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STRATIFYING SYSTEMS FOR EXACT CATEGORIES

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Abstract. In this paper, we develop the theory of stratifying systems in the context of exact categories as a generalisation of the notion of stratifying systems in module categories, which have been studied by different authors. We prove that attached to a stratifying system in an exact category $(\mathcal{A}, \mathcal{E})$ there is an standardly stratified algebra B such that the category $\mathscr{F}_F(\Theta)$, of F-filtered objects in the exact category $(\mathcal{A}, \mathcal{E})$ is equivalent to the category $\mathscr{F}(\Delta)$ of Δ-good modules associated to B. The theory we develop in exact categories, give us a way to produce standardly stratified algebras from module categories by just changing the exact structure on it. In this way, we can construct exact categories whose bounded derived category is equivalent to the bounded derived category of an standardly stratified algebra. Finally, applying the relative homological algebra developed by Auslander–Solberg, we can construct examples of stratifying systems that are not a stratifying system in the classical sense, so our approach really produces new stratifying systems.

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1. Introduction. The notion of exact category in the context of additive categories is due to Quillen, see [**28**]. In Quillen's definition we have to give a special class of short exact sequences satisfying a set of axioms in order to obtain an exact category. The interest of exact categories is for a variety of reasons. First of all, they are a natural generalisation of abelian categories. There are several reasons for going beyond abelian categories. The fact that we may choose an exact structure gives us more flexibility, which turns out to be useful (the Example 8.1). Even if one is working in abelian categories, one soon finds the need to consider other exact structures different from the canonical one, for instance, in relative homological algebra (see [**4–6**, **16**]). In representation theory, exact categories arise naturally (filtered modules by a set of modules [**29**], stable categories, etc.). Exact categories are also important in representation theory because of a celebrated theorem of Happel, which tells us that Frobenius exact categories produce triangulated categories by passing to the stable category (see [**15**]).

On the other hand, the standardly stratified algebras were defined by Ágoston, Dlab and Lukács in [**1**, **10**] as a generalisation of quasi-hereditary algebras. Standardly stratified algebras are interesting because of their relationship with tilting theory and Lie theory, see for example [**1**, **2**, **23**], and quasi-hereditary algebras and homological dimensions, see for example [**22**, **24**]. In an attempt to generalise the standard modules and their main properties, Erdmann and Sáenz developed the notion of Ext-injective stratifying system in [**13**]. In that paper, they generalise the characteristic tilting module obtained by Ringel in [**29**].

Later on, Marcos, Mendoza and Sáenz defined a different notion of stratifying system and proved that this notion is equivalent to that given by Erdmann and Sáenz [**20**, **21**]. In [**21**], it was proven that for a given a stratifying system $(\Theta, \leq)$ in $\mathrm{mod}(A)$, there exists a module Q, such that $\mathrm{End}(Q)^{op}$ is a standardly stratified algebra with the same order $\leq$. Furthermore, there exists an exact equivalence between the Θ-filtered modules in $\mathrm{mod}(A)$ and the Δ-good modules in $\mathrm{mod}(\mathrm{End}(Q)^{op})$.

Recently, in [**25**], the authors developed the stratifying systems in triangulated categories and gave applications to exceptional sequences and derived equivalences. So, it seems to us interesting to look at what happens in the context of exact categories.

The aim of this paper is to generalise the stratifying systems to exact categories in order to produce new stratifying systems and as a result a new method to construct standardly stratified algebras. Our approach gives us more flexibility since we can change the canonical exact structure in the category of modules and obtain new exact categories where our approach works. We have preferred to give direct proofs to our results without using any embedding into abelian categories since we think that is more illustrative.

2. Preliminaries. In this paper, we work on an exact category in the sense of Quillen [**28**]. We will work with the axioms given by B. Keller in [**17**, Appendix A], which are equivalent to that given by Quillen in the case of an additive category with splitting idempotents (see apenddix of [**12**]). For the convenience of the reader, we recall the basic definitions. In all that follows, $\mathcal{A}$ stands for an additive category. It is said that $\mathcal{A}$ is a category with *splitting idempotents* if for each idempotent $e = e^2 \in \mathrm{Hom}_{\mathcal{A}}(X, X)$ there are morphisms $\mu : Y \longrightarrow X$ and $\rho : X \longrightarrow Y$, such that $\mu\rho = e$ and $\rho\mu = 1_Y$.

Definition 2.1 [**17**, Appendix A]. A pair (i, d) of composable morphisms $X \xrightarrow{i} Y \xrightarrow{d} Z$ in $\mathcal{A}$ is an *exact pair*, if i is the kernel of d and d is the cokernel of i.

Definition 2.2 [**17**, Appendix A]. Let $\mathcal{E}$ be a class of exact pairs $X \xrightarrow{i} Y \xrightarrow{d} Z$, which is closed under isomorphisms. The morphisms i and d appearing in a pair (i, d) in $\mathcal{E}$ are called an *inflation* and a *deflation* of $\mathcal{E}$, respectively. A pair $(i, d) \in \mathcal{E}$ is called a *conflation*. It is said that the class $\mathcal{E}$ is an *exact structure* on $\mathcal{A}$ and the pair $(\mathcal{A}, \mathcal{E})$ is an *exact category*, if the following axioms hold:

(E0) The identity morphism of the zero object $1_0 : 0 \longrightarrow 0$ is a deflation;
(E1) The composition of two deflations is a deflation;
(E2) For every morphism $f : Z' \longrightarrow Z$ and any deflation $d : Y \longrightarrow Z$, there is a pullback diagram

$$\begin{array}{ccc} Y' & \xrightarrow{d'} & Z' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{d} & Z, \end{array}$$

such that d' is a deflation.

(E2)^{op} For every morphism $f : X \longrightarrow X'$ and any inflation $i : X \longrightarrow Y$, there is a pushout diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ X' & \xrightarrow{i'} & Y', \end{array}$$

such that i' is an inflation.

REMARK 2.3. Let $\eta : X \xrightarrow{i} Y \xrightarrow{d} Z$ be a conflation. If the second square in the following commutative diagram is a pullback in $\mathcal{A}$,

$$\begin{array}{ccccc} X & \xrightarrow{i'} & Y' & \xrightarrow{d'} & Z' \\ \| & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{i} & Y & \xrightarrow{d} & Z \end{array}$$

then $\eta' : X \xrightarrow{i'} Y' \xrightarrow{d'} Z'$ is a conflation, which will be denoted by ηf. Dually, if $h : X \longrightarrow X'$ is a morphism in $\mathcal{A}$, $h\eta$ will denote the conflation obtained by a pushout of η with h.

3. Relative theory produced by subfunctors of $\mathrm{Ext}_{\mathcal{E}}(-,-)$**.** Throughout the remaining of this paper we fix an exact category $(\mathcal{A}, \mathcal{E})$ with splitting idempotents. For given objects A, $C \in \mathcal{A}$, we denote by $\mathrm{Ext}_{\mathcal{E}}(C, A)$ the set of all exact pairs $A \xrightarrow{i} B \xrightarrow{d} C$ in $\mathcal{E}$ modulo, the usual equivalence relation. That is, two conflations (i, d) and (i', d') are equivalent if there is a commutative diagram as follows:

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{d} & C \\ \| & & \downarrow{\scriptstyle f} & & \| \\ A & \xrightarrow{i'} & B' & \xrightarrow{d'} & C. \end{array}$$

REMARK 3.1.

(a) It can be shown that, in the above diagram, the morphism f is an isomorphism (see [**12**, Appendix]).

(b) If $\eta : A \xrightarrow{i} B \xrightarrow{d} C$ and $\eta' : A' \xrightarrow{i'} B' \xrightarrow{d'} C'$ are conflations, then $\eta \oplus \eta' : A \oplus A' \xrightarrow{i \oplus i'} B \oplus B' \xrightarrow{d \oplus d'} C \oplus C'$ is a conflation.

(c) Since an additive category has finite direct sums, there exists the diagonal and codiagonal maps.

Using the facts that in an exact category $(\mathcal{A}, \mathcal{E})$, the pullback and pushout of a conflation is a conflation (that is, E2 and E2^{op}) and the remark above, it can be proved that the set $\mathrm{Ext}_{\mathcal{E}}(C, A)$ is an abelian group under Baer's sum.

Let $\alpha : C' \longrightarrow C$ and $\beta : A \longrightarrow A'$ be morphisms in $\mathcal{A}$. So, we have a map $\mathrm{Ext}_{\mathcal{E}}(\alpha, \beta) : \mathrm{Ext}_{\mathcal{E}}(C, A) \longrightarrow \mathrm{Ext}_{\mathcal{E}}(C', A')$ given by $\mathrm{Ext}_{\mathcal{E}}(\alpha, \beta)(x) = \beta(x\alpha)$. Observe that

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$\beta(x\alpha) = (\beta x)\alpha$, and furthermore, we have a functor $\mathrm{Ext}_{\mathcal{E}}(-,-) : \mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathrm{Ab}$, where Ab stands for the category of abelian groups.

REMARK 3.2. The functor $\mathrm{Ext}_{\mathcal{E}}(-,-) : \mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathrm{Ab}$ is additive.

DEFINITION 3.3 [**12**]. Let F be a subfunctor of $\mathrm{Ext}_{\mathcal{E}}(-,-) : \mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathrm{Ab}$ (that is, $F(C, A)$ is a subgroup of $\mathrm{Ext}_{\mathcal{E}}(C, A)$, $\forall A, C \in \mathcal{A}$). A conflation $\eta : A \xrightarrow{i} B \xrightarrow{d} C$ is said to be an *F-exact pair* if $\eta \in F(C, A)$. In this case, the map d is said to be an *F-epimorphism* and i is an *F-monomorphism*. The class of all the F exact pairs is denoted by $\mathcal{E}_F$. Observe that $\mathcal{E}_F \subseteq \mathcal{E}$ and $\mathcal{E}_F$ is closed under isomorphisms.

F will always denote a subfunctor of $\mathrm{Ext}_{\mathcal{E}}(-,-)$, where $(\mathcal{A}, \mathcal{E})$ is an exact category with splitting idempotents.

REMARK 3.4. Let F be a subfunctor of $\mathrm{Ext}_{\mathcal{E}}(-,-) : \mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathrm{Ab}$ and $\alpha : C' \longrightarrow C$, $\beta : A \longrightarrow A'$ in $\mathcal{A}$. If $\eta \in F(C, A)$, then $F(\alpha, \beta)(\eta) = \beta(\eta\alpha) \in F(C', A')$. In particular, the class of all the F-exact pairs is closed under pullbacks and pushouts. That is, if η is F-exact, then $\eta\alpha$ and $\beta\eta$ are F-exact.

Usually, for simplicity, we shall denote the abelian group $\mathrm{Hom}_{\mathcal{A}}(A, C)$ by ${}_{\mathcal{A}}(A, C)$.

LEMMA 3.5. *Let F be an additive subfunctor of* $\mathrm{Ext}_{\mathcal{E}}(-,-) : \mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathrm{Ab}$ *and let $\eta : A \xrightarrow{i} B \xrightarrow{d} C$ be an F-exact pair. Then, for each $C' \in \mathcal{A}$, there is an exact sequence in* Ab

$$0 \to {}_{\mathcal{A}}(C', A) \to {}_{\mathcal{A}}(C', B) \to {}_{\mathcal{A}}(C', C) \xrightarrow{\delta} F(C', A) \xrightarrow{F(1_{C'}, i)} F(C', B) \to$$

$$\xrightarrow{F(1_{C'}, d)} F(C', C),$$

where $\delta(f) := \eta f$.

Moreover, if $F = \mathrm{Ext}_{\mathcal{E}}(-,-)$, then the above long exact sequence, can be extended to the next group, namely, $\mathrm{Ext}_{\mathcal{E}}(C', C)$.

Proof. The well-known proof for the classical situation can be adapted to this setting. □

DEFINITION 3.6. Let $T : \mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathrm{Ab}$ be an additive functor and let F be an additive subfunctor of $\mathrm{Ext}_{\mathcal{E}}(-,-)$. It is said that

(a) T is $\mathcal{E}_F$*-closed on the right,* if for every $A \in \mathcal{A}$ and any exact pair $X \xrightarrow{i} Y \xrightarrow{d} Z$ in $\mathcal{E}_F$, the sequence $T(A, X) \xrightarrow{T(A,i)} T(A, Y) \xrightarrow{T(A,d)} T(A, Z)$ is exact in Ab.

(b) T is $\mathcal{E}_F$*-closed on the left,* if for every $A \in \mathcal{A}$ and any exact pair $X \xrightarrow{i} Y \xrightarrow{d} Z$ in $\mathcal{E}_F$, the sequence $T(Z, A) \xrightarrow{T(d,A)} T(Y, A) \xrightarrow{T(i,A)} T(X, A)$ is exact in Ab.

(c) T is $\mathcal{E}_F$*-closed,* if it is $\mathcal{E}_F$-closed on the left and on the right.

In the case $F = \mathrm{Ext}_{\mathcal{E}}(-,-)$, we said that T is $\mathcal{E}$-closed on the left (resp. right).

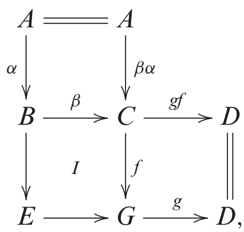
REMARK 3.7. We have preferred to call a functor $T : \mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathrm{Ab}$ a functor $\mathcal{E}_F$-closed, instead of $\mathcal{E}_F$-exact (that would be more natural), because $\mathcal{E}_F$ is not necessarily an exact structure on $\mathcal{A}$. Moreover, this definition generalises the one given in [**7**], that is, from the abelian to exact categories. Next, we will see when $\mathcal{E}_F$ is an exact structure on $\mathcal{A}$.

Proposition 3.8 [**12**, Proposition 1.4]. *Let F be an additive subfunctor of* $\mathrm{Ext}_{\mathcal{E}}(-,-) : \mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathrm{Ab}$. *Then, the following conditions are equivalent:*

(*a*) *The class $\mathcal{E}_F$ is an exact structure on $\mathcal{A}$.*
(*b*) *F is $\mathcal{E}_F$-closed on the left.*
(*c*) *F is $\mathcal{E}_F$-closed on the right.*
(*d*) *F is $\mathcal{E}_F$-closed.*

In the next sections, we will need the following useful lemmas.

Proposition 3.9. *Let F be an additive $\mathcal{E}_F$-closed subfunctor of* $\mathrm{Ext}_{\mathcal{E}}(-,-)$ *and let* $A \overset{\alpha}{\to} B \to E$ *and* $B \overset{\beta}{\to} C \to D$ *be F-exact pairs. Then, the above F-exact pairs can be completed to the following commutative diagram*



such that the squared marked with I is both a pushout and a pullback and all the rows and columns are F-exact pairs.

Proof. By [**17**, page 28 in Appendix A] and [**12**, page 31 in Appendix], it follows that I is a pullback and a pushout diagram. Since $\beta\alpha$ is an inflation, we conclude by 3.4 that all the rows and columns are F-exact. □

Lemma 3.10 [**12**, Lemma 1.2]. *Let F be a subfunctor of* $\mathrm{Ext}_{\mathcal{E}}(-,-) : \mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathrm{Sets}$. *Then, F is an additive subfunctor of* $\mathrm{Ext}_{\mathcal{E}} : \mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathrm{Ab}$, *if and only if the class of F-exact pairs $\mathcal{E}_F$ is closed under finite direct sums.*

Definition 3.11. Let $\mathcal{X}$ be a class of objects in the exact category $\mathcal{A}$. A morphism $f : X \to C$ in $\mathcal{A}$ is said to be an *$\mathcal{X}$-precover* of C if $X \in \mathcal{X}$ and $\mathrm{Hom}_{\mathcal{A}}(X', f) : \mathrm{Hom}_{\mathcal{A}}(X', X) \to \mathrm{Hom}_{\mathcal{A}}(X', C)$ is surjective $\forall X' \in \mathcal{X}$. An $\mathcal{X}$-precover $f : X \to C$ is a *cover* if whenever $fh = f$ we have that h is an automorphism.

We recall that an $\mathcal{X}$-precover is also known as a right $\mathcal{X}$-approximation and the notion of $\mathcal{X}$-cover as a minimal right $\mathcal{X}$-approximation.

In what follows, we recall some notions and elementary well-known facts about standardly stratified algebras. Let Λ be an artin R-algebra and $\mathrm{Mod}(\Lambda)$ be the category of all left Λ-modules. We denote by $\mathrm{mod}(\Lambda)$ the full subcategory of all finitely generated left Λ-modules, and by $\mathrm{proj}(\Lambda)$ the full subcategory of $\mathrm{mod}(\Lambda)$, whose objects are the projective Λ-modules. For $M, N \in \mathrm{mod}(\Lambda)$, the *trace* $\mathrm{Tr}_M(N)$ of M in N, is the Λ-submodule of N generated by the images of all the morphisms from M to N.

We next recall the definition (see [**1**, **10**, **11**, **29**]) of the class of the standard Λ-modules. Let n be the rank of the Grothendieck group $\mathrm{K}_0(\Lambda)$. We fix a linear order $\leq$ on the set $[1, n]$ and a representative set ${}_\Lambda P = \{{}_\Lambda P(i) : i \in [1, n]\}$ containing one module of each iso-class of indecomposable projective Λ-modules. The set of *standard Λ-modules* is ${}_\Lambda\Delta = \{{}_\Lambda\Delta(i) : i \in [1, n]\}$, where ${}_\Lambda\Delta(i) = {}_\Lambda P(i)/\mathrm{Tr}_{\oplus_{j>i} {}_\Lambda P(j)}({}_\Lambda P(i))$. Note that ${}_\Lambda\Delta(i)$ is the largest factor module of ${}_\Lambda P(i)$ with composition factors only amongst ${}_\Lambda S(j)$ for $j \leq i$.

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Let $\mathscr{F}({}_\Lambda\Delta)$ be the subcategory of $\mathrm{mod}(\Lambda)$ consisting of the Λ-modules having a ${}_\Lambda\Delta$-filtration, that is, a sequence of submodules $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_s = M$ with factors M_{i+1}/M_i isomorphic to a module in ${}_\Lambda\Delta$ for all i. The algebra Λ is said to be a *standarly stratified algebra*, with respect to the linear order $\leq$ on the set $[1, n]$, if $\mathrm{proj}(\Lambda) \subseteq \mathscr{F}({}_\Lambda\Delta)$ (see [**1**, **9**, **10**]). A *quasi-hereditary algebra* is a standardly stratified algebra $(\Lambda, \leq)$, such that $\mathrm{End}({}_\Lambda\Delta(i))$ is a division ring, for each $i \in [1, n]$.

4. Exact *R*-categories. Let R be a commutative ring. We recall that an *R-category* is a category $\mathcal{C}$ satisfying the following two conditions: (a) for each pair X, Y of objects in $\mathcal{C}$ the set of morphisms $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an R-module, and (b) the composition of morphisms is R-bilinear. An R-category $\mathcal{C}$ is called *Hom-finite* if $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a finitely generated R-module, for each $X, Y \in \mathcal{C}$.

A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between two R-categories, is said to be an *R-functor* if $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is a morphism of R-modules for each pair X, Y of objects in $\mathcal{C}$.

Let $\mathcal{C}$ be an additive category. It is said that $\mathcal{C}$ is *Krull–Schmidt* if any object $X \in \mathcal{C}$ has a finite decomposition $X = \oplus_{i=1}^{n} X_i$ satisfying that each X_i is indecomposable with local endomorphism ring $\mathrm{End}_{\mathcal{C}}(X_i)$.

Definition 4.1. An *artin exact R-category* is an R-category $\mathcal{C}$, for some artinian ring R, which is Hom-finite and Krull–Schmidt and exact for some structure $\mathcal{E}$

Let Λ be an artin R-algebra. It is well known that $\mathrm{mod}(\Lambda)$ is an artin exact R-category, by considering the class of all the exact sequences in $\mathrm{mod}(\Lambda)$.

Proposition 4.2. *Let $\mathcal{A}$ be an artin R-category, $A \in \mathcal{A}$, $\Gamma := \mathrm{End}_{\mathcal{A}}(A)^{op}$ and the evaluation functor at A, $e_A := \mathrm{Hom}_{\mathcal{T}}(A, -) : \mathcal{A} \longrightarrow \mathrm{Mod}(\Gamma)$. Then, the following conditions hold:*

(*a*) *Γ is an artin R-algebra.*

(*b*) $\mathrm{Im}(e_A) \subseteq \mathrm{mod}(\Gamma)$, *then* $e_A := \mathrm{Hom}_{\mathcal{T}}(A, -) : \mathcal{A} \longrightarrow \mathrm{mod}(\Gamma)$ *and induces an equivalence of categories* $\mathrm{add}(A) \xrightarrow{\simeq} \mathrm{proj}(\Gamma)$.

(*c*) $e_A : \mathrm{Hom}_{\mathcal{A}}(Z, X) \longrightarrow \mathrm{Hom}_{\Gamma}(e_A(Z), e_A(X))$ *is an isomorphism of R-modules,* $\forall Z \in \mathrm{add}(A)$ *and* $\forall X \in \mathcal{A}$.

Proof. The proof done by M. Auslander (see [**3**]), can be easily extended to the context of an artin R-category. □

Note that item (b) of the previous proposition is false if $\mathcal{A}$ is not an artin R-category. Indeed, taking $\mathcal{A} = \mathrm{Mod}(\mathbb{Z})$ and $A = \mathbb{Z}$ we have that $\mathrm{End}_{\mathcal{A}}(\mathbb{Z}, \mathbb{Z}) \simeq \mathbb{Z}$ and the functor $e_{\mathbb{Z}}$ does not satisfy that $\mathrm{Im}(e_{\mathbb{Z}}) \subseteq \mathrm{mod}(\mathbb{Z})$.

Definition 4.3. Let $(\mathcal{A}, \mathcal{E})$ be an artin exact R-category. $\mathcal{A}$ is said to be *$\mathrm{Ext}_{\mathcal{E}}$-finite* if $\mathrm{Ext}_{\mathcal{E}}(C, A)$ is a finitely generated R-module for all $A, C \in \mathcal{A}$. In this setting, an additive subfunctor F of $\mathrm{Ext}_{\mathcal{E}}(-, -)$ is an *R-subfunctor* if $F(C, A)$ is an R-submodule of $\mathrm{Ext}_{\mathcal{E}}(C, A)$, $\forall C, A \in \mathcal{A}$.

Let Λ be an artin R-algebra. It is well known that $\mathrm{mod}(\Lambda)$ is an artin exact R-category which is $\mathrm{Ext}_{\Lambda}(-, -)$-finite; and moreover, every additive subfunctor of $\mathrm{Ext}_{\Lambda}(-, -)$ is an R-subfunctor. The R-module structure on $\mathrm{Ext}_{\Lambda}(M, N)$ is defined as follows: Let $r \in R$, $\eta \in \mathrm{Ext}_{\Lambda}(M, N)$ and set $f_r : M \longrightarrow M$ the morphism given by $f_r(m) = rm$, define $r\eta := \eta f_r = \mathrm{Ext}(f_r, N)(\eta)$.

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LEMMA 4.4. *Let* $(\mathcal{A}, \mathcal{E})$ *be an artin exact* R*-category, which is* $\mathrm{Ext}_{\mathcal{E}}$*-finite and let* F *be an* $\mathcal{E}_F$*-closed* R*-subfunctor of* $\mathrm{Ext}_{\mathcal{E}}(-,-) : \mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathrm{Ab}$. *Then,* $F(C, A)$ *is a finitely generated* R*-module for all* $A, C \in \mathcal{A}$.

Proof. Since R is artinian, it is also Noetherian, and then every submodule of a finitely generated module is finitely generated. □

The following construction is usually called the universal extension.

LEMMA 4.5. *Let* $(\mathcal{A}, \mathcal{E})$ *be an artin exact* R*-category, which is* $Ext_{\mathcal{E}}$*-finite and let* F *be an* $\mathcal{E}_F$*-closed* R*-subfunctor of* $\mathrm{Ext}_{\mathcal{E}}(-,-)$. *Then, for any* $A, C \in \mathrm{Obj}(\mathcal{A})$, *such that* $F(C, A) \neq 0$, *the following conditions hold:*

(*a*) *There is a non-splitting* F*-exact pair in* $\mathcal{A}$

$$\eta_{C,A} : \quad A^n \xrightarrow{f} E \xrightarrow{g} C,$$

such that $\delta : \mathrm{Hom}_{\mathcal{A}}(A^n, A) \longrightarrow F(C, A)$ *is surjective, where* $n := \ell_R(F(C, A))$.

(*b*) *If* $F(A, A) = 0$, *then* $F(E, A) = 0$, *where* E *is defined in item (a).*

Proof. The proof given in [**25**, Lemma 3.4] can be adapted to the context of exact categories. □

LEMMA 4.6. *Let* $(\mathcal{A}, \mathcal{E})$ *be an artin exact* R*-category,* F *an additive* $\mathcal{E}_F$*-closed* R*-subfunctor of* $\mathrm{Ext}_{\mathcal{E}}(-,-)$ *and let* $\eta : A \xrightarrow{s} B \xrightarrow{g} C$ *be a non-splitting* F*-exact pair, such that* C *is an indecomposable object and* $\mathrm{Hom}_{\mathcal{A}}(A, C) = 0$. *Then, there exists a non-splitting* F*-exact pair* $A' \longrightarrow B' \longrightarrow C$, *such that* A' *is a direct summand of* A *and* B' *is an indecomposable direct summand of* B.

Proof. The proof given in [**25**, Proposition 3.5] can be adapted to the context of exact categories. □

5. Filtered objects in an exact category. Let $\mathcal{C}$ be a class of objects in $\mathcal{A}$ and let F be an additive subfunctor of $\mathrm{Ext}_{\mathcal{E}}(-,-)$. It is said that an object $M \in \mathcal{A}$ admits an $F-$*filtration* in $\mathcal{C}$, if there is a family of F-exact pairs $\{\xi_i : M_{i-1} \to M_i \to X_i\}_{i=1}^{m}$, such that $M_0 = 0$, $M_m = M$ and $X_i \in \mathcal{C}$ for $i = 1, \ldots, m$. We denote by $\mathscr{F}_F(\mathcal{C})$ the class of objects M in $\mathcal{A}$, which admits an F-filtration by objects in $\mathcal{C}$.

Let $M \in \mathscr{F}_F(\mathcal{C})$ and let $\xi : \{\xi_i : M_{i-1} \to M_i \to X_i\}_{i=1}^{m}$ be an F-filtration in $\mathcal{C}$ of M. In this case, the F-*length relative* to ξ of M is $\ell_{F,\xi}(M) := m$ and the F-**length** of M is $\ell_F(M) := \min\{\ell_{F,\xi}(M) \mid \xi \text{ is a F-filtration of } M\}$.

We have the F-*left perpendicular* class of $\mathcal{C}$ to be

$${}^{F\perp}\mathcal{C} := \{X \in \mathcal{A} \mid F(X, C) = 0 \quad \forall\, C \in \mathcal{C}\}.$$

We also recall that a class $\mathcal{C} \subseteq \mathcal{A}$ is *closed under* F*-extensions,* if for every F-exact pair $A \xrightarrow{\alpha} B \xrightarrow{\alpha'} C$ with $A, C \in \mathcal{C}$, we have that $B \in \mathcal{C}$.

PROPOSITION 5.1. *Let* $\mathcal{C}$ *be a class of objects in an exact category* $(\mathcal{A}, \mathcal{E})$ *and let* F *be an additive* $\mathcal{E}_F$*-closed subfunctor of* $\mathrm{Ext}_{\mathcal{E}}(-,-)$. *Then, the class* $\mathscr{F}_F(\mathcal{C})$ *is closed under* F*-extensions. In particular, it is closed under finite direct sums.*

Proof. Let $A \longrightarrow B \longrightarrow C$ be an F-exact pair with A, $C \in \mathscr{F}_F(\mathcal{C})$. Proceeding by induction on $n = \ell_F(C)$, it can be shown that if A and C belong to $\mathscr{F}_F(\mathcal{C})$, then $B \in \mathscr{F}_F(\mathcal{C})$.

Now, we check that $\mathscr{F}_F(\mathcal{C})$ is closed under finite direct sums. Let $A, B \in \mathscr{F}_F(\mathcal{C})$. We have the following F-exact pair $A \longrightarrow A \oplus B \longrightarrow B$ since $F(B, A)$ contains the splitting conflations. Finally, using that $\mathscr{F}_F(\mathcal{C})$ is closed under F-extensions, we have that $A \oplus B \in \mathscr{F}_F(\mathcal{C})$. □

COROLLARY 5.2. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category, F be an additive $\mathcal{E}_F$-closed subfunctor of* $\mathrm{Ext}_{\mathcal{E}}(-,-)$ *and let $\mathcal{C} \subseteq \mathcal{A}$ be a class of objects in $\mathcal{A}$. Then, ${}^{F\perp}\mathcal{C} = {}^{F\perp}(\mathscr{F}_F(\mathcal{C}))$.*

Proof. It is enough to prove that ${}^{F\perp}\mathcal{C} \subseteq {}^{F\perp}(\mathscr{F}_F(\mathcal{C}))$ since the other inclusion follows from the fact that $\mathcal{C} \subseteq \mathscr{F}_F(\mathcal{C})$. Let $Y \in {}^{F\perp}\mathcal{C}$ and $Z \in \mathscr{F}_F(\mathcal{C})$. We have that $F(Y, Z) = 0$ since $F(Y, -)|_{\mathcal{C}} = 0$. Then, ${}^{F\perp}\mathcal{C} = {}^{F\perp}(\mathscr{F}_F(\mathcal{C}))$. □

LEMMA 5.3. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category and F an additive $\mathcal{E}_F$-closed subfunctor of* $\mathrm{Ext}_{\mathcal{E}}(-,-)$*. If there exists two F-exact pairs of the form $Z \xrightarrow{i} Y \xrightarrow{d} \theta_1$ and $Y \xrightarrow{i'} X \xrightarrow{d'} \theta_2$, such that $F(\theta_2, \theta_1) = 0$, then there exists two F-exact pairs of the form $Z \longrightarrow W \longrightarrow \theta_2$, $W \longrightarrow X \longrightarrow \theta_1$, such that the composition $Z \longrightarrow Y \longrightarrow X$ is equal to the composition $Z \longrightarrow W \longrightarrow X$.*

Proof. By 3.9, we have the following commutative diagram in $\mathcal{A}$

$$
\begin{array}{ccccc}
Z & = & Z & & \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'i} & & \\
Y & \xrightarrow{i'} & X & \xrightarrow{d'} & \theta_2 \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle \beta} & & \| \\
\theta_1 & \xrightarrow{\alpha_1} & C & \xrightarrow{\alpha_2} & \theta_2,
\end{array}
$$

where the rows and columns are F-exact pairs. Moreover, the F-exact pair $\eta' : \theta_1 \xrightarrow{\alpha_1} C \xrightarrow{\alpha_2} \theta_2$ splits since $F(\theta_2, \theta_1) = 0$. Then, we have the following splitting F-exact pair $\xi : \theta_2 \xrightarrow{\beta_2} C \xrightarrow{\beta_1} \theta_1$. By the dual of 3.9, we have the following commutative diagram in $\mathcal{A}$

$$
\begin{array}{ccccc}
Z & \longrightarrow & W & \longrightarrow & \theta_2 \\
\| & & \downarrow & & \downarrow{\scriptstyle \beta_2} \\
Z & \xrightarrow{i'i} & X & \xrightarrow{\beta} & C \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \beta_1} \\
 & & \theta_1 & = & \theta_1,
\end{array}
$$

where the rows and columns are F-exact pairs. Therefore, the required F-exact pairs are $\xi' : Z \longrightarrow W \longrightarrow \theta_2$ and $\xi'' : W \longrightarrow X \longrightarrow \theta_1$. Finally, the equality of the compositions follows from the above diagrams. □

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Let $\Theta = \{\Theta(i)\}_{i=1}^{t}$ be a family of objects in $\mathcal{A}$. For a given F-filtration $\xi = \{\xi_k : M_{k-1} \longrightarrow M_k \longrightarrow X_k\}_{k=1}^{n}$ of $M \in \mathscr{F}_F(\Theta)$, we denote by $[M : \Theta(i)]_\xi$ the ξ-*filtration multiplicity* of $\Theta(i)$ in M, that is, the cardinally of the set $\{k \in \mathbb{N} \mid X_k \simeq \Theta(i)\}$. In general, the filtration multiplicity could be depending on a given F-filtration. We set $\ell_{F,\xi}(M) = \sum_{k=1}^{t}[M : \Theta(i)]_\xi$.

Proposition 5.4. *Let F be an additive $\mathcal{E}_F$-closed subfunctor of $\mathrm{Ext}_{\mathcal{E}}(-,-)$ and let $\Theta = \{\Theta(i)\}_{i=1}^{t}$ be a family of objects in an exact category $(\mathcal{A}, \mathcal{E})$, such that $F(\Theta(j), \Theta(i)) = 0\ \forall j \geq i$. If ξ is an F-filtration of $M \in \mathscr{F}_F(\Theta)$, then there exists an F-filtration η of M in Θ and a family Ξ of F-exact pairs satisfying the following conditions:*

(*a*) *$m(i) := [M : \Theta(i)]_\xi = [M : \Theta(i)]_\eta$ for all $i \in [1, t]$,*

(*b*) *The F-filtration η is ordered; that is,*

$$\eta = \{\eta_i : M_{i-1} \longrightarrow M_i \longrightarrow \Theta(k_i)\}_{i=1}^{n}$$

with $M_0 := 0$ and $k_n \leq k_{n-1} \leq \cdots \leq k_1$.

(*c*) *$\Xi = \{\Xi_i : M'_{i-1} \longrightarrow M'_i \longrightarrow \Theta(\lambda_i)^{m(\lambda_i)}\}_{i=1}^{d}$, where $\{\Theta(\lambda_i)\}_{i=1}^{d}$ is the set consisting of the different $\Theta(j)$ appearing in the F-filtration ξ of M. Moreover, $M'_0 = 0$, $M'_d = M$ and $\lambda_d < \lambda_{d-1} < \cdots < \lambda_1$.*

Proof. Let ξ be an F-filtration of $M \in \mathscr{F}_F(\Theta)$. We can assume that $M \neq 0$.

We can prove (a) and (b), proceeding by induction on $n := \ell_{F,\xi}(M)$ and using 5.3 to rearrange the exact pairs obtained.

(c) follows from (b) by regrouping the k_i that are the same and rename them by λ_i, and using 3.9 in order to get the family of F-exact sequences. □

Definition 5.5. Let $(\mathcal{A}, \mathcal{E})$ be an exact category, F be an additive $\mathcal{E}_F$-closed subfunctor of $\mathrm{Ext}_{\mathcal{E}}(-,-)$ and let $\Theta = \{\Theta(i)\}_{i=1}^{n}$ be a family of objects in $\mathcal{A}$. The class of Θ-*projective* objects in $\mathcal{A}$ is the class $\mathcal{P}(\Theta) := {}^{F\perp}(\mathscr{F}(\Theta))$. Dually, the class of Θ-*injective* objects in $\mathcal{A}$ is the class $\mathcal{I}(\Theta) := (\mathscr{F}(\Theta))^{F\perp}$.

Note that by 5.2 and its dual, we have that $\mathcal{P}(\Theta) = {}^{F\perp}\Theta$ and $\mathcal{I}(\Theta) = \Theta^{F\perp}$.

6. Stratifying Systems.

In this section, we generalise the notion of stratifying systems [**20**, **21**, **26**], to the context of an artin exact R-category $(\mathcal{A}, \mathcal{E})$. In all that follows, we will consider an artin exact R-category $(\mathcal{A}, \mathcal{E})$ and also we fix F, an additive $\mathcal{E}_F$-closed R-subfunctor of $\mathrm{Ext}_{\mathcal{E}}(-,-)$.

Definition 6.1. An F-*system* Θ of size t in $(\mathcal{A}, \mathcal{E})$, consists of the following data.

(S1) $\Theta = \{\Theta(i)\}_{i=1}^{t}$ is a family of indecomposable objects in $\mathcal{A}$.

(S2) $\mathrm{Hom}_{\mathcal{A}}(\Theta(j), \Theta(i)) = 0$ for $j > i$.

(S3) $F(\Theta(j), \Theta(i)) = 0$ for $j \geq i$.

Definition 6.2. An F-*projective system* $(\Theta, \mathbf{Q})$ of size t in $(\mathcal{A}, \mathcal{E})$, consists of the following data:

(PS1) $\Theta = \{\Theta(i)\}_{i=1}^{t}$ is a family of non-zero objects in $\mathcal{A}$.

(PS2) $\mathrm{Hom}_{\mathcal{A}}(\Theta(j), \Theta(i)) = 0$, for $j > i$.

(PS3) $\mathbf{Q} = \{Q(i)\}_{i=1}^{t}$ is a family of indecomposable objects in $\mathcal{A}$, such that $Q := \bigoplus_{i=1}^{t} Q(i) \in {}^{F\perp}\Theta = \mathcal{P}(\Theta)$.

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(PS4) For every $i \in [1, t]$, there exists an F-exact pair in $\mathcal{A}$

$$K(i) \xrightarrow{\alpha_i} Q(i) \xrightarrow{\beta_i} \Theta(i),$$

such that $K(i) \in \mathscr{F}_F(\{\Theta(j) \mid j > i\})$.

REMARK 6.3. If $(\Theta, \mathbf{Q})$ is an F-projective system of size t in $(\mathcal{A}, \mathcal{E})$, then we have that $F(\mathbf{Q}, \mathscr{F}_F(\Theta)) = 0$.

The following lemma is straightforward.

LEMMA 6.4. *Let* $(\Theta, \mathbf{Q})$ *be an* F*-projective system of size* t *in* $(\mathcal{A}, \mathcal{E})$*. Then, the following conditions hold:*

(*a*) $\mathrm{Hom}_{\mathcal{A}}(K(j), \Theta(i)) = 0$ *for* $j \geq i$.

(*b*) *For all* $j \geq i$*, the morphism*

$$\mathrm{Hom}_{\mathcal{A}}(\beta_j, \Theta(i)) : \mathrm{Hom}_{\mathcal{A}}(\Theta(j), \Theta(i)) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(Q(j), \Theta(i))$$

is an isomorphism of abelian groups, and moreover $F(\Theta(j), \Theta(i)) = 0$.

(*c*) *The* R*-functor* $\mathrm{Hom}_{\mathcal{A}}(Q', -) : \mathscr{F}_F(\Theta) \longrightarrow \mathrm{mod}(\mathrm{End}_{\mathcal{A}}(Q)^{op})$ *is exact for any* $Q' \in \mathrm{add}(Q)$.

PROPOSITION 6.5. *Let* $(\Theta, \mathbf{Q})$ *be an* F*-projective system of size* t *in* $(\mathcal{A}, \mathcal{E})$*. Then, the following conditions hold:*

(*a*) *For any* $M \in \mathscr{F}_F(\Theta)$*, the filtration multiplicity* $[M : \Theta(i)]_\xi$ *of* $\Theta(i)$ *in* M *does not depend on the given* F*-filtration* ξ *of* M*, and hence it will be denoted by* $[M : \Theta(i)]$*. In particular, the length* $\ell_\Theta = \sum_{i=1}^{t}[M : \Theta(i)]$ *is well defined.*

(*b*) $Q(i) \not\simeq Q(j)$*, if* $i \neq j$.

Proof.

(a) Consider an F-filtration of $M \in \mathscr{F}_F(\Theta)$, $\{\xi_k : M_{k-1} \to M_k \to \Theta(j_k)\}_{k=1}^{m}$ with $M_0 = 0$ and $M_m = M$. Applying $\mathrm{Hom}_{\mathcal{A}}(Q(i), -)$ to each F-exact pair ξ_j and by setting $\langle X, Y\rangle := \ell_R(\mathrm{Hom}_{\mathcal{A}}(X, Y))$, we get that $c_i := \langle Q(i), M\rangle = \sum_{j=1}^{t}[M : \Theta(j)]_\xi \langle Q(i), \Theta(j)\rangle$. Consider the matrix $D := (d_{ij})$, where $d_{ij} := \langle Q(i), \Theta(j)\rangle$. By 6.4 (b) and the condition (PS2) of 6.2, we have that D is an upper triangular matrix with $d_{ii} \neq 0$ for all i, and thus $\det(D) \neq 0$. By using the column vectors $X := ([M : \Theta(1)]_\xi, [M : \Theta(2)]_\xi, \ldots, [M : \Theta(t)]_\xi)^t$ and $C := (c_1, c_2, \ldots, c_t)^t$ the above equalities can be written as a matrix equation $D \cdot X = C$. Since $\det(D) \neq 0$, we obtain that $X = D^{-1} \cdot C$, and hence $[M : \Theta(j)]_\xi$ only depends on the numbers $c_i = \langle Q(i), M\rangle$ and $d_{i,j} = \langle Q(i), \Theta(j)\rangle$.

(b) Let $i \neq j$. We can assume that $j > i$. By (a) and the condition (PS4) of 6.2, it follows that $[Q(i) : \Theta(i)] = 1$ and $[Q(j) : \Theta(i)] = 0$, and thus $Q(i) \not\simeq Q(j)$. □

PROPOSITION 6.6. *Let* $(\Theta, \mathbf{Q})$ *be an* F*-projective system of size* t *in* $(\mathcal{A}, \mathcal{E})$*. Then, the following statements hold:*

(*a*) *For each* $i \in [1, t]$*, the morphism* $\beta_i : Q(i) \longrightarrow \Theta(i)$*, appearing in the* F*-exact pair* η_i *from the condition (PS4) of 6.2, is a* $\mathcal{P}(\Theta)$*-cover of* $\Theta(i)$.

(*b*) *Let* $(\Theta, \mathbf{Q}')$ *be another* F*-projective system of size* t *in* $\mathcal{A}$*. Then,* $\mathbf{Q}' \simeq \mathbf{Q}$.

Proof. The same proof given in [**25**, Proposition 5.6] □

Let Θ be an F-system in $(\mathcal{A}, \mathcal{E})$. A natural question, here, is to ask for the existence of a family $\mathbf{Q} = \{Q(i)\}_{i=1}^{t}$ of objects in $\mathcal{A}$ such that $(\Theta, \mathbf{Q})$ is an F-projective system.

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In order to do that, we will need the following result, which is a generalisation of the process of standardisation due originally to Dlab and Ringel (see [**11**]).

THEOREM 6.7. *Let* $(\mathcal{A}, \mathcal{E})$ *be* Ext*-finite,* Θ *an* F*-system of size* $t > 1$ *in* $(\mathcal{A}, \mathcal{E})$ *and* $i \in [1, t]$*. Then, for each* $k \in [1, t-i]$ *there exists an* F*-exact pair in* $\mathcal{A}$

$$\xi_k : \quad V_k \longrightarrow U_k \longrightarrow \Theta(i),$$

satisfying the following conditions :

(*a*) U_k *is indecomposable,*

(*b*) $V_k \in \mathscr{F}_F(\{\Theta(j) \mid i < j \leq i+k\})$,

(*c*) $F(U_k, \Theta(j)) = 0$ *for* $j \in [i, i+k]$.

Proof. We will proceed by induction on k.

Let $k = 1$. By definition we have that $\mathrm{Hom}_{\mathcal{A}}(\Theta(i+1), \Theta(i)) = 0$. If $F(\Theta(i), \Theta(i+1)) = 0$, the desired F-exact pair is

$$0 \longrightarrow \Theta(i) \xrightarrow{1} \Theta(i).$$

Suppose that $F(\Theta(i), \Theta(i+1)) \neq 0$. By 4.5, there exists a non-splitting F-exact pair

$$\xi : \quad \Theta(i+1)^n \longrightarrow E \longrightarrow \Theta(i),$$

and moreover, we have that $F(E, \Theta(i+1)) = 0$. By applying $\mathrm{Hom}_{\mathcal{A}}(-, \Theta(i))$ to ξ, we get the exact sequence

$$F(\Theta(i), \Theta(i)) \longrightarrow F(E, \Theta(i)) \longrightarrow F(\Theta(i+1)^n, \Theta(i)).$$

Since $F(\Theta(i), \Theta(i)) = F(\Theta(i+1)^n, \Theta(i)) = 0$, we conclude that $F(E, \Theta(i)) = 0$. Moreover, since $\mathrm{Hom}_{\mathcal{A}}(\Theta(i+1)^n, \Theta(i)) = 0$, it follows by 4.6, the existence of an F-exact pair

$$\xi' : \Theta(i+1)^m \longrightarrow U_1 \longrightarrow \Theta(i)$$

with $m \leq n$ and U_1 an indecomposable direct summand of E. Thus, $\xi_1 := \xi'$ satisfies the required conditions. Suppose now that there exists an F-exact pair

$$\xi_k : \quad V_k \longrightarrow U_k \longrightarrow \Theta(i)$$

satisfying the above required properties. We will construct the F-exact pair ξ_{k+1} from ξ_k as follows.

If $F(U_k, \Theta(i+k+1)) = 0$, the F-exact pair $\xi_{k+1} := \xi_k$ is the desired one.

Suppose that $F(U_k, \Theta(i+k+1)) \neq 0$. By 4.5, there exists a non-splitting F-exact pair

$$\eta : \quad \Theta(i+k+1)^a \longrightarrow U \longrightarrow U_k$$

with $F(U, \Theta(i+k+1)) = 0$. Applying $\mathrm{Hom}_{\mathcal{A}}(-, \Theta(j))$ to η with $j \in [i, i+k+1]$, we can conclude that $F(U, \Theta(j)) = 0$ for $j \in [i, i+k+1]$. It follows that $\mathrm{Hom}_{\mathcal{A}}(\Theta(i+k+1)^a, U_k) = 0$ since $U_k \in \mathscr{F}_F(\{\Theta(j) \mid i \leq j \leq i+k\})$. Therefore, by 4.6, there exists an F-exact pair

$$\eta' : \quad \Theta(i+k+1)^d \longrightarrow U_{k+1} \longrightarrow U_k$$

with $d \leq a$ and U_{k+1} an indecomposable direct summand of U. By the dual of 3.9, we get the following commutative diagram in $\mathcal{A}$

$$\begin{array}{ccccc}
\Theta(i+k+1)^d & \longrightarrow & V_{k+1} & \longrightarrow & V_k \\
\| & & \downarrow{\scriptstyle \mu_{k+1}} & & \downarrow{\scriptstyle \mu_k} \\
\Theta(i+k+1)^d & \longrightarrow & U_{k+1} & \longrightarrow & U_k \\
 & & \downarrow & & \downarrow \\
 & & \Theta(i) & = & \Theta(i),
\end{array}$$

where the rows and columns are F-exact pairs. Using the fact that $V_k \in \mathscr{F}_F(\{\Theta(j) \mid i < j \leq i+k\})$ it follows by 5.1, that $V_{k+1} \in \mathscr{F}_F(\{\Theta(j) \mid i < j \leq i+k+1\})$. Moreover, $F(U_{k+1}, \Theta(j)) = 0$ for $j \in [i, i+k+1]$ since U_{k+1} is an indecomposable direct summand of U. Hence, the desired F-exact pair is the first column of the preceding diagram; that is, $\xi_{k+1} : V_{k+1} \xrightarrow{\mu_{k+1}} U_{k+1} \longrightarrow \Theta(i)$. □

COROLLARY 6.8. *Let $(\mathcal{A}, \mathcal{E})$ be Ext-finite and Θ an F-system of size t in $(\mathcal{A}, \mathcal{E})$. Then, there exists a unique, up to isomorphism, family $\boldsymbol{Q}$ of objects in $\mathcal{A}$ such that $(\Theta, \boldsymbol{Q})$ is an F-projective system of size t in $(\mathcal{A}, \mathcal{E})$.*

Proof. For each $i < t$, we set $\eta_i := \xi_{t-i}$, where ξ_{t-i} is the F-exact pair of 6.7,

$$\xi_{t-i} : \quad V_{t-i} \longrightarrow U_{t-i} \longrightarrow \Theta(i).$$

Let $K(i) := V_{t-i}$ and $Q(i) := U_{t-i}$. For $i = t$, we consider the F-exact pair $0 \longrightarrow \Theta(t) \xrightarrow{1} \Theta(t)$ and we set $Q(t) := \Theta(t)$ and $K(t) := 0$. Finally, the uniqueness follows from 6.6. □

DEFINITION 6.9. Let $(\Theta, \mathbf{Q})$ an F-projective system of size t in $(\mathcal{A}, \mathcal{E})$. The Θ-*support* of $M \in \mathscr{F}_F(\Theta)$ is the set $\mathrm{Supp}_\Theta(M) := \{i \in [1, t] \mid [M : \Theta(i)] \neq 0\}$.

For $0 \neq M \in \mathscr{F}_F(\Theta)$, let $\max(M)$ ($\min(M)$) denote the maximum (minimum) of $\mathrm{Supp}_\Theta(M)$.

THEOREM 6.10. *Let $(\Theta, \boldsymbol{Q})$ be an F-projective system of size t in $(\mathcal{A}, \mathcal{E})$, and $0 \neq M \in \mathscr{F}_F(\Theta)$ with $i := \min(M)$. Then, there exists an F-exact pair in $\mathcal{A}$,*

$$N \longrightarrow Q_0(M) \xrightarrow{\epsilon_M} M,$$

satisfying the following conditions:

(*a*) *$N \in \mathscr{F}_F(\Theta)$ and $Q_0(M) \in \mathrm{add}(\bigoplus_{j \geq i} Q(j))$,*
(*b*) *$\min(M) < \min(N)$ if $M \neq 0$,*
(*c*) *$\epsilon_M : Q_0(M) \longrightarrow M$ is a $\mathcal{P}(\Theta)$-precover of M.*

Proof. Let $M \neq 0$. By 6.4 (b) and 5.4 (c), there is an F-exact pair $N \xrightarrow{\varphi} M \xrightarrow{\psi} \Theta(i)^{m_i}$ with $N \in \mathscr{F}_F(\Theta)$ and $\min(M) < \min(N)$, where $m_i = [M : \Theta(i)]$. We will proceed by reverse induction on $i = \min(M)$. If $i = \min(M) = t$, we have that $N = 0$ and hence the desired F-exact pair is $\eta : 0 \longrightarrow \Theta(t)^{m_i} \xrightarrow{1} \Theta(t)^{m_i}$ since $Q(t) \simeq \Theta(t)$ and $\eta \in F(\Theta(t)^{m_i}, 0)$.

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Let $i = \min(M) < t$. If $N = 0$, we have that $M \simeq \Theta(i)^{m_i}$. By 3.10, 6.2, 5.1 and 6.6 (a), the following F-exact pair is the desired one

$$K(i)^{m_i} \longrightarrow Q(i)^{m_i} \longrightarrow \Theta(i)^{m_i}.$$

Suppose that $N \neq 0$. Since $\min(M) < \min(N)$, by induction, there exists an F-exact pair

$$N' \longrightarrow Q_0(N) \xrightarrow{\epsilon_N} N,$$

such that $k := \min(N) < \min(N') =: i'$, $Q_0(N) \in \mathrm{add}(\bigoplus_{j \geq k} Q(j))$ and $\epsilon_N : Q_0(N) \longrightarrow N$ is a $\mathcal{P}(\Theta)$-precover of N. Consider the following commutative diagram in $\mathcal{A}$

$$\begin{array}{ccccc}
 & & K(i)^{m_i} & = & K(i)^{m_i} \\
 & & \downarrow & & \downarrow \\
N & \xrightarrow{i_1} & E & \xrightarrow{p_2} & Q(i)^{m_i} \\
\| & & \downarrow{\scriptstyle \theta} & X & \downarrow{\scriptstyle \beta_i^{mi}} \\
N & \xrightarrow{\varphi} & M & \xrightarrow{\psi} & \Theta(i)^{m_i},
\end{array}$$

where the square marked with X is a pullback and all the rows and columns are F-exact (see 3.4 (a)). Since $N \in \mathscr{F}_F(\Theta)$, we have that $F(Q(i), N) = 0$. Thus, the first row in the preceding diagram splits; therefore, there exists $i_2 : Q(i)^{m_i} \longrightarrow E$, such that $\beta_i^{m_i} = \psi\theta i_2$. By the horseshoe lemma, we have the following F-exact diagram

$$\begin{array}{ccccc}
N' & \longrightarrow & P & \longrightarrow & K(i)^{m_i} \\
\downarrow & & \downarrow & & \downarrow \\
Q_0(N) & \xrightarrow{j_1} & Q_0(M) & \xrightarrow{\pi_2} & Q(i)^{m_i} \\
\downarrow{\scriptstyle \epsilon_N} & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \beta_i^{mi}} \\
N & \xrightarrow{\varphi} & M & \xrightarrow{\psi} & \Theta(i)^{m_i},
\end{array}$$

where $\alpha := \theta i_2$, $\epsilon := (\varphi\epsilon_N, \alpha)$, $Q_0(M) := Q_0(N) \bigoplus Q(i)^{m_i}$, $j_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\pi_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}$. It is easy to see that the following F-exact pair $\zeta :\quad P \longrightarrow Q_0(M) \xrightarrow{\epsilon} M$ is the desired.

Finally, we show that ϵ is an $\mathcal{P}(\Theta)$-precover of M. Indeed, let $X \in {}^{F\perp}\Theta$. By applying the functor $\mathrm{Hom}_{\mathcal{A}}(X, -)$ to ζ, we get the following exact sequence $\mathrm{Hom}_{\mathcal{A}}(X, Q_0(M)) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, M) \longrightarrow F(X, P)$. By 5.2, we conclude that ${}^{F\perp}\Theta = {}^{F\perp}(\mathscr{F}_F(\Theta))$. Then, $F(X, P) = 0$ since $P \in \mathscr{F}_F(\Theta)$. Hence, ϵ is an $\mathcal{P}(\Theta)$-precover of M. $\square$

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7. The standardly stratified algebra associated to an F-projective system and its derived category. In this section, we will consider an artin exact R-category $(\mathcal{A}, \mathcal{E})$ and also we fix, F, an additive $\mathcal{E}_F$-closed R-subfunctor of $\mathrm{Ext}_{\mathcal{E}}(-,-)$.

PROPOSITION 7.1. *Let* $(\Theta, \boldsymbol{Q})$ *be an* F*-projective system of size* t *in* $(\mathcal{A}, \mathcal{E})$*,* $A := \mathrm{End}_{\mathcal{A}}(Q)^{op}$*,* $e_Q := \mathrm{Hom}_{\mathcal{A}}(Q, -) : \mathcal{A} \longrightarrow \mathrm{mod}(A)$ *and* ${}_AP(i) := e_Q(Q(i))$ *for each* $i \in [1, t]$*. Then, the following statements hold:*

(*a*) *The family* ${}_AP := \{{}_AP(i) \mid i \in [1, t]\}$ *is a representative set of the indecomposable projective* A*-modules.*

(*b*) $e_Q(\Theta(i)) \simeq {}_A\Delta(i)$*,* $\forall i \in [1, t]$*.*

(*c*) A *is a standardly stratified algebra with the usual order on* $[1, t]$*, that is,* $\mathrm{proj}(A) \subseteq \mathscr{F}({}_A\Delta)$*.*

(*d*) *The restriction* $e_Q : \mathscr{F}_F(\Theta) \longrightarrow \mathscr{F}({}_A\Delta)$ *is an exact equivalence of* R*-categories.*

Proof. By 6.4(c), the functor $e_Q = \mathrm{Hom}_{\mathcal{A}}(Q, -) : \mathcal{A} \longrightarrow \mathrm{mod}(A)$ is an exact functor.

(a) It follows by 6.5 (b) and 4.2.

(b) and (c) Let $i \in [1, t]$. By 6.2 (PS4) and 6.10, we have two F-exact pairs $\eta_i : K(i) \xrightarrow{\alpha_i} Q(i) \longrightarrow \Theta(i)$, $\eta'_i : K' \longrightarrow Q' \xrightarrow{\lambda_i} K(i)$. Applying the functor $e_Q = \mathrm{Hom}_{\mathcal{A}}(Q, -)$ to the F-exact pairs η_i and η'_i we get the following exact sequence in $\mathrm{mod}(A)$

$$\epsilon_i : e_Q(Q') \xrightarrow{e_{Q(\gamma_i)}} {}_A P(i) \longrightarrow e_Q(\Theta(i)) \longrightarrow 0,$$

where $\gamma_i := \alpha_i \lambda_i$. It is easy to show that

$$\mathrm{Im}(e_Q(\gamma_i)) = \mathrm{Tr}_{\oplus_{j>i A}P(j)}({}_AP(i)) =: U(i).$$

Therefore, $e_Q(\Theta(i)) \simeq_A \Delta(i)$.
On the other hand, from the following exact sequence

$$0 \longrightarrow U(i) \longrightarrow_A P(i) \longrightarrow e_Q(\Theta(i)) \longrightarrow 0,$$

we conclude that ${}_AP(i) \in \mathscr{F}({}_A\Delta)$ since $U(i) \in \mathscr{F}({}_A\Delta)$, $e_Q(\Theta(i)) \simeq_A \Delta(i)$ and because $\mathscr{F}({}_A\Delta)$ is closed under extensions. This shows that A is an standardly stratified algebra.

(d) Since $e_Q = \mathrm{Hom}_{\mathcal{A}}(Q, -) : \mathcal{A} \longrightarrow \mathrm{mod}(A)$ is an exact functor, we have to prove that $\mathrm{Im}(e_Q) \subseteq \mathscr{F}({}_A\Delta)$ and that the restriction $e_Q : \mathscr{F}_F(\Theta) \longrightarrow \mathscr{F}({}_A\Delta)$ is full, faithful and dense.
First of all we will see that $\mathrm{Im}(e_Q) \subseteq \mathscr{F}({}_A\Delta)$. Let $0 \neq M \in \mathscr{F}_F(\Theta)$. We prove by induction on $\ell_F(M)$ that $e_Q(M) \in \mathscr{F}({}_A\Delta)$.

If $\ell_F(M) = 1$ then $M \simeq \Theta(i)$ for some i. By (b), $e_Q(M) \simeq {}_A\Delta(i) \in \mathscr{F}({}_A\Delta)$.
Let $\ell_F(M) > 1$. Since $M \in \mathscr{F}_F(\Theta)$, there exists a Θ-filtration

$$\eta := \{\eta_l : \quad M_{l-1} \longrightarrow M_l \longrightarrow \Theta(j_l)\}_{l=1}^{m},$$

such that $M_0 := 0$ and $M_m = M$. We consider the F-exact pair $\eta_m : M_{m-1} \longrightarrow M \longrightarrow \Theta(j_m)$ with $M_{m-1} \in \mathscr{F}(\Theta)$ and $\ell_F(M_{m-1}) = \ell_F(M) - 1$. By induction and since $\mathscr{F}({}_A\Delta)$ is closed under extensions, we conclude that $e_Q(M) \in \mathscr{F}({}_A\Delta)$.

Now, we prove that $\mathfrak{e}_Q$ is fully faithful and dense. Let $M, N \in \mathscr{F}(\Theta)$. By 6.10 we have an F-exact sequence $\eta : Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$ with $Q_0, Q_1 \in \mathrm{add}(Q)$. Applying $\mathfrak{e}_Q$ to η, we get the exact sequence $\mathfrak{e}_Q(Q_1) \xrightarrow{f} \mathfrak{e}_Q(Q_0) \longrightarrow \mathfrak{e}_Q(M) \longrightarrow 0$ in $\mathrm{mod}(A)$. Then, we have the following exact and commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & {}_{\mathcal{A}}(M, N) & \longrightarrow & {}_{\mathcal{A}}(Q_0, N) & \longrightarrow & {}_{\mathcal{A}}(Q_1, N) \\
 & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} \\
0 & \longrightarrow & {}_{A}(\mathfrak{e}_Q(M), \mathfrak{e}_Q(N)) & \longrightarrow & {}_{A}(\mathfrak{e}_Q(Q_0), \mathfrak{e}_Q(N)) & \longrightarrow & {}_{A}(\mathfrak{e}_Q(Q_1), \mathfrak{e}_Q(N)),
\end{array}$$

where α_2 and α_3 are isomorphisms since $Q_0, Q_1 \in \mathrm{add}(Q)$ (see 4.2). By the five lemma, α_1 is an isomorphism, this shows that $\mathfrak{e}_Q$ is full and faithful.

Finally, we will see that $\mathfrak{e}_Q$ is dense. Indeed, let $0 \neq M \in \mathscr{F}({}_A\Delta)$. We proceed by induction on $\ell_{{}_A\Delta}(M)$. If $\ell_{{}_A\Delta}(M) = 1$, then $M \simeq {}_A\Delta(i) \simeq \mathfrak{e}_Q(\Theta(i))$ for some i.

Let $\ell_{{}_A\Delta}(M) = m > 1$ be. Then, there exists an exact sequence in $\mathrm{mod}(A)$ $0 \longrightarrow {}_A\Delta(i) \longrightarrow M \longrightarrow M/{}_A\Delta(i) \longrightarrow 0$ with $\ell_{{}_A\Delta}(M/{}_A\Delta(i)) = \ell_{{}_A\Delta}(M) - 1$ for some i. By induction there exists $0 \neq Z \in \mathscr{F}_F(\Theta)$, such that $\mathfrak{e}_Q(Z) \simeq M/{}_A\Delta(i)$. By 6.10, there is an F-exact pair $\eta_Z : \quad Z' \xrightarrow{u} Q_0(Z) \xrightarrow{\epsilon_Z} Z$ with $Z' \in \mathscr{F}(\Theta)$; therefore, we get the following exact sequence in $\mathrm{mod}(A)$: $0 \longrightarrow \mathfrak{e}_Q(Z') \xrightarrow{\mathfrak{e}_Q(u)} \mathfrak{e}_Q(Q_0(Z)) \xrightarrow{\mathfrak{e}_Q(\epsilon_Z)} \mathfrak{e}_Q(Z) \longrightarrow 0$. Thus, we have the following commutative exact diagram in $\mathrm{mod}(A)$. Since $\mathfrak{e}_Q(Q_0(Z))$ is projective, the exact sequence η splits and hence we conclude that $C \simeq {}_A\Delta(i) \bigoplus \mathfrak{e}_Q(Q_0(Z)) \simeq \mathfrak{e}_Q(\Theta(i) \bigoplus Q_0(Z))$, $i_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $p_2 = (0, 1)$. Then, $\mu = \begin{pmatrix} \varphi \\ \mathfrak{e}_Q(u) \end{pmatrix}$ with $\varphi : \mathfrak{e}_Q(Z') \longrightarrow \mathfrak{e}_Q(\Theta(i))$. Since the restriction $\mathfrak{e}_Q\,|_{\mathscr{F}_F(\Theta)}$ is full, there exists $h : Z' \longrightarrow \Theta(i)$, such that $\mathfrak{e}_Q(h) = \varphi$ and then $\mu = \begin{pmatrix} \mathfrak{e}_Q(h) \\ \mathfrak{e}_Q(u) \end{pmatrix} = \mathfrak{e}_Q \begin{pmatrix} h \\ u \end{pmatrix}$. Since $(0, 1) \begin{pmatrix} h \\ u \end{pmatrix} = u$ and u is an inflation, we obtain that $\Psi := \begin{pmatrix} h \\ u \end{pmatrix}$ is an inflation. Moreover Ψ is an F-monomorphism since u is an F-monomorphism and F is a subfunctor of $\mathrm{Ext}_{\mathcal{E}}(-, -)$. Then, there is an F-exact pair $\xi : \quad Z' \xrightarrow{\Psi} \Theta(i) \bigoplus Q_0(Z) \longrightarrow X$ in $\mathcal{A}$. By [**17**, page 28 in Appendix], we can construct the following commutative diagram, where all the rows and the first column are F-exact and I is pullback

$$\begin{array}{ccccc}
 & & \Theta(i) & = & \Theta(i) \\
 & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle j} \\
Z' & \xrightarrow{\Psi} & \Theta(i) \bigoplus Q_0(Z) & \xrightarrow{\varphi} & X \\
\| & & \downarrow{\scriptstyle i} & I & \downarrow{\scriptstyle \alpha} \\
Z' & \xrightarrow{u} & Q_0(Z) & \xrightarrow{\epsilon_Z} & Z
\end{array}$$

with $i := (0, 1)$. We have that $\epsilon_Z i = \alpha\varphi$ is an F-epimorphism since F is a closed subfunctor. It follows that α is an F-epimorphism. From the fact that $j = \mathrm{Ker}(\alpha)$, we have that the second column, from the last diagram, is an F-exact pair. We note that $X \in \mathscr{F}_F(\Theta)$ since Z, $\Theta(i) \in \mathscr{F}_F(\Theta)$ and

$\mathscr{F}_F(\Theta)$ is closed under F-extensions. By applying e_Q to ξ, we get the exact sequence in $\mathrm{mod}(A)$ $0 \longrightarrow \mathrm{e}_Q(Z') \xrightarrow{\mathrm{e}_{Q(\Psi)}} \mathrm{e}_Q(\Theta(i) \bigoplus Q_0(Z)) \longrightarrow \mathrm{e}_Q(X) \longrightarrow 0.$ But $\mathrm{e}_Q(\Psi) = \mu$, then $\mathrm{e}_Q(X) \simeq \mathrm{Coker}(\mu) = M$. Proving that e_Q is dense. □

Corollary 7.2. *Let Θ be an F-system of size t in $(\mathcal{A}, \mathcal{E})$. Then, there exists an F-projective system $(\Theta, \mathbf{Q})$ of size t, such that the following statements hold:*

(*a*) *$\mathscr{F}_F(\Theta)$ is closed under direct summands.*

(*b*) *For any object $M \in \mathscr{F}_F(\Theta)$, there exists an F-exact pair $Z \longrightarrow Q_M \longrightarrow M$ in $\mathscr{F}_F(\Theta)$, such that $Q_M \longrightarrow M$ is an* $\mathrm{add}(Q)$*-cover of M and* $\mathrm{min}(M) < \mathrm{min}(Z)$ *if $M \neq 0$ with $Q := \bigoplus_{i=1}^{t} Q(i)$.*

Proof. By 6.8, there exists $(\Theta, \mathbf{Q})$. Let $A := \mathrm{End}_{\mathcal{A}}(Q)^{op}$. We know by 7.1 that $\mathrm{e}_Q : \mathscr{F}_F(\Theta) \longrightarrow \mathscr{F}({}_A\Delta)$ is an exact equivalence, A is an standardly stratified algebra with the usual order on $[1, t]$ and $\mathrm{e}_Q(\Theta(i)) \simeq_A \Delta(i)$, $\forall i$. Then, $\mathscr{F}_F(\Theta)$ is closed under direct summands since $\mathscr{F}({}_A\Delta)$ is closed under direct summands (see [**2**, Theorem 1.6]). □

According to Thomason, we can construct the bounded derived category of an exact category with splitting idempotents (see, for example, [**27**]). The following result gives us a description of the bounded derived category of $\mathscr{F}_F(\Theta)$.

Theorem 7.3. *Let Θ be an F-system of size t in an Ext-finite artin exact R-category $(\mathcal{A}, \mathcal{E})$. There exists a standardly stratified algebra A and a equivalence as triangulated categories*

$$\mathrm{RHom}(Q, -) : \mathrm{D}^b(\mathscr{F}_F(\Theta)) \longrightarrow \mathrm{D}^b(\mathrm{mod}(A)).$$

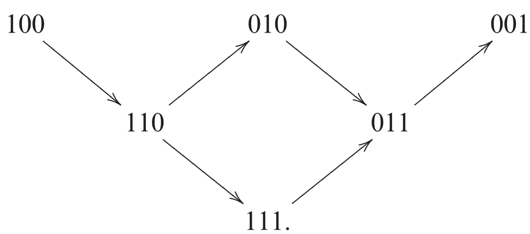
Proof. By 6.8, there exists a F-projective system $(\Theta, \mathbf{Q})$. The equivalence given in 7.1(d) extends to the level of bounded derived categories

$$\mathrm{RHom}(Q, -) : \mathrm{D}^b(\mathscr{F}_F(\Theta)) \longrightarrow \mathrm{D}^b(\mathscr{F}({}_A\Delta)).$$

By [**25**, Lemma 7.1], we have that $\mathrm{D}^b(\mathscr{F}({}_A\Delta)) \simeq \mathrm{D}^b(\mathrm{mod}(A))$ as triangulated categories since $A = \mathrm{End}(Q)^{op}$ is standardly stratified. Then, the result follows. □

8. Examples.

8.1. Example 1: F-system that is not stratifying system in the classical sense. For this, we are going to use the Auslander–Solberg relative theory developed in [**4**]. Let k be a field, consider the following quiver $Q :\ 1 \longleftarrow 2 \longleftarrow 3$ and its path algebra $\Lambda = kQ$. The Auslander–Reiten quiver of Λ is



Consider the following set of indecomposable modules: $\Theta(1) := P(1) = 100$, $\Theta(2) := P(2) = 110$, $\Theta(3) := P(3) = 111$, $\Theta(4) := 010$.

We consider $(\mathrm{mod}(\Lambda), \mathcal{E})$, where $\mathcal{E}$ is the class of all exact sequences in $\mathrm{mod}(\Lambda)$. Then, we have that $(\mathrm{mod}(\Lambda), \mathcal{E})$ is an artin exact k-category, which is Ext_Λ-finite. It is easy to see that $\mathrm{Hom}_\Lambda(\Theta(j), \Theta(i)) = 0$ for $j > i$.

Since $\mathrm{gldim}(\Lambda) = 1$, we have that $\mathrm{Ext}^1_\Lambda(M, N) \simeq \mathrm{DHom}_\Lambda(N, \tau M)$ as k-vector spaces for all $M, N \in \mathrm{mod}(\Lambda)$. Then, we have that

1. $\mathrm{Ext}^1_\Lambda(\Theta(4), \Theta(4)) = 0$,
2. $\mathrm{Ext}^1_\Lambda(\Theta(4), \Theta(3)) = 0$,
3. $\mathrm{Ext}^1_\Lambda(\Theta(4), \Theta(2)) = 0$,
4. $\mathrm{Ext}^1_\Lambda(\Theta(4), \Theta(1)) \neq 0$.

We also have that $\mathrm{Ext}^1_\Lambda(\Theta(j), \Theta(i)) = 0$ if $j = 1, 2, 3$ since $\Theta(j)$ is projective. Hence, the set $\{\Theta(i)\}_{i=1}^4$ is not a stratifying system in the classical sense since $\mathrm{Ext}^1_\Lambda(\Theta(4), \Theta(1)) \neq 0$.

Let's see that $\Theta = \{\Theta(i)\}_{i=1}^4$ is an F-stratifying system for some additive subfunctor F, which is $\mathcal{E}_F$-closed of $\mathrm{Ext}^1_\Lambda(-, -)$.

Let's take $\mathcal{X} = \mathrm{add}(100 \oplus 110 \oplus 111 \oplus 010)$ and consider $F := F_\mathcal{X}$, where $F_\mathcal{X}$ is defined as follows:

$$F_\mathcal{X}(C, A) := \{0 \to A \to B \to C \to 0 \mid \mathrm{Hom}(-, B)|_\mathcal{X} \to \mathrm{Hom}(-, C)|_\mathcal{X} \to 0\}$$

and $F_\mathcal{X}(\alpha, \beta) = \mathrm{Ext}_\Lambda(\alpha, \beta)$.

It can be proved that $F_\mathcal{X}$ is an additive subfunctor of $\mathrm{Ext}^1_\Lambda(-, -)$ which is $\mathcal{E}_F$-closed.

Since $\mathcal{P}(\Lambda) \subseteq \mathcal{X}$, we have that the projectives relative to the functor $F_\mathcal{X}$ are precisely $\mathcal{X}$, that is, $\mathcal{P}(F) = \mathcal{X}$ (see, for example, [**4**, Proposition 1.10]).

We assert that $\{\Theta(i)\}_{i=1}^4$ is an $F_\mathcal{X}$-stratifying system.

Indeed, is enough to show that $F(\Theta(4), \Theta(1)) = 0$. But this follows from the fact that $\Theta(4) \in \mathcal{P}(F) = \mathcal{X}$.

Hence, $\{\Theta(i)\}_{i=1}^4$ is an $F_\mathcal{X}$-stratifying system in $\mathrm{mod}(\Lambda)$, which is not a stratifying system in the classical sense. It can be shown that in this case $Q(i) = \Theta(i)\ \forall i$ and then $Q = \bigoplus_{i=1}^4 \Theta(i)$. Therefore,

$$A^{op} := \mathrm{End}(Q)^{op} = \begin{pmatrix} k & k & k & 0 \\ 0 & k & k & k \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{pmatrix}.$$

Therefore, by theorem 7.3 it follows that

$$\mathrm{D}^b(\mathscr{F}_F(\Theta)) \simeq \mathrm{D}^b(\mathrm{mod}(A^{op})).$$

8.2. Example 2. The notion of exceptional sequences originates from the study of vector bundles (see, for instance, [**14**, **30**]).

Let K be an algebraically closed field. Let $\mathcal{H}$ be a hereditary, exact (or abelian) K-linear category with finite dimensional morphisms and extensions spaces. Recall that an object M in $\mathcal{H}$ is called *exceptional* if $\mathrm{Ext}^1_\mathcal{H}(M, M) = 0$ and $\mathrm{End}_\mathcal{H}(M)$ is a division ring, hence $\mathrm{End}_\mathcal{H}(M) \simeq K$ since K is algebraically closed. Furthermore, a sequence $X = (X_1, \ldots, X_t)$ is an *exceptional sequence* if it is a sequence of exceptional objects satisfying $\mathrm{Hom}_\mathcal{H}(X_j, X_i) = 0$ for $j > i$ and $\mathrm{Ext}^1_\mathcal{H}(X_j, X_i) = 0$ for $j \geq i$. An exceptional

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sequence $(X_1, \dots, X_n)$ is called *complete*, if n is the rank of the Grothendieck group of $\mathcal{H}$. An exceptional sequence $X = (X_1, X_2)$ is called an *exceptional pair*.

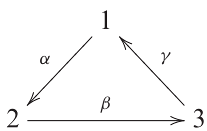
It is worth to mention that exceptional pairs has been amply studied (see, for example, [**19**]). Given an exceptional pair (X_1, X_2) in a hereditary, abelian K-linear category $\mathcal{H}$ with finite dimensional morphisms, in [**19**], H. Lenzing and H. Meltzer classified the smallest subcategory of $\mathcal{H}$ containing X_1 and X_2 (denoted by $\mathcal{C}(X_1, X_2)$), which is closed under extensions, kernels and cokernels. They proved that if $r = \dim_K \mathrm{Hom}_{\mathcal{H}}(X_1, X_2) \geq 1$ there are exactly 3 possible types for $\mathcal{C}(X_1, X_2)$, all derived equivalent to $\mathrm{mod}(H_r)$ over the r-Kronecker algebra H_r (for more details see [**19**]).

Therefore, we have that exceptional sequences are example of F-systems in $\mathcal{H}$ with $F = \mathrm{Ext}_{\mathcal{H}}(-, -)$. We note that the example 1 give us an hereditary exact category and $\Theta = (\Theta(1), \Theta(2), \Theta(3), \Theta(4))$ is an exceptional sequence in $(\mathrm{mod}(\Lambda), \mathcal{E})$. In the case of a hereditary algebra A, the following result is known.

Lemma 8.1. [**8**, Lemma 2] *Let A be a hereditary K-algebra. Then, a sequence of modules $X = (X_1, \dots, X_t)$ is exceptional, if and only if $X = \{X_i\}_{i=1}^t$ is a stratifying system in* $\mathrm{mod}(A)$.

In [**31**], the author computed the number of complete exceptional sequences for algebras Λ associated to quivers of Dynkin type and by the above result the number of stratifying systems of size n, where n is the number of isoclasses simples in $\mathrm{mod}(\Lambda)$.

The following example give us an exceptional sequence in a non-hereditary algebra and it was communicated to H. Krause by Martin Kalck (see [**18**, Example 5.9]). Let K be a field and consider the finite dimensional K-algebra Λ given by the quiver



with the relations $\gamma\beta = 0 = \alpha\gamma$. Let us denote by S_i the simple associated to the vertex i and let P_i the projective cover of S_i. Then, (S_1, P_2, P_3) is an exceptional sequence in $\mathrm{mod}(\Lambda)$ and $\mathscr{F}(S_1, P_2, P_3) = \mathrm{add}(S_1 \oplus P_2 \oplus P_3)$.

As a consequence of our results, we have the following result for exceptional sequences.

Proposition 8.2. *Let $\mathcal{H}$ be an hereditary exact K-category with finite dimensional morphisms and extensions spaces and let $X = (X_1, \dots, X_t)$ an exceptional sequence. Then, there exists a quasi-hereditary algebra A and equivalence*

$$e_Q : \mathscr{F}(X) \longrightarrow \mathscr{F}({}_A\Delta)$$

of K-categories, which extends to an equivalence of triangulated categories

$$\mathrm{D}^b(\mathscr{F}(X)) \longrightarrow \mathrm{D}^b(\mathrm{mod}(A)).$$

Proof. This follows from 7.1 and 7.3. □

8.3. Example 3. Let $\mathcal{A}$ be an abelian category; $\mathcal{A}$ is called a length category if every object in $\mathcal{A}$ has a finite composition series. Recall that a projective object P in

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an abelian category is a generator, if for every object X, there exists an epimorphism $P^r \longrightarrow X$ for some integer n. Next, recall the following definition.

DEFINITION 8.3 [**18**, Definition 3.1]. Let $\mathcal{A}$ be an abelian length category having only finitely many isoclasses of simple objects. Then, $\mathcal{A}$ is called a *highest weight category* if there are finitely many exact sequences

$$0 \longrightarrow U_i \longrightarrow P_i \longrightarrow \Delta_i \longrightarrow 0 \quad (1 \leq i \leq n)$$

in $\mathcal{A}$ satisfying the following:

(a) $\mathrm{End}_{\mathcal{A}}(\Delta_i)$ is a division ring for all i.
(b) $\mathrm{Hom}_{\mathcal{A}}(\Delta_j, \Delta_i) = 0$ for all $j > i$.
(c) $U_i \in \mathscr{F}(\Delta_{i+1}, \dots, \Delta_n)$ for all i.
(d) $P = \oplus_{i=1}^n P_i$ is a projective generator of $\mathcal{A}$. The objects $\Delta_1, \dots, \Delta_n$ are called *standard objects*.

If the objects P_i in the previous definition are indecomposable, we get that the standard objects $\Delta := \{\Delta(i)\}_{i=1}^n$ form an F-projective system, where $F = \mathrm{Ext}^1_{\mathcal{A}}(-,-)$.
It is worth to mention that in the study of highest weight categories, the theory of recollements arises in a natural way (see, for example, [**18**]).
As a corollary of the results in this paper we get the following.

PROPOSITION 8.4. *Let $\mathcal{A}$ be an abelian length K-category, Krull–Schmidt and with finite dimensional morphisms and extensions spaces, which is a highest weight category with standard objects $\Delta = \{\Delta_1, \dots, \Delta_n\}$. If the objects P_i in the definition 8.3 are indecomposable, then $A := \mathrm{End}_{\mathcal{A}}(P)^{op}$ is a quasi-hereditary algebra, where $P = \oplus_{i=1}^n P(i)$ and there exists an equivalence*

$$e_P : \mathscr{F}(\Delta) \longrightarrow \mathscr{F}({}_A\Delta)$$

of K-categories that extends to an equivalence of triangulated categories

$$\mathrm{D}^b(\mathscr{F}(\Delta)) \longrightarrow \mathrm{D}^b(\mathrm{mod}(A)).$$

Proof. This follows from 7.1 and 7.3. □

8.4. Example 4. Let $\mathcal{A}$ be an abelian category and $\mathrm{Ch}(\mathcal{A})$ be the category of chain complexes over the category $\mathcal{A}$. Let $\mathcal{E}$ be the class of all exact sequences in $\mathrm{Ch}(\mathcal{A})$

$$0 \longrightarrow A^\bullet \xrightarrow{\alpha} B^\bullet \xrightarrow{\beta} C^\bullet \longrightarrow 0\,,$$

such that $0 \longrightarrow A^i \xrightarrow{\alpha_i} B^i \xrightarrow{\beta_i} C^i \longrightarrow 0$ are split exact sequences for all i. It is well known that $(\mathrm{Ch}(\mathcal{A}), \mathcal{E})$ is an exact category (in fact it is a Frobenius category).
Consider the canonical functor $i_0 : \mathcal{A} \longrightarrow \mathrm{Ch}(\mathcal{A})$ that sends $M \in \mathcal{A}$ to the stalk complex $\underline{M}$ concentrated in degree zero, then i_0 is additive full and faithful. Therefore, $\mathrm{Hom}_{\mathcal{A}}(M, N) \simeq \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(\underline{M}, \underline{N})$.
Now, consider the canonical translation functor $[1] : \mathrm{Ch}(\mathcal{A}) \longrightarrow \mathrm{Ch}(\mathcal{A})$, such that $(M^\bullet[1])^i := M^{i+1}$ and let $\mathrm{K}(\mathcal{A})$ the homotopy category of $\mathrm{Ch}(\mathcal{A})$. It is well known that

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there exists an isomorphism of abelian groups

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(M^\bullet, N^\bullet[1]) \simeq \mathrm{Ext}_{\mathcal{E}}(M^\bullet, N^\bullet).$$

Let $\Theta = \{\Theta(i)\}_{i=1}^{t}$ be and stratifying system in $\mathcal{A}$. We assert that $\underline{\Theta} := \{\underline{\Theta(i)}\}_{i=1}^{t}$ is an $\mathrm{Ext}_{\mathcal{E}}$-system in $(\mathrm{Ch}(\mathcal{A}), \mathcal{E})$.

Indeed, $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(\underline{\Theta(j)}, \underline{\Theta(i)}) \simeq \mathrm{Hom}_{\mathcal{A}}(\Theta(j), \Theta(i)) = 0$ for $j > i$ and

$$\mathrm{Ext}_{\mathcal{E}}(\underline{\Theta(j)}, \underline{\Theta(i)}) = \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(\underline{\Theta(j)}, \underline{\Theta(i)}[1]) = 0.$$

In this way we can produce a lot of $\mathrm{Ext}_{\mathcal{E}}$-systems in $(\mathrm{Ch}(\mathcal{A}), \mathcal{E})$.

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REFERENCES

1. I. Ágoston, V. Dlab and E. Lukács, Stratified algebras, *Math. Rep. Acad. Sci. Can.* **20**(1) (1998), 22–28.

2. I. Ágoston, D. Happel, E. Lukács and L. Unger, Standardly stratified algebras and tilting, *J. Algebra* **226**(1) (2000), 144–160.

3. M. Auslander, Representation dimension of Artin algebras, reprint of the 1971 original, in *Selected works of Maurice Auslander*. Queen Mary College mathematics notes (American Mathematical Society, Providence, RI, 1999), 505–574.

4. M. Auslander and Ø. Solberg, Relative homology and representation theory I, relative homology and homologically finite subcategories, *Commun. Algebra* **21**(9) (1993), 2995–3031.

5. M. Auslander and Ø Solberg, Relative homology and representation theory II, relative cotilting theory, *Commun. Algebra* **21**(9) (1993), 3033–3079.

6. T. Bühler, Exact categories, *Expo. Math.* **28** (2010), 1–69.

7. M. C. R. Butler and G. Horrocks, Classes of extensions and resolutions, *Philos. Trans. R. Soc. Lond. Ser. A* **254** (1961), 155–222.

8. P. Cadavid and E. N. Marcos, Stratifying systems over the hereditary path algebra with quiver $\mathbb{A}_{p,q}$, *São Paolo J. Math. Sci.* **10** (2016), 73–90.

9. E. Cline, B. J. Parshall and L. L. Scott, Stratifying endomorphism algebras, *Mem. AMS* **591** (1996), 1–119.

10. V. Dlab, Quasi-hereditary algebras revisited, *Analele Stiintifice Univ. Ovidius Constanta* **4** (1996), 43–54.

11. V. Dlab and C. M. Ringel, The module theoretical approach to Quasi-hereditary algebras, in *Representations of algebras and related topics*, London Mathematical Society lecture note series 168 (Cambridge University Press, New York, 1992), 200–224.

12. P. Dräxler, I. Reiten, S. Smaløand ØSolberg, Exact categories and vector espace categories, *Trans. Amer. Math. Soc*. **351** (1999), 647–682.

13. K. Erdmann and C. Sáenz, On standardly stratified algebras, *Commun. Algebra* **32** (2003), 3429–3446.

14. A. L. Gorodentsev and A. N. Rudakov, Exceptional vector bundles on projective spaces, *Duke Math. J.* **54** (1987), 115–130.

15. D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, London Mathematical Society lecture note series 119 (Cambridge University Press, New York, 1988).

16. G. Hochschild, Relative homological algebra, *Trans. Amer. Math. Soc.* **82** (1956), 246–269.

17. B. Keller, Chain complexes and stable categories, *Manuscr. Math.* **67** (1990), 379–417.

18. H. Krause, Highest weight categories and recollements, *Ann. Inst. Fourier* **67** (2017), 2679–2701.

19. H. Lenzing and H. Meltzer, Exceptional pairs in hereditary categories, *Commun. Algebra* **37**(8) (2009), 2547–2556.

20. E. N. Marcos, O. Mendoza and C. Sáenz, Stratifying systems via relative simple modules, *J. Algebra* **280** (2004), 472–487.

21. E. N. Marcos, O. Mendoza and C. Sáenz, Stratifying systems via relative projective modules, *Commun. Algebra* **33** (2005), 1559–1573.

22. V. Mazorchuk, On finitistic dimension of stratified algebras, *Algebra Discrete Math.* **3**(3) (2004), 77–88.

23. V. Mazorchuk, Stratified algebras arising in Lie theory, in *Representations of finite dimensional algebras and related topics in Lie theory and geometry, Fields Institute Communications* (Dlab V. and Ringel C. M., Editors), vol. 40, (American Mathematical Society, Providence, RI, 2004), 245–260.

24. V. Mazorchuk and P. Parker, On the relation between finitistic and good filtration dimensions, *Commun. Algebra* **32**(5) (2004), 1903–1917.

25. O. Mendoza and V. Santiago, Homological systems in triangulated categories, *Appl. Categorical Struct.* **24**(1) (2016), 1–35.

26. O. Mendoza, C. Sáenz and C. Xi, Homological systems in module categories over pre-ordered sets, *Quart. J. Math.* **60** (2009), 75–103.

27. A. Neeman, The derived category of an exact category, *J. Algebra* **135** (1990), 388–394.

28. D. Quillen, Higher algebraic K-theory I, in *Algebraic K-theory I. Proceedings of the Conference Held at Battelle Memorial Institute, Seattle, WA, 1972*. Lecture notes in mathematics, vol. 341 (Springer, Berlin, 1973), 85–147.

29. C. M. Ringel, The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences, *Math. Z.* **208** (1991), 209–223.

30. A. N. Rudakov, *Helices and vector bundles: Seminaire Rudakov*, London Mathematical Society Lecture Note Series, volume 148 (Cambridge University Press, New York, 1990).

31. U. Seidel, Exceptional sequences for quivers of Dynkin type, *Commun. Algebra* **29**(3) (2001), 1373–1386.