial 6-torsion element of $J^2(X_0(37)^3_{\mathbb{C}})$.*

1.2 Application to Chow–Heegner points

Chow–Heegner points were introduced by Bertolini et al. [2] as a generalization of the construction of Heegner points. The idea is to produce rational points on elliptic curves by pushing forward algebraic cycles on higher dimensional varieties using suitable correspondences, or generalized modular parametrizations, as they are referred to in [2].

Let f be a normalized cuspidal eigenform of weight 2 and level $\Gamma_0(p)$ with rational Fourier coefficients. Denote by E_f the optimal elliptic curve over $\mathbb{Q}$ of conductor p associated with f by Eichler and Shimura [27]. Using an auxiliary normalized cuspidal eigenform g of weight 2 and level $\Gamma_0(p)$, it is possible to construct a correspondence $\Pi_g^f \in \mathrm{CH}^2(X^3 \times E_f)(\mathbb{Q})$, which gives rise via push-forward to a generalized modular parametrization, that is, a natural transformation

$$\Pi_{g,*}^f : \mathrm{CH}^2(X^3)_0 \longrightarrow \mathrm{CH}^1(E_f)_0 = E_f.$$

The Chow–Heegner point associated with the modified diagonal cycle based at a point $e \in X(\bar{\mathbb{Q}})$ is then defined as

$$P_g^f(e) := \Pi_{g,*}^f(\Delta_{\mathrm{GKS}}(e)) \in E_f(\bar{\mathbb{Q}}).$$

Darmon et al. [7] have studied such points, in the broader context of Shimura curves over totally real fields, notably by computing their images under the complex Abel–Jacobi map in terms of iterated integrals. Methods have been developed by Darmon et al. [5] to numerically calculate such points in the case of modular curves.

Let $F := g \otimes g \otimes f$. We exhibit a correspondence mapping $\Delta_{\mathrm{GKS}}^F(e)$ to $P_g^f(e)$. When the global root number $W(F)$ is -1, Darmon et al. [7] have studied the nontorsion properties of $P_g^f(\xi_\infty)$, building on [28]. In the complementary situation when $W(F) = +1$, we use Theorem 1.1 and functoriality of Abel–Jacobi maps with respect to correspondences to deduce the following:

Theorem 1.3 *Let f and g be as above, and let $F = g \otimes g \otimes f$. If $W(F) = +1$, then the Chow–Heegner point $P_g^f(e)$ is torsion in $E_f(\mathbb{Q})$, for all $e \in X(\mathbb{Q})$.*

Theorem 1.3 with $e = \xi_\infty$ recovers a result of Daub [8, Theorem 3.3.8] by a different method in the case of prime level. Similar arguments should work for f not rational.

1.3 Strategy of the proof

The key ingredient in the proof of Theorem 1.1 is the Atkin–Lehner involution w_p of X. The global root number of $W(F)$ is the product of the global root numbers of f_1, f_2, and f_3, which are each equal to the negative of their w_p-eigenvalue. As a consequence, the assumption that $W(F)$ equals +1 translates into information about the action of $w_p \times w_p \times w_p$ on F, and consequently on $\mathrm{AJ}_{X^3}(\Delta_{\mathrm{GKS}}^F(e))$, as the latter lies in the F-isotypic Hecke component of the intermediate Jacobian by functoriality of the Abel–Jacobi map with respect to correspondences. The work of Mazur [22]

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provides necessary information about the rational points $X(\mathbb{Q})$ and the action of w_p on them.

1.4 Composite square-free level

The arguments of this paper carry over to the more general setting where the level N is composite, but square-free. This is the situation initially considered in the work of Gross and Kudla [11]. It becomes necessary to replace eigenforms by newforms.

Let f_1, f_2, f_3 be three normalized newforms of weight 2 and level $\Gamma_0(N)$, and let $F := f_1 \otimes f_2 \otimes f_3$. The level being square-free guarantees that the local root numbers $W_p(F)$ for $p \mid N$ are the products of the local root numbers at p of f_1, f_2, and f_3, which are each the negative of their w_p-eigenvalue. The Atkin–Lehner correspondences w_p, $p \mid N$, commute with the good Hecke correspondences T_n (i.e., with $(n, N) = 1$), and this is sufficient for our purposes (see Remark 2.1). Assume that there exists $p \mid N$ for which $W_p(F) = -1$. Using multiplicity one for newforms, this assumption can be parlayed into information about the torsion properties of the images of modified diagonal cycles under Abel–Jacobi maps, as long as one has sufficient understanding of the action of the Atkin–Lehner involution w_p on the rational points of $X_0(N)$. The only rational points on composite level modular curves $X_0(N)$ of genus ≥ 2 are the rational cusps [18]. It is known that the subgroup of the Jacobian $J_0(N)$ generated by the cusps is torsion by the Manin–Drinfeld theorem [20]. It follows that Theorem 1.1 remains true for normalized newforms f_1, f_2, f_3 of composite square-free level under the assumption $W_p(F) = -1$ for some $p \mid N$.

The proof of Theorem 1.3 adapts verbatim to the setting of composite square-free level, provided that the eigenforms are newforms and $W_p(F) = -1$ for some $p \mid N$. This recovers [8, Theorem 3.3.8] by a different approach.

Examining Stein's Table 2 in [5, Appendix], we obtain results similar to Theorem 1.2, e.g., in the following cases:

- $N = 57$: f corresponds to the elliptic curve with Cremona label 57c, and g corresponds to the curves with labels 57a or 57b.
- $N = 58$: f corresponds to the elliptic curve with Cremona label 58b, and g corresponds to the curve with label 58a.

1.5 Related work

The approach taken in this paper is explicit and elementary, exploiting the connection between triple product root numbers and eigenvalues of Atkin–Lehner involutions. A more powerful approach is considered in the work of Yuan et al. [28], using Prasad's dichotomy for the existence of trilinear forms on automorphic representations. Forthcoming work of Qiu and Zhang [24] further develops this approach and gives applications.

1.6 Outline

Background on cusp forms of weight 2 is recalled in Section 2. Section 3 recalls facts about the triple product L-function and states the Beilinson–Bloch conjecture in this

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setting. Section 4 constitutes the proof of Theorem 1.1. The application to Chow–Heegner points is given in Section 5. Theorem 1.2 is proved in Section 6.

1.7 Notational conventions

Fix a complex embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$, as well as p-adic embeddings $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ for each rational prime p. In this way, all finite extensions of $\mathbb{Q}$ are viewed simultaneously as subfields of $\mathbb{C}$ and $\mathbb{C}_p$. For a field extension F of $\mathbb{Q}$, the subscript F on a group (resp. $\mathbb{Q}$-algebra) will denote the tensor product with F over $\mathbb{Z}$ (resp. $\mathbb{Q}$). For any field K, we fix an algebraic closure $\bar{K}$. By a variety X over K, we shall mean an integral separated scheme of finite type over K. A subvariety is an integral separated closed subscheme. If F is a field extension of K, X_F will denote the base change of X to $\mathrm{Spec}(F)$. An algebraic cycle of codimension r on X is a finite $\mathbb{Z}$-linear combination of subvarieties of $X_{\bar{K}}$ of codimension r. The Chow group of codimension r algebraic cycles modulo $\bar{K}$-rational equivalence will be denoted $\mathrm{CH}^r(X)$. It will be viewed as a functor from the category of field extension of K contained in $\bar{K}$ to the category of abelian groups given by the rule

$$\bar{K}/F/K \mapsto \mathrm{CH}^r(X)(F) := \{[Z] \in \mathrm{CH}^r(X) \colon \sigma(Z) \sim_{\mathrm{rat}} Z, \forall \sigma \in \mathrm{Aut}_F(\bar{K})\}.$$

This convention is borrowed from [7] and differs from the more classical notation of [10]. Given two varieties X and Y over K, we write $\mathrm{Corr}^r(X,Y) := \mathrm{CH}^{\dim X + r}(X \times Y)$.

2 Cusp forms

Let $p > 3$ be a rational prime. Let $Y := Y_0(p)$ be the modular curve over $\mathbb{Q}$ for the congruence subgroup $\Gamma_0(p) \subset \mathrm{SL}_2(\mathbb{Z})$ consisting of matrices which are upper-triangular modulo p. It admits a canonical proper desingularization $Y_0(p) \hookrightarrow X_0(p)$, obtained over the complex numbers by adjoining the cusps. The curve $X := X_0(p)$ is a geometrically connected, smooth, and proper curve over $\mathbb{Q}$. It is the coarse moduli scheme representing pairs (E, H) consisting of a generalized elliptic curve E over a $\mathbb{Q}$-scheme S, together with a cyclic subgroup scheme H of order p. It admits a uniformization by the extended Poincaré upper half-plane

$$\mathcal{H}^* \longrightarrow X(\mathbb{C}), \qquad \tau \mapsto (\mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}, \langle 1/p + \mathbb{Z} \oplus \tau\mathbb{Z} \rangle), \tag{2.1}$$

which identifies $X(\mathbb{C})$ with the quotient $\Gamma_0(p)\backslash\mathcal{H}^*$. There are two cusps ξ_∞ and ξ_0 on X, which correspond via (2.1) to the points $i\infty$ and 0 of $\mathcal{H}^*$. The genus g_X of X is given by the formula

$$g_X = \begin{cases} \lfloor \frac{p+1}{12} \rfloor - 1, & \text{if } p \equiv 1 (\mathrm{mod}\ 12), \\ \lfloor \frac{p+1}{12} \rfloor, & \text{otherwise}. \end{cases} \tag{2.2}$$

The space $S_2(\Gamma_0(N))$ of weight 2 cusp forms of level $\Gamma_0(p)$ is naturally identified with the space of global sections of the sheaf of regular differential 1-forms on X via the isomorphism

$$S_2(\Gamma_0(p)) \xrightarrow{\sim} H^0(X_{\mathbb{C}}, \Omega^1_X), \qquad f \mapsto \omega_f := 2\pi i f(\tau) d\tau. \tag{2.3}$$

In particular, the dimension of $S_2(\Gamma_0(p))$ is equal to g_X.

2.1 Hecke operators

The curve X is equipped with the usual collection of Hecke correspondences, which act on cohomology and give rise to operators on $S_2(\Gamma_0(p))$ via (2.3). These correspondences and their induced operators are denoted by U_p and T_n, for integers $n \geq 1$ coprime to p. Their precise definition can be found in [1, (3.1)].

The curve X also comes equipped with the Atkin–Lehner involution w_p. It is defined, following the moduli description, by mapping a p-isogeny $\phi : E \longrightarrow E'$ of elliptic curves to its dual isogeny $\phi^\vee : E' \longrightarrow E$. In terms of covering spaces, using (2.1), it is given by $\tau \mapsto -\frac{1}{p\tau}$, where $\tau \in \mathcal{H}$. This involution is defined over $\mathbb{Q}$ and thus maps $\mathbb{Q}$-rational points of X to $\mathbb{Q}$-rational points. It induces, via (2.3), an operator on $S_2(\Gamma_0(p))$, which we also denote by w_p.

The operators T_m, with $(m, p) = 1$, on $S_2(\Gamma_0(p))$ commute with the operators T_n, U_p and w_p [1, Lemma 17]. Let $\mathbb{T} := \mathbb{T}(p)$ denote the $\mathbb{Q}$-algebra generated by the operators T_n, with $(n, p) = 1$. The space of cusp forms $S_2(\Gamma_0(p))$ admits a basis of eigenforms for $\mathbb{T}$ [1, Theorem 2].

Let $f = \sum_{n \geq 1} a_n(f) q^n \in S_2(\Gamma_0(p))$ be a normalized eigenform, in the sense that $a_1(f) = 1$. Because the level is prime, there are no oldforms. As a consequence, we have the equality of operators $U_p = -w_p$. In particular, the operators U_p and w_p commute. Note that this is only the case for general composite level when restricting to newforms [1, Lemma 17]. It follows that $w_p(f) = -a_p(f) f$. In particular, we have $a_p(f) \in \{\pm 1\}$.

The normalized eigenform f determines a surjective homomorphism $\lambda_f : \mathbb{T} \longrightarrow K_f$ of algebras by sending T_n to $a_n(f)$. Here, K_f is the totally real finite extension of $\mathbb{Q}$ generated by the Fourier coefficients $a_n(f)$ of f.

Let $S_2(\Gamma_0(p))_f$ denote the f-isotypic component of $S_2(\Gamma_0(p))$ consisting of cusp forms f' in $S_2(\Gamma_0(p))$ such that $T(f') = \lambda_f(T) f'$, for all $T \in \mathbb{T}$. By the theorem of multiplicity one [1, Lemma 20 and 21] of Atkin and Lehner for newforms, the space $S_2(\Gamma_0(p))_f$ is one-dimensional over $\mathbb{C}$. We have the spectral decomposition

$$S_2(\Gamma_0(p)) = \bigoplus_h S_2(\Gamma_0(p))_h,$$

where the sum is taken over all normalized eigenforms $h \in S_2(\Gamma_0(p))$. Since the dual space $S_2(\Gamma_0(p))^\vee$ is a free $\mathbb{T}_{\mathbb{C}}$-module of rank one by multiplicity one, we similarly obtain a decomposition

$$\mathbb{T}_{\mathbb{C}} = \bigoplus_h \mathbb{T}_{\mathbb{C},h},$$

where $\mathbb{T}_{\mathbb{C},h}$ denotes the algebra of Hecke operators T_n, with $(n, p) = 1$, acting on $S_2(\Gamma_0(p))_h$, which is again a $\mathbb{C}$-vector space of dimension one.

Let $[f]$ denote the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ orbit of f. Form the complex vector space $\bigoplus_{g \in [f]} S_2(\Gamma_0(p))_g$ of dimension $d_f := [K_f : \mathbb{Q}]$, and consider the $\mathbb{Q}$-subspace

$S_2(\Gamma_0(p))_{[f]}$ of forms with rational coefficients. This $\mathbb{Q}$-vector space is stable under the action of $\mathbb{T}_{\mathbb{Q}}$, and we let $\mathbb{T}_{\mathbb{Q},[f]}$ denote the $\mathbb{Q}$-algebra generated by the Hecke operators acting on $S_2(\Gamma_0(p))_{[f]}$. We then have the decomposition

$$\mathbb{T} = \bigoplus_{[h]} \mathbb{T}_{\mathbb{Q},[h]} \simeq \bigoplus_{[h]} K_h, \tag{2.4}$$

where the sum is taken over all $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ conjugacy classes of normalized eigenforms in $S_2(\Gamma_0(p))$.

Let $\mathrm{End}_{\mathbb{Q}}(J)$ denote the ring of endomorphisms defined over $\mathbb{Q}$ of the Jacobian variety $J := \mathrm{Pic}^0_{X/\mathbb{Q}}$ of X, and let $\mathrm{End}^0_{\mathbb{Q}}(J) := \mathrm{End}_{\mathbb{Q}}(J) \otimes \mathbb{Q}$. As p is prime, we have $\mathrm{End}^0_{\mathbb{Q}}(J) = \mathbb{T}$ [25, Corollary 3.3]. To summarize, we have the decomposition

$$\mathrm{End}^0_{\mathbb{Q}}(J) = \mathbb{T} \simeq \bigoplus_{[h]} K_h. \tag{2.5}$$

It will be useful to remark that there is a natural ring isomorphism

$$\mathrm{End}^0_{\mathbb{Q}}(J) \simeq (\mathrm{CH}^1(X^2)(\mathbb{Q})_{\mathbb{Q}})/(\mathrm{pr}_1^* \mathrm{CH}^1(X)(\mathbb{Q})_{\mathbb{Q}} + \mathrm{pr}_2^* \mathrm{CH}^1(X)(\mathbb{Q})_{\mathbb{Q}}), \tag{2.6}$$

by [3, Theorem 11.5.1]. See Section 1.7 for our conventions about Chow groups.

Remark 2.1 The exposition is simplified by the assumption that the level is prime, which implies that there are no oldforms. We refer to [6, Section 1.6] for the decomposition (2.4) in the more general setting of composite level N. In this case, the algebra $\mathrm{End}^0_{\mathbb{Q}}(J)$ is a product of matrix algebras. It contains $\mathbb{T}$ as its center and the full Hecke algebra as a maximal commutative subalgebra. Moreover, $\mathrm{End}^0_{\mathbb{Q}}(J)$ is generated as a $\mathbb{Q}$-algebra by $\mathbb{T}$, together with certain degeneracy operators [17, Theorem 1].

2.2 Hecke projectors

Let $f = \sum_{n\geq 1} a_n(f) q^n \in S_2(\Gamma_0(p))$ be a normalized eigenform. Let $V := S_2(\Gamma_0(p))^\vee$ be the $\mathbb{C}$-dual of $S_2(\Gamma_0(p))$. The complex points of the Jacobian $J_{\mathbb{C}}$ are

$$J_{\mathbb{C}}(\mathbb{C}) = H^0(X_{\mathbb{C}}, \Omega^1_X)^\vee / \mathrm{Im}\, H_1(X_{\mathbb{C}}(\mathbb{C}), \mathbb{Z}),$$

where $\Lambda := \mathrm{Im}\, H_1(X_{\mathbb{C}}(\mathbb{C}), \mathbb{Z})$ is viewed as a lattice via integration of differential forms. By (2.3), we thus have an identification $J_{\mathbb{C}}(\mathbb{C}) = V/\Lambda$ as a g_X-dimensional complex torus, where we recall that g_X is the genus of X. Let V_f be the subspace of V on which $\mathbb{T}$ acts via the homomorphism $\lambda_f : \mathbb{T} \longrightarrow K_f$, and let $\mathrm{pr}_f : V \longrightarrow V_f$ be the orthogonal projection with respect to the Petersson scalar product. The projector pr_f naturally belongs to $\mathbb{T}_{K_f} = \mathbb{T} \otimes_{\mathbb{Q}} K_f$, and by (2.5) and (2.6) we may view pr_f as an idempotent element

$$[t_f] \in (\mathrm{CH}^1(X^2)(\mathbb{Q})_{K_f})/(\mathrm{pr}_1^* \mathrm{CH}^1(X)(\mathbb{Q})_{K_f} + \mathrm{pr}_2^* \mathrm{CH}^1(X)(\mathbb{Q})_{K_f}), \tag{2.7}$$

where t_f denotes some lift of pr_f to $\mathrm{CH}^1(X^2)(\mathbb{Q})_{K_f}$. The correspondence t_f is some choice of K_f-linear combination of Hecke correspondences which induces the projection on cohomology onto the f-isotypic component.

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If we let $V_{[f]} := \bigoplus_{g\in[f]} V_g$ and $\mathrm{pr}_{[f]} := \sum_{g\in[f]} \mathrm{pr}_g$, then $\mathrm{pr}_{[f]}$ is the orthogonal projection $V \longrightarrow V_{[f]}$ with respect to the Petersson scalar product. Note that $\mathrm{pr}_{[f]}$ naturally belongs to the Hecke algebra $\mathbb{T}$, and corresponds under (2.5) to the idempotent element $e_{[f]}$ in $\bigoplus_{[h]} K_h$ which has 1 as $[f]$-coordinate and 0 as $[h]$-coordinate for $[h] \neq [f]$. By (2.6), we may view $\mathrm{pr}_{[f]}$ as an idempotent element

$$[t_{[f]}] \in (\mathrm{CH}^1(X^2)(\mathbb{Q})_{\mathbb{Q}})/(\mathrm{pr}_1^* \mathrm{CH}^1(X)(\mathbb{Q})_{\mathbb{Q}} + \mathrm{pr}_2^* \mathrm{CH}^1(X)(\mathbb{Q})_{\mathbb{Q}}), \tag{2.8}$$

where $t_{[f]}$ denotes some lift of $\mathrm{pr}_{[f]}$ to $\mathrm{CH}^1(X^2)(\mathbb{Q})_{\mathbb{Q}}$.

Let I_f be the ideal $\ker(\lambda_f) \cap \mathbb{T}_{\mathbb{Z}}$ of the integral Hecke algebra $\mathbb{T}_{\mathbb{Z}}$ (the $\mathbb{Z}$-algebra generated by the Hecke operators T_n, with $(n, p) = 1$). The image $I_f(J)$ is a subabelian variety which is stable under $\mathbb{T}_{\mathbb{Z}}$ and defined over $\mathbb{Q}$. The abelian variety associated with the Galois orbit $[f]$ by Eichler and Shimura [27] is defined as the quotient $A_{[f]} := J/I_f(J)$. Let $m_{[f]} \in \mathbb{N}$ be the denominator of $\mathrm{pr}_{[f]} \in \mathbb{T}$, i.e., the smallest positive integer such that $\pi_{[f]} := m_{[f]}\,\mathrm{pr}_{[f]}$ belongs to $\mathbb{T}_{\mathbb{Z}}$. Then $A_{[f]}$ is isomorphic over $\mathbb{C}$ to the complex torus $V_{[f]}/\pi_{[f]}(\Lambda)$, and the map $\pi_{[f]} : V/\Lambda \longrightarrow V_{[f]}/\pi_{[f]}(\Lambda)$ corresponds to the natural quotient map $\pi_{[f]} : J \longrightarrow A_{[f]}$ of abelian varieties over $\mathbb{Q}$ with kernel $I_f(J)$ [6, Lemma 1.46].

3 Triple products

Let $f_1 = \sum_{n\geq 1} a_n(f_1)q^n$, $f_2 = \sum_{n\geq 1} a_n(f_2)q^n$, and $f_3 = \sum_{n\geq 1} a_n(f_3)q^n$ be three normalized cuspidal eigenforms of weight 2 and level $\Gamma_0(p)$, and let $F := f_1 \otimes f_2 \otimes f_3$ be the associated cusp form of weight $(2, 2, 2)$ for $\Gamma_0(p)^3$. Let $K_F = K_{f_1} \cdot K_{f_2} \cdot K_{f_3}$ denote the compositum of the Hecke fields of the forms f_1, f_2, and f_3.

3.1 Triple product *L*-functions

For $i \in \{1, 2, 3\}$ and a prime ℓ, let λ_i be the prime ideal of K_{f_i} above ℓ determined by the embeddings fixed in Section 1.7. Denote by K_{f_i,λ_i} the completion of K_{f_i} with respect to λ_i, and let $V_\ell(f_i) : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathbf{GL}_2(K_{f_i,\lambda_i})$ be the two-dimensional ℓ-adic Galois representation associated to f_i [6, Theorem 3.1]. We remark that given any choice of correspondence t_{f_i} as in (2.7), the representation $V_\ell(f_i)$ admits a description as $(t_{f_i})_* H^1_{\mathrm{et}}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$ followed by the map induced by the projection $K_{f_i} \otimes \mathbb{Q}_\ell \longrightarrow K_{f_i,\lambda_i}$.

The triple product L-function $L(F, s) = L(f_1, f_2, f_3, s)$ is the L-function associated with the compatible family of eight-dimensional ℓ-adic representations

$$V_\ell(F) := V_\ell(f_1) \otimes V_\ell(f_2) \otimes V_\ell(f_3).$$

It admits a description as an Euler product converging absolutely for $\mathscr{R}(s) > 5/2$. The Euler factors are given explicitly in [11, (1.7) and (1.8)].

Define the local L-factor at infinity following the general recipe of [9] by

$$L_\infty(F, s) = 2^4(2\pi)^{3-4s}\Gamma(s-1)^3\Gamma(s).$$

The completed L-function $\Lambda^*(F, s) := (p^5)^{\frac{s}{2}} L_\infty(F, s)L(F, s)$ admits an analytic continuation to the entire complex plane and satisfies the functional equation

$$\Lambda^*(F, s) = W(F) \cdot \Lambda^*(F, 4 - s), \tag{3.1}$$

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where $W(F) \in \{\pm 1\}$ is the global root number of F [11, Proposition 1.1]. The global root number, as stated in [11, Section 1], is given by

$$W(F) = a_p(f_1)a_p(f_2)a_p(f_3). \tag{3.2}$$

A detailed proof of this can for instance be found in [19, Proposition 4.5].

3.2 The Beilinson–Bloch conjecture

The center of symmetry of the functional equation (3.1) is the point $s = 2$ at which $L(F, s)$ has no pole. Moreover, $L_\infty(F, s)$ has neither zero nor pole at $s = 2$, so the center is a critical point, and we have

$$W(F) = (-1)^{\mathrm{ord}_{s=2} L(F,s)}. \tag{3.3}$$

For $i \in \{1, 2, 3\}$, let t_{f_i} be a choice of self-correspondence of X lifting the f_i-Hecke projector pr_{f_i} (2.7). Define a self-correspondence of X^3 by

$$t_F := t_{f_1} \otimes t_{f_2} \otimes t_{f_3} = \mathrm{pr}_{14}^*(t_{f_1}) \cdot \mathrm{pr}_{25}^*(t_{f_2}) \cdot \mathrm{pr}_{36}^*(t_{f_3}) \in \mathrm{Corr}^0(X^3, X^3)(\mathbb{Q})_{K_F}, \tag{3.4}$$

where $\mathrm{pr}_{ij} : X^6 \longrightarrow X^2$ denotes the natural projection to the ith and jth components.

Remark 3.1 The correspondence t_F is some choice of K_F-linear combination of tensor products of Hecke correspondences projecting to the 1-dimensional F-isotypic component of the $(\mathbb{T}^{\otimes 3} \otimes \mathbb{R})$-module $H^0(X^3, \Omega^3_{X^3}) \otimes \mathbb{R} = H^0(X, \Omega^1_X)^{\otimes 3} \otimes \mathbb{R}$.

The Beilinson–Bloch conjecture [4] predicts in this setting that

$$\mathrm{ord}_{s=2} L(F, s) = \dim_{K_F} (t_F)_*(\mathrm{CH}^2(X^3)_0(\mathbb{Q})_{K_F}). \tag{3.5}$$

In the case when $W(F) = +1$, Gross and Kudla proved a formula for the central value $L(F, 2)$, expressing it as a product of a complex period and an algebraic number [11, Proposition 10.8]. This algebraic number admits an explicit description in terms of the coefficients of the Jacquet–Langlands transfers of f_1, f_2, and f_3 to the definite quaternion algebra ramified at p and ∞.

In the case when $W(F) = -1$, the L-function $L(F, s)$ vanishes to odd order at its centre $s = 2$. By (3.5), we expect $(t_F)_*(\mathrm{CH}^2(X^3)_0(\mathbb{Q})_{K_F})$ to have dimension greater or equal to 1. A natural element of $\mathrm{CH}^2(X^3)_0(\mathbb{Q})$ to consider is the modified diagonal cycle, also referred to as the Gross–Kudla–Schoen cycle, which we now define.

Let Δ denote the image of X under the diagonal embedding $X \longrightarrow X^3$, i.e.,

$$\Delta = \{(x, x, x) \mid x \in X\} \subset X^3. \tag{3.6}$$

In order to get a null-homologous cycle, we apply a projector to Δ following [11, 12].

Definition 3.1 Let C be a smooth, projective, and geometrically connected curve over a number field k, and let e be a point in $X(\bar{k})$. For any nonempty subset T of $\{1, 2, 3\}$, let T' denote the complementary set. Write $p_T : C^3 \longrightarrow C^{|T|}$ for the natural projection map and let $q_T(e) : C^{|T|} \longrightarrow C^3$ denote the inclusion obtained by filling in the missing coordinates using the point e. Let $P_T(e)$ denote the graph of $q_T(e) \circ p_T$

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viewed as a codimension 3 cycle on the product $C^3 \times C^3$. Define the Gross–Kudla–Schoen projector

$$P_{\mathrm{GKS}}(e) := \sum_T (-1)^{|T'|} P_T(e) \in \mathrm{CH}^3(C^3 \times C^3)(\bar{k}),$$

where the sum is taken over all subsets of $\{1,2,3\}$. This is an idempotent in the ring of correspondences of C^3 with the property that it annihilates the cohomology groups $H^i(C^3_{\mathbb{C}}(\mathbb{C}),\mathbb{Z})$, for $i \in \{4,5,6\}$, and maps $H^3(C^3_{\mathbb{C}}(\mathbb{C}),\mathbb{Z})$ onto the Künneth summand $H^1(C_{\mathbb{C}}(\mathbb{C}),\mathbb{Z})^{\otimes 3}$ [12, Corollary 2.6].

Given a point $e \in X(\bar{\mathbb{Q}})$, the Gross–Kudla–Schoen cycle with base point e is

$$\Delta_{\mathrm{GKS}}(e) := P_{\mathrm{GKS}}(e)_*(\Delta) \in \mathrm{CH}^2(X^3)_0(\bar{\mathbb{Q}}). \tag{3.7}$$

Note that the cycle $\Delta_{\mathrm{GKS}}(e)$ is in fact null-homologous since $P_{\mathrm{GKS}}(e)$ annihilates $H^4(X^3_{\mathbb{C}}(\mathbb{C}),\mathbb{Z})$, the target of the cycle class map. Define the "F-isotypic component" of the Gross–Kudla–Schoen cycle by $(t_F)_*\Delta_{\mathrm{GKS}}(e) \in \mathrm{CH}^2(X^3)_0(\bar{\mathbb{Q}})_{K_F}$.

Remark 3.2 Although t_F is not unique, the difference $t_F - t'_F$ of two such projectors annihilates $H^0(X^3_{\mathbb{C}}, \Omega^3_{X^3})$. Conditional on the nondegeneracy of the Beilinson–Bloch height pairing for X^2, this implies that $(t_F)_*\Delta_{\mathrm{GKS}}(e) = (t'_F)_*\Delta_{\mathrm{GKS}}(e)$. Unconditionally, the Beilinson–Bloch height of $(t_F)_*\Delta_{\mathrm{GKS}}(e)$ is independent of the choice of t_F [12, Proposition 8.3, Notes 8.5 and 8.6].

Gross and Kudla [11, Conjecture 13.2] conjectured the formula

$$\frac{L'(F,2)}{\Omega_F} = \langle (t_F)_*(\Delta_{\mathrm{GKS}}(\xi_\infty)), (t_F)_*(\Delta_{\mathrm{GKS}}(\xi_\infty))\rangle^{\mathrm{BB}}, \tag{3.8}$$

where $\langle\ ,\ \rangle^{\mathrm{BB}} : \mathrm{CH}^2(X^3)_0(\mathbb{Q})_{\mathbb{R}} \times \mathrm{CH}^2(X^3)_0(\mathbb{Q})_{\mathbb{R}} \longrightarrow \mathbb{R}$ denotes the Beilinson–Bloch height pairing [11, (13.9)], and $\Omega_F := \|\omega_{f_1}\|^2 \cdot \|\omega_{f_2}\|^2 \cdot \|\omega_{f_3}\|^2/(4\pi p)$ is the complex period of F with $\|\cdot\|$ denoting the Petersson norm. A proof of (3.8) due to Yuan et al. was announced in [28].

4 Abel–Jacobi maps

Let f_1, f_2, f_3 be three normalized eigenforms in $S_2(\Gamma_0(p))$, and let $F = f_1 \otimes f_2 \otimes f_3$. We work under the following assumption on the sign of the functional equation (3.1).

Assumption 4.1 $W(F) = +1$.

Under Assumption 4.1, the L-function $L(F,s)$ vanishes to even order at the central critical point $s = 2$, by (3.3), and the Beilinson–Bloch conjecture (3.5) predicts that the algebraic rank of the F-isotypic component of $\mathrm{CH}^2(X^3)_0(\mathbb{Q})$ is even. Comparing with the situation of Heegner points on modular curves studied in [13], it seems reasonable to expect that the F-isotypic component of $\Delta_{\mathrm{GKS}}(e)$ is trivial, for all $e \in X(\mathbb{Q})$. While this appears to be difficult to show directly in the Chow group, we

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can prove the corresponding statement for the image of the cycle under the complex Abel–Jacobi map

$$\mathrm{AJ}_{X^3} : \mathrm{CH}^2(X^3_{\mathbb{C}})_0(\mathbb{C}) \longrightarrow J^2(X^3_{\mathbb{C}}) := \frac{(\mathrm{Fil}^2 H^3_{\mathrm{dR}}(X^3_{\mathbb{C}}))^\vee}{\mathrm{Im}\, H_3(X^3_{\mathbb{C}}(\mathbb{C}), \mathbb{Z})}, \tag{4.1}$$

whose target is the second Griffiths intermediate Jacobian of $X^3_{\mathbb{C}}$. This map is a higher dimensional generalization of the familiar Abel–Jacobi isomorphism for curves. It is defined by the integration formula

$$\mathrm{AJ}_{X^3}(Z)(\alpha) := \int_{\partial^{-1}(Z)} \alpha, \quad \text{for all } \alpha \in \mathrm{Fil}^2 H^3_{\mathrm{dR}}(X^3_{\mathbb{C}}),$$

where $\partial^{-1}(Z)$ denotes any continuous 3-chain in $X^3_{\mathbb{C}}(\mathbb{C})$ whose image under the boundary map ∂ is Z.

Definition 4.1 Given a point e in $X(\bar{\mathbb{Q}})$ and a choice of correspondence t_F (3.4) projecting to the F-isotypic component of $H^0(X^3, \Omega^3_{X^3}) \otimes \mathbb{R}$, define the F-isotypic component of the Abel–Jacobi image of the Gross–Kudla–Schoen cycle by

$$\mathrm{AJ}^F_{X^3}(\Delta_{\mathrm{GKS}}(e)) := \mathrm{AJ}_{X^3}((t_F)_*(\Delta_{\mathrm{GKS}}(e))) \in J^2(X^3_{\mathbb{C}})_{K_F}.$$

Remark 4.2 Definition 4.1 is independent of the choice of t_F, as AJ_{X^3} is functorial and any two such projectors act similarly on cohomology.

Henceforth, we fix a choice of projector $t_F = t_{f_1} \otimes t_{f_2} \otimes t_{f_3}$. The aim of this section is to prove the main result:

Theorem 4.3 *Let f_1, f_2, and f_3 be three normalized eigenforms in $S_2(\Gamma_0(p))$, denote by $F = f_1 \otimes f_2 \otimes f_3$ their triple product, and suppose that F satisfies Assumption 4.1. Then $\mathrm{AJ}^F_{X^3}(\Delta_{\mathrm{GKS}}(e)) = 0$ in $J^2(X^3_{\mathbb{C}})_{K_F}$, for all $e \in X(\mathbb{Q})$.*

Remark 4.4 Similar arguments to the ones presented in the proof of Theorem 4.3 below can be used to prove that the image of $(t_F)_*(\Delta_{\mathrm{GKS}}(e))$ under Bloch's [4] ℓ-adic étale Abel–Jacobi map

$$\mathrm{AJ}^{\mathrm{et}}_{X^3} : \mathrm{CH}^2(X^3)_0(\mathbb{Q}) \longrightarrow H^1(\mathbb{Q}, H^3_{\mathrm{et}}(X^3_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(2))) \tag{4.2}$$

is torsion, when the global root number is $W(F) = +1$. It is conjectured that for any smooth proper variety over a number field, and for any prime ℓ, the ℓ-adic Abel–Jacobi maps in any codimension are injective up to torsion [16, Conjecture 9.15]. Thus, conditional on this conjecture, $(t_F)_*(\Delta_{\mathrm{GKS}}(e))$ is trivial in the Chow group $\mathrm{CH}^2(X^3)_0(\mathbb{Q})_{K_F}$.

The rest of this section constitutes the proof of Theorem 4.3. We distinguish different situations depending on the genus g_X of X, which we recall is given by the formula (2.2). The curve X has genus zero exactly when $p \in \{2, 3, 5, 7, 13\}$. In this case, the space of cusp forms $S_2(\Gamma_0(p))$ is trivial, and there is no triple product L-function to consider in the first place. We have $\Delta_{\mathrm{GKS}}(e) = 0$ in $\mathrm{CH}^2(X^3)_0(\mathbb{Q})$, as the cycle class map is injective in this case [12, Proposition 4.1].

4.1 The genus one case

Suppose that $g_X = 1$, i.e., $p \in \{11, 17, 19\}$. In this case, X is an elliptic curve over $\mathbb{Q}$ of Mordell–Weil rank 0. For all $e \in X(\mathbb{Q})$, we have $6\Delta_{\mathrm{GKS}}(e) = 0$ in $\mathrm{CH}^2(X^3)_0(\mathbb{Q})$ [12, Corollary 4.7]. On the L-function side, $f_1 = f_2 = f_3 = f$ is the normalized eigenform corresponding to the elliptic curve X. By [11, (11.8)] the triple product L-function decomposes as

$$L(F, s) = L(\mathrm{Sym}^3 f, s)L(f, s-1)^2.$$

Note that $W(F) = a_p(f)^3 = a_p(f) = W(f) = +1$ by (3.2) and the fact that the sign of the functional equation of $L(f, s)$ centered at $s = 1$ is equal to +1, since X has Mordell–Weil rank 0. For each $p \in \{11, 17, 19\}$, we have $L(F, 2) \neq 0$ [11, Tables 12.5–12.7]. In other words, $\mathrm{ord}_{s=2}(L(F, s)) = 0$. The fact that $\Delta_{\mathrm{GKS}}(e)$ is torsion in the Chow group is therefore consistent with the Beilinson–Bloch conjecture (3.5).

4.2 The higher genus case

Suppose that $g_X \geq 2$. It will be convenient to sometimes view the Atkin–Lehner involution w_p of Section 2.1 as a correspondence by taking its graph. By slight abuse of notation, we will write $w_p \in \mathrm{Corr}^0(X, X)(\mathbb{Q})$. The operator w_p naturally belongs to the Hecke algebra $\mathbb{T}$ by (2.5), and commutes with the Hecke operators. The modular forms f_j, with $j \in \{1, 2, 3\}$, are eigenforms for the operator w_p with eigenvalues given by $-a_p(f_j)$ respectively (see Section 2.1).

Consider the involution $u_p := w_p \times w_p \times w_p$ of X^3. By taking its graph, it may be viewed as a correspondence, and we write again $u_p \in \mathrm{Corr}^0(X^3, X^3)(\mathbb{Q})$, by slight abuse of notation. Note that, as correspondences, we have

$$u_p = w_p \otimes w_p \otimes w_p := \mathrm{pr}_{14}^*(w_p) \cdot \mathrm{pr}_{25}^*(w_p) \cdot \mathrm{pr}_{36}^*(w_p) \in \mathrm{Corr}^0(X^3, X^3)(\mathbb{Q}).$$

The map u_p induces an involution on cohomology via pull-back, hence an involution on the space of cusp forms of weight $(2, 2, 2)$ for $\Gamma_0(p)^3$. By (3.2), we see that

$$u_p^*(F) = -W(F) \cdot F. \tag{4.3}$$

Lemma 4.5 *We have $(u_p)_*(\Delta_{\mathrm{GKS}}(e)) = \Delta_{\mathrm{GKS}}(w_p(e))$, for any $e \in X(\bar{\mathbb{Q}})$.*

Proof The induced map $(u_p)_* : \mathrm{CH}^2(X^3) \longrightarrow \mathrm{CH}^2(X^3)$ on Chow groups simply maps a cycle to its image under u_p. We have $u_p(\Delta) = \Delta$, since u_p is an automorphism of X^3. However, $u_p(P_T(e)_*(\Delta)) = P_T(w_p(e))_*(\Delta)$ for any proper subset T of $\{1, 2, 3\}$. ■

Proposition 4.6 *Let f_1, f_2, and f_3 be three normalized eigenforms in $S_2(\Gamma_0(p))$, denote by $F = f_1 \otimes f_2 \otimes f_3$ their triple product, and suppose that F satisfies Assumption 4.1. For any point $e \in X(\bar{\mathbb{Q}})$, we have $\mathrm{AJ}_{X^3}^F(\Delta_{\mathrm{GKS}}(e)) = -\mathrm{AJ}_{X^3}^F(\Delta_{\mathrm{GKS}}(w_p(e)))$.*

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Proof By functoriality of Abel–Jacobi maps with respect to correspondences, we have

$$\mathrm{AJ}_{X^3}((u_p)_*(t_F)_*(\Delta_{\mathrm{GKS}}(e))) = (u_p^*)^\vee\, \mathrm{AJ}^F_{X^3}(\Delta_{\mathrm{GKS}}(e)). \tag{4.4}$$

For $i \in \{1,2,3\}$, w_p commutes with t_{f_i} as self-correspondences of X up to vertical and horizontal divisors, by (2.5) and (2.6). This implies that

$$\begin{aligned} u_p \circ t_F &= (w_p \circ t_{f_1}) \otimes (w_p \circ t_{f_2}) \otimes (w_p \circ t_{f_3}) \\ &= (t'_{f_1} \circ w_p) \otimes (t'_{f_2} \circ w_p) \otimes (t'_{f_3} \circ w_p) = t'_F \circ u_p, \end{aligned}$$

where $t'_F = t'_{f_1} \otimes t'_{f_2} \otimes t'_{f_3}$ is possibly another F-isotypic projector. In particular, using Lemma 4.5, we obtain

$$(u_p)_*(t_F)_*(\Delta_{\mathrm{GKS}}(e)) = (t'_F)_*(u_p)_*(\Delta_{\mathrm{GKS}}(e)) = (t'_F)_*(\Delta_{\mathrm{GKS}}(w_p(e))).$$

The left hand side of (4.4) is thus equal to $\mathrm{AJ}^F_{X^3}(\Delta_{\mathrm{GKS}}(w_p(e)))$ by Remark 4.2.

On the other hand, $\mathrm{AJ}^F_{X^3}(\Delta_{\mathrm{GKS}}(e))$ lies in $(t_F^*)^\vee(J^2(X^3_{\mathbb{C}}))$ by functoriality of the complex Abel–Jacobi map with respect to correspondences, that is, in the F-isotypic Hecke component of the intermediate Jacobian. The triple product Hecke algebra $\mathbb{T}^{\otimes 3}$ acts via correspondences on the latter by multiplication by the Hecke eigenvalues of F. For any $\alpha \in \mathrm{Fil}^2 H^3_{\mathrm{dR}}(X^3_{\mathbb{C}})$, we have the equality

$$(u_p^*)^\vee\, \mathrm{AJ}^F_{X^3}(\Delta_{\mathrm{GKS}}(e))(\alpha) = \mathrm{AJ}_{X^3}(\Delta_{\mathrm{GKS}}(e))(u_p^*(t_F^*(\alpha))).$$

The operator u_p in $\mathbb{T}^{\otimes 3}$ acts via pull-back on the F-isotypic component $(t_F)^* H^3_{\mathrm{dR}}(X^3_{\mathbb{C}})$ as multiplication by $-W(F)$ by (4.3). In particular, $u_p^*(t_F^*(\alpha)) = -W(F)t_F^*(\alpha)$. By Assumption 4.1, the right hand side of (4.4) is thus $-\mathrm{AJ}^F_{X^3}(\Delta_{\mathrm{GKS}}(e))$. ∎

Mazur proved, for $g_X \geq 2$ and $p \notin \{37,43,67,163\}$, that $X(\mathbb{Q}) = \{\xi_\infty, \xi_0\}$, where we recall that ξ_∞ and ξ_0 denote the two cusps of X [22, Theorem 1]. Moreover, the modular curve $X_0(37)$ has two noncuspidal $\mathbb{Q}$-rational points, while $X_0(p)$ has a unique noncuspidal $\mathbb{Q}$-rational point, for $p \in \{43,67,163\}$.

Corollary 4.7 *Let f_1, f_2, and f_3 be three normalized eigenforms in $S_2(\Gamma_0(p))$, denote by $F = f_1 \otimes f_2 \otimes f_3$ their triple product, and suppose that F satisfies Assumption 4.1. If p belongs to $\{43,67,163\}$, and e denotes the unique noncuspidal $\mathbb{Q}$-rational point of X, then $\mathrm{AJ}^F_{X^3}(\Delta_{\mathrm{GKS}}(e)) = 0$.*

Proof The involution w_p maps $\mathbb{Q}$-rational points to $\mathbb{Q}$-rational points and permutes the two cusps ξ_∞ to ξ_0. It therefore fixes the noncuspidal point e, and the result follows from Proposition 4.6. ∎

Corollary 4.8 *Let f_1, f_2, and f_3 be three normalized eigenforms in $S_2(\Gamma_0(p))$, denote by $F = f_1 \otimes f_2 \otimes f_3$ their triple product, and suppose that F satisfies Assumption 4.1. If $g_X \geq 2$, then $\mathrm{AJ}^F_{X^3}(\Delta_{\mathrm{GKS}}(\xi_\infty)) = \mathrm{AJ}^F_{X^3}(\Delta_{\mathrm{GKS}}(\xi_0)) = 0$.*

Proof Gross and Schoen [12, Proposition 3.6] have constructed a correspondence Ξ in $\mathrm{Corr}^1(X, X^3)(\mathbb{Q})$ with the property that the natural transformation induced by push-forward

$$\Xi_* : \mathrm{CH}^1(X) = \mathrm{Pic}(X) \longrightarrow \mathrm{CH}^2(X^3) \tag{4.5}$$

sends the rational equivalence class of a divisor $\sum m(e)e$ to $\sum m(e)\Delta_{\mathrm{GKS}}(e)$. In particular, the cycle $\Delta_{\mathrm{GKS}}(\xi_\infty) - \Delta_{\mathrm{GKS}}(\xi_0)$ in $\mathrm{CH}^2(X^3)_0(\mathbb{Q})$ depends only on the class of the degree zero divisor $(\xi_\infty) - (\xi_0)$ in $\mathrm{CH}^1(X)_0(\mathbb{Q}) = J(\mathbb{Q})$. By Manin–Drinfeld [20], the divisor $(\xi_\infty) - (\xi_0)$ is torsion in the Jacobian J. It follows that $\Delta_{\mathrm{GKS}}(\xi_\infty) - \Delta_{\mathrm{GKS}}(\xi_0)$ is torsion in $\mathrm{CH}^2(X^3)_0(\mathbb{Q})$, and in particular $\mathrm{AJ}^F_{X^3}(\Delta_{\mathrm{GKS}}(\xi_\infty)) - \mathrm{AJ}^F_{X^3}(\Delta_{\mathrm{GKS}}(\xi_0)) = 0$ in $J^2(X^3_{\mathbb{C}})_{K_F}$. The involution w_p permutes the cusps ξ_∞ and ξ_0. By Proposition 4.6, we thus have the equality $\mathrm{AJ}^F_{X^3}(\Delta_{\mathrm{GKS}}(\xi_\infty)) = -\mathrm{AJ}^F_{X^3}(\Delta_{\mathrm{GKS}}(\xi_0))$, and the proof is complete. ∎

4.3 The case $p = 37$

To complete the proof of Theorem 4.3, the only remaining case is the one where $p = 37$ and the chosen base point is a noncuspidal $\mathbb{Q}$-rational point. The curve $X_0(37)$ has been extensively studied by Mazur and Swinnerton-Dyer [23, Section 5]. It has genus 2 and is therefore hyperelliptic. Its hyperelliptic involution will be denoted by S. In particular, for all points e in $X_0(37)(\bar{\mathbb{Q}})$, we have $6\Delta_{\mathrm{GKS}}(e) = 0$ in the Griffiths group $\mathrm{Gr}^2(X_0(37)^3)$ of null-homologous algebraic cycles modulo algebraic equivalence [12, Corollary 4.9]. The involution S is distinct from the Atkin–Lehner involution w_{37}, as the quotient $X_0(37)/w_{37}$ has genus 1. Since S commutes with every automorphism of $X_0(37)$ [23, p. 27], it commutes in particular with w_{37}, and we can define another involution $T = S \circ w_{37} = w_{37} \circ S$. Let $\gamma_0 = T(\xi_0)$ and $\gamma_\infty = T(\xi_\infty)$ be the images of the two cusps by T. By [23, Proposition 2], we have

$$X_0(37)(\mathbb{Q}) = \{\xi_0, \xi_\infty, \gamma_0, \gamma_\infty\} \qquad \text{and} \qquad w_{37}(\gamma_0) = \gamma_\infty. \tag{4.6}$$

We now complete the proof of Theorem 4.3.

Corollary 4.9 *Let f_1, f_2, and f_3 be three normalized eigenforms in $S_2(\Gamma_0(37))$, denote by $F = f_1 \otimes f_2 \otimes f_3$ their triple product, and suppose that F satisfies Assumption 4.1. Then*

$$\mathrm{AJ}^F_{X_0(37)^3}(\Delta_{\mathrm{GKS}}(\gamma_0)) = \mathrm{AJ}^F_{X_0(37)^3}(\Delta_{\mathrm{GKS}}(\gamma_\infty)) = 0.$$

Proof By (4.6), the Atkin–Lehner involution w_{37} interchanges γ_0 and γ_∞. By Proposition 4.6, we have $\mathrm{AJ}^F_{X_0(37)^3}(\Delta_{\mathrm{GKS}}(\gamma_0)) = -\mathrm{AJ}^F_{X_0(37)^3}(\Delta_{\mathrm{GKS}}(\gamma_\infty))$. The element

$$2\,\mathrm{AJ}^F_{X_0(37)^3}(\Delta_{\mathrm{GKS}}(\gamma_0)) = \mathrm{AJ}_{X_0(37)^3}((t_F)_*(\Delta_{\mathrm{GKS}}(\gamma_0) - \Delta_{\mathrm{GKS}}(\gamma_\infty)))$$

in $J^2(X_0(37)^3_{\mathbb{C}})_{K_F}$ depends only on the class of $(\gamma_0) - (\gamma_\infty)$ in $J_0(37)(\mathbb{Q})$ by the existence of (4.5). But this class is the image of the class of $(\xi_0) - (\xi_\infty)$ by the involution of $J_0(37)$ obtained from T by push-forward. The latter class is torsion by the Manin–Drinfeld theorem [20]. ∎

5 Chow–Heegner points

Let f be a normalized eigenform in $S_2(\Gamma_0(p))$ with rational coefficients, and let E_f be the optimal elliptic curve quotient of J associated with f by the Eichler–Shimura construction [27]. Following Section 2.2, denote by $\pi_f : J \longrightarrow E_f$ the natural quotient map with connected kernel. It is induced by the element

$$[m_f t_f] \in \mathrm{CH}^1(X^2)(\mathbb{Q})/(\mathrm{pr}_1^* \mathrm{CH}^1(X)(\mathbb{Q}) + \mathrm{pr}_2^* \mathrm{CH}^1(X)(\mathbb{Q})),$$

where $m_f \in \mathbb{N}$ denotes the denominator of $\mathrm{pr}_f \in \mathbb{T}$.

Remark 5.1 To the best of the author's knowledge, it is unknown whether there are finitely or infinitely many elliptic curves over $\mathbb{Q}$ of prime conductor. It is a result of Setzer [26, Theorem 2] that, given a prime p distinct from 2, 3, and 17, there is an elliptic curve of conductor p over $\mathbb{Q}$ with a rational 2-torsion point if and only if $p = u^2 + 64$ for some rational integer u. A conjecture of Hardy and Littlewood [14, Conjecture F] implies that there are infinitely many values of u such that $u^2 + 64$ is prime. Thus, conditional on this conjecture of Hardy and Littlewood, there are infinitely many primes p which occur as the conductor of an elliptic curve over $\mathbb{Q}$. This is explained in detail in the preprint [15].

Let g be an auxiliary normalized eigenform in $S_2(\Gamma_0(p))$. Following the notations of Section 2.2, recall that $\mathrm{pr}_{[g]} \in \mathbb{T}$ denotes the $[g]$-isotypic Hecke projector. Define the $[g]$-isotypic component $\mathrm{End}^0_{\mathbb{Q}}(J)[g] := \mathrm{pr}_{[g]} \cdot \mathrm{End}^0_{\mathbb{Q}}(J)$ and let $\mathrm{CH}^1(X^2)[g]_{\mathbb{Q}}$ be the group of cycles mapping to $\mathrm{End}^0_{\mathbb{Q}}(J)[g]$ under (2.6) modulo vertical and horizontal divisors. Let $t_{[g]}$ be an element of $\mathrm{CH}^1(X^2)[g]_{\mathbb{Q}}$ mapping to $\mathrm{pr}_{[g]}$.

For any correspondence $Z \in \mathrm{CH}^1(X^2)(\mathbb{Q})$, define

$$\Pi_Z := \mathrm{pr}_{12}^*(Z) \cdot \mathrm{pr}_{34}^*(\Delta) \in \mathrm{CH}^2(X^4)(\mathbb{Q}),$$

where $\Delta \in \mathrm{CH}^1(X^2)(\mathbb{Q})$ is the diagonal cycle. It induces a push-forward map

$$\Pi_{Z,*} : \mathrm{CH}^2(X^3)_0(L) \longrightarrow \mathrm{CH}^1(X)_0(L) = J(L)$$

for any field extension L of $\mathbb{Q}$. For $e \in X(\bar{\mathbb{Q}})$, define the point

$$P_Z(e) := \Pi_{Z,*}(\Delta_{\mathrm{GKS}}(e)) \in J(\bar{\mathbb{Q}}).$$

Remark 5.2 The association of a point in J to a self-correspondence is well-defined modulo vertical and horizontal divisors [8, Ex. 3.1.7]. Associate to $Z \in \mathrm{CH}^1(X^2)(\mathbb{Q})_{\mathbb{Q}}$ a point $P_Z(e) := P_{mZ}(e) \otimes 1/m \in J(\bar{\mathbb{Q}})_{\mathbb{Q}}$, where $m \in \mathbb{N}$ such that $mZ \in \mathrm{CH}^1(X^2)(\mathbb{Q})$.

By composing correspondences, we can define

$$\Pi_{Z,t_f} := (m_f t_f) \circ \Pi_Z = \mathrm{pr}_{12}^*(Z) \cdot \mathrm{pr}_{34}^*(m_f t_f) \in \mathrm{Corr}^{-1}(X^3, X)(\mathbb{Q}). \quad (5.1)$$

This induces, in the terminology of [2], a generalized modular parametrization

$$\Pi_Z^f := \Pi_{Z,t_f,*} = \pi_f \circ \Pi_{Z,*} : \mathrm{CH}^2(X^3)_0(L) \longrightarrow E_f(L),$$

independent of the choice of t_f. Given $e \in X(\bar{\mathbb{Q}})$, we define the Chow–Heegner point

$$P_Z^f(e) := \Pi_Z^f(\Delta_{\mathrm{GKS}}(e)) = \pi_f(P_Z(e)) \in E_f(\bar{\mathbb{Q}}).$$

By Remark 5.2, we can define the Chow–Heegner point associated with f and $[g]$ by

$$P_{[g]}^f(e) := P_{t_{[g]}}^f(e) \in E_f(\bar{\mathbb{Q}})_{\mathbb{Q}}.$$

Concretely, we have

$$P_{[g]}^f(e) = \pi_f(\Pi_{m_{[g]}t_{[g]},*}(\Delta_{\mathrm{GKS}}(e))) \otimes 1/m_{[g]} \in E_f(\bar{\mathbb{Q}})_{\mathbb{Q}},$$

where $m_{[g]}$ is the denominator of $\mathrm{pr}_{[g]}$.

Building on the work of Yuan et al. [28], Darmon et al. proved the following in [7]:

Theorem 5.3 *Assume that $g \neq f$, $W(f) = -1$, and $W(\mathrm{Sym}^2 g \otimes f) = +1$. The subspace*

$$\langle P_T^f(\xi_\infty) : T \in \mathrm{CH}^1(X^2)[g]_{\mathbb{Q}} \rangle \subset E_f(\mathbb{Q})_{\mathbb{Q}}$$

is nonzero if and only if

$$\mathrm{ord}_{s=1} L(f,s) = 1 \quad \text{and} \quad \mathrm{ord}_{s=2} L(\mathrm{Sym}^2(g^\sigma) \otimes f, s) = 0, \quad \forall\, \sigma : K_g \hookrightarrow \mathbb{C}.$$

Proof This is a particular case of [7, Theorem 3.7]. ∎

Remark 5.4 The triple product L-function attached to (g,g,f) decomposes as

$$L(g,g,f,s) = L(f,s-1)L(\mathrm{Sym}^2 g \otimes f, s),$$

and therefore the assumptions of Theorem 5.3 imply in particular that $W(g,g,f) = -1$.

Remark 5.5 When g equals f, $(t_f^{\otimes 3})_*(\Delta_{\mathrm{GKS}}(e))$ is the Gross–Kudla–Schoen cycle in $\mathrm{CH}^2(E_f^3)_0(\mathbb{Q})$ based at $\pi_f(e)$, which is torsion by [12, Corollary 4.7]. The resulting Chow–Heegner point is then trivial by (5.2), whence the assumption in Theorem 5.3.

In the complementary setting where $W(g,g,f) = +1$, we now prove the following:

Theorem 5.6 *If E_f admits split multiplicative reduction at p, then $P_{[g]}^f(e)$ is trivial in $E_f(\mathbb{Q})_{\mathbb{Q}}$, for all $e \in X(\mathbb{Q})$. Equivalently, $m_{[g]}^2 P_{[g]}^f(e)$ is torsion in $E_f(\mathbb{Q})$, for all e in $X(\mathbb{Q})$.*

Proof Following Section 2.2, we have $t_{[g]} = \sum_{h\in[g]} t_h$, and thus

$$t_{[g]} \otimes t_{[g]} \otimes t_f = \sum_{h_1,h_2\in[g]} t_{h_1} \otimes t_{h_2} \otimes t_f.$$

By (3.2), for any $h_1, h_2 \in [g]$, the global root number of the triple product L-function $L(h_1, h_2, f, s)$ is given by $W(h_1, h_2, f) = a_p(h_1)a_p(h_2)a_p(f)$. The pth Fourier coefficient of a normalized cuspidal eigenform is the negative of the w_p-eigenvalue of the form, hence it belongs to $\{\pm 1\}$. In particular, since this coefficient belongs to $\mathbb{Q}$, it is fixed by the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, and thus $a_p(g) = a_p(h_1) = a_p(h_2) \in \{\pm 1\}$. It follows that $W(h_1, h_2, f) = a_p(f) = a_p(E_f)$. We have $a_p(E_f) = 1$, since E_f admits split multiplicative reduction at p, and the triple (h_1, h_2, f) satisfies Assumption 4.1. By Theorem 4.3, for any $e \in X(\mathbb{Q})$, $\mathrm{AJ}_{X^3}((t_{h_1} \otimes t_{h_2} \otimes t_f)_*(\Delta_{\mathrm{GKS}}(e)))$ is trivial in the intermediate Jacobian. Thus, $\mathrm{AJ}_{X^3}((t_{[g]} \otimes t_{[g]} \otimes t_f)_*(\Delta_{\mathrm{GKS}}(e)))$ is trivial in $J^2(X^3_{\mathbb{C}})_{\mathbb{Q}}$, or equivalently, $\mathrm{AJ}_{X^3}((m_{[g]}t_{[g]} \otimes m_{[g]}t_{[g]} \otimes m_f t_f)_*(\Delta_{\mathrm{GKS}}(e)))$ is torsion in $J^2(X^3_{\mathbb{C}})$.

Define the cycle $\Pi := \mathrm{pr}_{12}^*(\Delta) \cdot \mathrm{pr}_{34}^*(\Delta) \in \mathrm{CH}^2(X^4)(\mathbb{Q})$. Viewing $m_{[g]}t_{[g]} \otimes m_{[g]}t_{[g]} \otimes m_f t_f$ in $\mathrm{Corr}^0(X^3, X^3)(\mathbb{Q})$ and Π in $\mathrm{Corr}^{-1}(X^3, X)(\mathbb{Q})$, we compute

$$\begin{aligned}\Pi \circ (m_{[g]}t_{[g]} \otimes m_{[g]}t_{[g]} \otimes m_f t_f) &= \mathrm{pr}_{12}^*(m_{[g]}t_{[g]} \circ m_{[g]}t_{[g]}) \cdot \mathrm{pr}_{34}^*(m_f t_f) \\ &= m_{[g]}\Pi_{m_{[g]}t'_{[g]}, t_f},\end{aligned}$$

as elements of $\mathrm{Corr}^{-1}(X^3, X)(\mathbb{Q})$, where $t'_{[g]}$ is possibly another $[g]$-projector arising from the fact that $t_{[g]}$ is an idempotent element of the ring of self-correspondences modulo vertical and horizontal divisors. We deduce the equality of points in $E_f(\mathbb{Q})_{\mathbb{Q}}$

$$\Pi_*(m_{[g]}t_{[g]} \otimes m_{[g]}t_{[g]} \otimes m_f t_f)_*(\Delta_{\mathrm{GKS}}(e)) \otimes 1/m_{[g]}^2 = P^f_{[g]}(e). \tag{5.2}$$

By functoriality of Abel–Jacobi maps with respect to correspondences, the diagram

$$\begin{array}{ccc} \mathrm{CH}^2(X^3_{\mathbb{C}})_0(\mathbb{C}) & \xrightarrow{\mathrm{AJ}_{X^3}} & J^2(X^3_{\mathbb{C}}) \\ {\scriptstyle \Pi_*}\downarrow & & \downarrow{\scriptstyle (\Pi^*)^\vee} \\ E_f(\mathbb{C}) & \xrightarrow[\mathrm{AJ}_{E_f}]{\sim} & J^1(E_{f,\mathbb{C}}) \end{array} \tag{5.3}$$

commutes. Here, $J^1(E_{f,\mathbb{C}}) = H^0(E_f(\mathbb{C}), \Omega^1_{E_f})^\vee / \mathrm{Im}\, H_1(E_f(\mathbb{C}), \mathbb{Z})$ is the Jacobian of E_f, and AJ_{E_f} is the classical Abel–Jacobi isomorphism for the elliptic curve E_f given by

$$\mathrm{AJ}_{E_f}(P)(\alpha) := \int_{\mathcal{O}}^P \alpha, \qquad \text{for all } \alpha \in H^0(E_f(\mathbb{C}), \Omega^1_{E_f}),$$

where $\mathcal{O}$ is the origin of E_f. By (5.2) and (5.3), we have the equality in $J^1(E_{f,\mathbb{C}})$

$$\mathrm{AJ}_{E_f}(m_{[g]}^2 P^f_{[g]}(e)) = (\Pi^*)^\vee\, \mathrm{AJ}_{X^3}((m_{[g]}t_{[g]} \otimes m_{[g]}t_{[g]} \otimes m_f t_f)_*(\Delta_{\mathrm{GKS}}(e))). \tag{5.4}$$

The result follows from the facts that $\mathrm{AJ}_{X^3}((m_{[g]}t_{[g]} \otimes m_{[g]}t_{[g]} \otimes m_f t_f)_*(\Delta_{\mathrm{GKS}}(e)))$ is torsion and AJ_{E_f} is an isomorphism. ∎

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Remark 5.7 Theorem 5.6 with $e = \xi_\infty$ is a special case of [8, Theorem 3.3.8]. In his thesis [8], Daub proved more generally for composite level N that if the local root number $W_p(g, g, f) = -1$ for some $p \mid N$, then the resulting Chow–Heegner points based at ξ_∞ are torsion. His proof relies on an identification of these points with Zhang points [29]. As explained in the introduction (Section 1.4), our method works for composite level N.

6 Example of a nontrivial torsion element

Techniques were developed in [5] to numerically calculate Chow–Heegner points associated with modified diagonal cycles. The algorithms are based on a formula for the image of these cycles under the complex Abel–Jacobi map (4.1) proved in [7]. Most of the examples calculated in [5] concern the situation where the elliptic curve E_f has algebraic rank equal to 1. In particular, the global root number of E_f is −1, and this is not the setting studied in the present paper. However, in the appendix of [5] by Stein, some examples are computed for which the rank of E_f is 0. In particular, we deduce the following:

Theorem 6.1 *Let f and g be the normalized eigenforms of weight 2 and level $\Gamma_0(37)$ corresponding to the elliptic curves with Cremona labels 37b and 37a, and define $F := g \otimes g \otimes f$. Then $\mathrm{AJ}_{X_0(37)^3}((2t_g \otimes 2t_g \otimes 2t_f)_*(\Delta_{\mathrm{GKS}}(\xi_\infty)))$ is a nontrivial 6-torsion element of $J^2(X_0(37)^3_{\mathbb{C}})$.*

Proof In [5, Appendix], it is verified numerically in this case that $m_g P_g^f(\xi_\infty)$ is a point of order 3 in $E_f(\mathbb{Q})$. By inspecting the first few Fourier coefficients of f and g, we see that $m_g = m_f = 2$ (see [5, Section 5.1]). The point $4P_g^f(\xi_\infty) \in E_f(\mathbb{Q})$ has order 3, and by (5.4) $\mathrm{AJ}_{X_0(37)^3}((2t_g \otimes 2t_g \otimes 2t_f)_*(\Delta_{\mathrm{GKS}}(\xi_\infty)))$ is thus nontrivial in $J^2(X_0(37)^3_{\mathbb{C}})$.

The element $2\,\mathrm{AJ}_{X_0(37)^3}((2t_g \otimes 2t_g \otimes 2t_f)_*(\Delta_{\mathrm{GKS}}(\xi_\infty)))$ is equal by Proposition 4.6 to $\mathrm{AJ}_{X_0(37)^3}((2t_g \otimes 2t_g \otimes 2t_f)_*(\Delta_{\mathrm{GKS}}(\xi_\infty) - \Delta_{\mathrm{GKS}}(\xi_0)))$, and depends only on the class of $(\xi_\infty) - (\xi_0)$ in $J_0(37)(\mathbb{Q})$ by existence of (4.5). The latter has order 3 [21, Theorem 1]. ■

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