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SECOND VARIATION OF THE "TOTAL SCALAR CURVATURE" ON CONTACT MANIFOLDS

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ABSTRACT. Let M^{2n+1} be a compact contact manifold and $\mathcal{A}$ the set of associated metrics. Using the scalar curvature R and the $*$-scalar curvature R^*, in [5] we defined the "total scalar curvature", by $I(g) = \int_M \frac{1}{2}(R + R^* + 4n(n+1))\,dV$ and showed that the critical points of $I(g)$ on $\mathcal{A}$ are the K-contact metrics, *i.e.* metrics for which the characteristic vector field is Killing. In this paper we compute the second variation of $I(g)$ and prove that the index of $I(g)$ and of $-I(g)$ are both positive at each critical point. As an application we show that the classical total scalar curvature $A(g) = \int_M R\,dV_g$ restricted to $\mathcal{A}$ cannot have a local minimum at any Sasakian metric.

1. **Introduction.** Let M be a compact contact $(2n+1)$-manifold with global contact form η and characteristic vector field ξ. For an associated metric g, let R and R^* denote the scalar and $*$-scalar curvatures respectively. In [5] we considered the integral $I(g) = \int_M \frac{1}{2}\big(R + R^* + 4n(n+1)\big)\,dV$ as a functional on the set $\mathcal{A}$ of all associated metrics of the given contact form on M. We call the integral $\int_M \frac{1}{2}\big(R + R^* + 4n(n+1)\big)\,dV$ the "total scalar curvature". One of the main results of [5] is that the critical points of $I(g)$ are those metrics for which ξ generates a 1-parameter group of isometries, *i.e.* K-contact metrics. The constant $4n(n+1)$ is included only to give $W = \frac{1}{2}\big(R + R^* + 4n(n+1)\big)$ as a natural generalization of the Webster scalar curvature of a CR-structure as introduced in dimension 3 by Chern and Hamilton [6].

For the classical integral functional, $A(g) = \int_M R\,dV_g$, on the set of all Riemannian metrics with the same total volume, a critical point is an Einstein metric. In [7] Y. Muto proved the following result.

THEOREM (MUTO). *The index of $A(g)$ and the index of $-A(g)$ are both positive at each critical point.*

In this paper we compute the second variation of $I(g)$ and prove the following.

THEOREM 1. *The index of $I(g)$ and the index of $-I(g)$ are both positive at each critical point.*

In dimension 3, $I(g)$ agrees with the functional $E_W(g) = \int_M W\,dV_g$ of Chern and Hamilton in [6] and hence we have the following corollary.

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COROLLARY. *The index of $E_W(g)$ and the index of $-E_W(g)$ are both positive at each critical point.*

In [4] the classical integral $A(g) = \int_M R\,dV_g$ was considered as a functional on $\mathcal{A}$. Since $\mathcal{A}$ is a smaller set of metrics, one would expect a weaker critical point condition than the Einstein condition; indeed the critical point condition is that the Ricci operator Q and the fundamental collineation ϕ of the contact metric structure commute when restricted to the contact subbundle [4]. The 3-dimensional case was studied in [9]. Since $Q\phi - \phi Q = 0$ on a Sasakian manifold (see *e.g.* [1, p. 76]), Sasakian metrics, when they exist, are critical points. As a application of Theorem 1 we will also prove the following.

THEOREM 2. *The functional $A(g)$ restricted to $\mathcal{A}$ cannot have a local minimum at any Sasakian metric.*

The "total scalar curvature" is also of interest in symplectic geometry as well. It was shown in [3] that $\int_M R + R^*\,dV$ is a symplectic invariant and to within a constant is the cup product

$$\big(c_1(M) \cup [\Omega]^{n-1}\big)([M])$$

where $c_1(M)$ is the first Chern class of M.

2. **Preliminaries.** By a *contact manifold* we mean a $(2n+1)$-dimensional C^∞ manifold M together with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. Given a contact form η, it is well known that there exists a unique vector field ξ, called the *characteristic vector field*, such that $d\eta(\xi, X) = 0$ and normalized by $\eta(\xi) = 1$.

A Riemannian metric g is an *associated metric* for a contact form η, if there exists a tensor field ϕ of type (1,1) such that $\phi^2 = -I + \eta \otimes \xi$, $\eta(X) = g(\xi, X)$ and $d\eta(X, Y) = g(X, \phi Y)$. We refer to (η, g) or (ϕ, ξ, η, g) as a *contact metric structure*. For a given contact form η, the set $\mathcal{A}$ of all associated metrics is infinite dimensional. Moreover all associated metrics have the same volume element, $dV = \frac{(-1)^n}{2^n n!}\eta \wedge (d\eta)^n$.

Given a contact metric structure (ϕ, ξ, η, g) define a tensor field h by $h = \frac{1}{2}\mathcal{L}_\xi \phi$, where $\mathcal{L}$ denotes Lie differentiation. The operator h enjoys many interesting properties. h is a symmetric operator, h anti-commutes with ϕ, $h\xi = 0$ and h vanishes if and only if ξ is Killing, *i.e.* ξ generates a 1-parameter group of isometries. When ξ is Killing, the contact metric structure is said to be *K-contact*. Moreover h is related to the covariant derivative of ξ by

(2.1) $$\nabla_X \xi = -\phi X - \phi h X$$

and h is related to the Ricci curvature in the direction ξ by

(2.2) $$\mathrm{Ric}(\xi) = 2n - \mathrm{tr}\, h^2.$$

A contact metric structure on M naturally gives rise to an almost complex structure on the product $M \times \mathbf{R}$ and if this almost complex structure is integrable, the given contact metric structure is *Sasakian*. Equivalently a contact metric structure is Sasakian if and only if

$$(\nabla_X\phi)Y = g(X,Y)\xi - \eta(Y)X$$

from which

$$\nabla_X\xi = -\phi X.$$

In particular a Sasakian manifold is always K-contact and in dimension 3 a K-contact manifold is Sasakian. For a general reference to these ideas, see *e.g.* [1].

On a contact metric manifold the $*$-*Ricci tensor* and the $*$-*scalar curvature* are defined by

$$R^*_{ij} = R_{iklt}\phi^{kl}\phi_j{}^t, \qquad R^* = R^{*i}_i.$$

In [8] Olszak showed that

$$R - R^* - 4n^2 = -\frac{1}{2}|\nabla\phi|^2 + 2n - \operatorname{tr} h^2 \le 0 \tag{2.3}$$

with equality holding if and only if the structure is Sasakian.

Let M be a contact manifold and $g(t)$ a path in $\mathcal{A}$. We denote by D the tangent to the path and we use the same letter to denote a tensor field of type (1,1) and of type (0,2). When differentiating $I(g)$ along a path and evaluating at $t=0$ to obtain the critical point condition as in [5], we regard D as independent of t, *i.e.* $g(t)_{ij} = g_{ij} + tD_{ij} + t^2E_{ij} + \cdots$. When taking the second derivative of $I(g)$, we regard the first derivative as having been computed at an arbitrary point of the path and $D_{ij} = \frac{\partial g_{ij}}{\partial t}$ and $E_{ij} = \frac{\partial^2 g_{ij}}{\partial t^2}$. The usage should be clear from the context.

LEMMA 1. *The tangent space to $\mathcal{A}$ at $g \in \mathcal{A}$ consists of the symmetric tensor fields D such that*

$$D\xi = 0, \quad D\phi + \phi D = 0. \tag{2.4}$$

Now the approach to the critical point problem is to differentiate the functional $I(g)$ along a path of metrics and set $I'(0) = 0$. So let $g(t)$ be a path of metrics in $\mathcal{A}$ and set

$$D_{ij} = \frac{\partial g_{ij}}{\partial t}\Big|_{t=0}$$

its tangent vector at $g = g(0)$. We define two other tensor fields by

$$D_{ji}{}^h = \frac{1}{2}(\nabla_j D_i{}^h + \nabla_i D_j{}^h - \nabla^h D_{ji}),$$
$$D_{kji}{}^h = \nabla_k D_{ji}{}^h - \nabla_j D_{ki}{}^h$$

where ∇ denotes the Levi-Civita connection of $g(0)$ and we note that

$$D_{ji}{}^h = \frac{\partial \Gamma^h_{ji}}{\partial t}\Big|_{t=0}, \qquad D_{kji}{}^h = \frac{\partial R_{kji}{}^h}{\partial t}\Big|_{t=0}$$

where Γ^h_{ji} and $R_{kji}{}^h$ denote the Christoffel symbols and curvature tensor of $g(t)$. A key point in critical point problems such as this is the following lemma [4, 5].

LEMMA 2. *Let T be a second order symmetric tensor field on M. Then* $\int_M T^{ij} D_{ij}\, dV = 0$ *for all tensor fields satisfying (2.4) if and only if* ϕ *and T commute when restricted to the contact subbundle.*

By the *index* of $I(g)$ (or of $A(g)$ or other functional) at a critical point $g = g(0)$, we mean the dimension, including infinity, of the space of tangents to paths through g for which $I''(0) < 0$.

3. **Proofs.** To set the stage for the proof of Theorem 1 we review briefly the proof of the result in [5] that the critical points of $I(g)$ are the K-contact metrics. Clearly it is enough to consider $\int_M R + R^*\, dV$. R and R^* were differentiated separately in [4], and hence we have

$$\frac{d}{dt}\int_M R + R^*\, dV\,\Big|_{t=0} = \int_M \{-R^{jl} - 2nh^{jl} - \nabla_i(\phi^{kl}\nabla_k\phi^{ij}) - R^{*jl}\} D_{jl}\, dV.$$

Extensive computation using various properties of associated metrics, reduces the integral to

$$-4n\int_M h^{jl} D_{jl}\, dV \tag{3.1}$$

Then from Lemma 2 and $h\xi = 0$, the critical point condition becomes $\phi h - h\phi = 0$; but $\phi h + h\phi = 0$ and hence $h = 0$ as desired.

The major effort of the proof of the present theorem is to regard the integral on the right hand side of (3.1) as the derivative of $I(g)$ at an arbitrary metric, differentiate again and evaluate at a critical point.

Recall that $h = \frac{1}{2}\mathcal{L}_\xi\phi$ or in coordinate form

$$h^j{}_k = \frac{1}{2}(\xi^i\nabla_i\phi^j{}_k - \phi^i{}_k\nabla_i\xi^j + \phi^j{}_i\nabla_k\xi^i).$$

Also since $D_{ji}{}^h$ is the t-derivative of the Levi-Civita connection, we have

$$\frac{d}{dt}\nabla_i\phi^j{}_k = \nabla_i\frac{d}{dt}\phi^j{}_k - D_{ik}{}^l\phi^j{}_l + D_{il}{}^j\phi^l{}_k, \qquad \frac{d}{dt}\nabla_i\xi^j = D_{il}{}^j\xi^l.$$

Therefore after some cancelation

$$\frac{d}{dt}h^j{}_k = \frac{1}{2}\big(\xi^i\nabla_i(-D^{jm}\phi_{mk}) + D^{im}\phi_{mk}\nabla_i\xi^j - D^{jm}\phi_{mi}\nabla_k\xi^i\big)$$

Using the basic properties $\nabla_\xi\phi = 0$ and (2.1) this becomes

$$\frac{d}{dt}h^j{}_k = -\frac{1}{2}\xi^i\phi_{mk}\nabla_i D^{jm} - D^j{}_k + \frac{1}{2}(h^j{}_m D^m{}_k - D^j{}_m h^m{}_k)$$

Thus for the derivative of the integrand we have

$$\begin{aligned}\frac{d}{dt}\Big(h^{jl}\frac{\partial g_{jl}}{\partial t}\Big) &= \frac{d}{dt}\Big(h^j{}_k g^{kl}\frac{\partial g_{jl}}{\partial t}\Big)\\ &= -\frac{1}{2}\xi^i\phi_m{}^l D_{jl}\nabla_i D^{jm} - |D|^2 - \frac{1}{2}(D^j{}_m h^{ml} + h^j{}_m D^{ml})D_{jl} + 2h^{jl}E_{jl}.\end{aligned}$$

Now evaluating at $t = 0$ we have $h = 0$ and the second derivative of $I(g)$ at $t = 0$ becomes

$$(3.2) \qquad I''(0) = \int_M 2n\xi^i \phi_m{}^l D_{jl} \nabla_i D^{jm} + 4n|D|^2 \, dV.$$

Moreover when $h = 0$, $\nabla_\xi D = \mathcal{L}_\xi D + 2D\phi$ or $\phi D \nabla_\xi D = \phi D \mathcal{L}_\xi D - 2D^2$. Thus from (3.2) we have the following proposition.

PROPOSITION. *At a critical metric*

$$I''(0) = 2n \int_M tr(\phi D \mathcal{L}_\xi D) \, dV.$$

PROOF OF THEOREM 1. Let $X_1, \ldots, X_{2n}, \xi$ be a local ϕ-basis defined on a neighborhood $\mathcal{U}$ (*i.e.* $X_1, \ldots, X_{2n}, \xi$ is an orthonormal basis with respect to g and $X_{2i} = \phi X_{2i-1}$) and note that the first vector field X_1 may be any unit vector field on $\mathcal{U}$ orthogonal to ξ. Let f be a C^∞ function with compact support in $\mathcal{U}$ and define a path of metrics $g(t)$ as follows. Make no change in g outside $\mathcal{U}$ and within $\mathcal{U}$ change g only in the planes spanned by X_1 and X_2 by the matrix

$$\begin{pmatrix} 1 + t^2 f^2 & tf \\ tf & 1 \end{pmatrix}.$$

It is easy to check that $g(t) \in \mathcal{A}$ and clearly the only non-zero components of D are $D_{12} = D_{21} = f$. Denoting the first vector field in the ϕ-basis by X, we have

$$\phi D(\mathcal{L}_\xi D)X = (\xi f)DX + fD[\xi, X] - \phi D^2[\xi, X].$$

Similarly computing $\phi D(\mathcal{L}_\xi D)\phi X$ and then taking inner products we have

$$\mathrm{tr}(\phi D \mathcal{L}_\xi D) = 4f^2 g([\xi, X], \phi X) = 4f^2 d\eta([\xi, X], X) = -2f^2 \eta\big([[\xi, X], X]\big).$$

Therefore

$$I''(0) = -4n \int f^2 \eta\big([[\xi, X], X]\big) \, dV$$

where X may be regarded as any unit vector field on $\mathcal{U}$ belonging to the contact subbundle. Thus the proof reduces to finding unit vector fields belonging to the contact subbundle on $\mathcal{U}$ for which $\eta\big([[\xi, X], X]\big)$ has either sign.

To construct the desired vector fields, let (x^i, y^i, z) be Darboux coordinates on $\mathcal{U}$, *i.e.* $\eta = \frac{1}{2}(dz - \sum_{i=1}^n y^i dx^i)$ on $\mathcal{U}$, $\xi = 2\frac{\partial}{\partial z}$. Set $X = F\frac{\partial}{\partial y^1} + G(\frac{\partial}{\partial x^1} + y^1 \frac{\partial}{\partial z})$. Then X belongs to the contact subbundle; computing $\eta\big([[\xi, X], X]\big)$ we have

$$\eta\big([[\xi, X], X]\big) = -F\frac{\partial G}{\partial z} + G\frac{\partial F}{\partial z}.$$

Thus the problem reduces to choosing F and G such that X is unit and such that $G\frac{\partial F}{\partial z} - F\frac{\partial G}{\partial z}$ can have either sign. Now with respect to Darboux coordinates the last row (and column)

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of the matrix of components of an associated metric g is $(-\frac{y^1}{4},\dots,-\frac{y^n}{4},0,\dots,0,\frac{1}{4})$. Thus the requirement that X be unit becomes

$$1 = g_{11}G^2 - \frac{(y^1)^2}{4}G^2 + 2g_{1\,n+1}FG + g_{n+1\,n+1}F^2.$$

Now choose two functions of (x^i, y^i, z), to be denoted by θ, such that $\frac{\partial\theta}{\partial z}$ is positive for one function and negative for the other and such that

$$\left(g_{11} - \frac{(y^1)^2}{4}\right) + 2g_{1\,n+1}\theta + g_{n+1\,n+1}\theta^2 > 0.$$

Let G be a solution of $G^2\big((g_{11} - \frac{(y^1)^2}{4}) + 2g_{1\,n+1}\theta + g_{n+1\,n+1}\theta^2\big) = 1$ and set $F = \theta G$. Then $X = F\frac{\partial}{\partial y^1} + G(\frac{\partial}{\partial x^1} + y^1\frac{\partial}{\partial z})$ is a unit vector field and $G\frac{\partial F}{\partial z} - F\frac{\partial G}{\partial z} = G^2\frac{\partial\theta}{\partial z}$ can be chosen to have either sign.

As an application of Theorem 1 we now prove Theorem 2.

PROOF OF THEOREM 2. Suppose that g_0 is a Sasakian metric and a local minimum of $A(g)$ restricted to $\mathcal{A}$. Then there exists a neighborhood $\mathcal{U}$ of $g_0 \in \mathcal{A}$ on which $A(g_0) \leq A(g)$. Since all associated metrics have the same volume element, $\int_M R_0 + 2n\,dV \leq]$ $\int_M R + 2n\,dV$ for every $g \in \mathcal{U}$. On the other hand from Olszak's formula (2.3),

$$W = \frac{1}{2}\big(R + R^* + 4n(n+1)\big) \geq R + 2n$$

with equality if and only if the metric is Sasakian. Thus we have

$$I(g_0) = \int_M W_0\,dV = \int_M R_0 + 2n\,dV \leq \int_M R + 2n\,dV \leq \int_M W\,dV = I(g)$$

for every $g \in \mathcal{U}$, that is g_0 is a local minimum for $I(g)$ contradicting Theorem 1.

REMARK. Recently the second author in [10] considered the functional $F(g) = \int_M \frac{1}{2}(R + R_1)\,dV$ where $R_1 = R^* + 2n\,\mathrm{Ric}(\xi)$. For a 3-dimensional contact manifold $F(g) = A(g)$. In general the critical point condition was shown in [10] to be simply $\nabla_\xi h = 0$. From (2.2),

$$R + R_1 = R + R^* + 2n(2n - \operatorname{tr} h^2) \leq R + R^* + 4n^2$$

with equality if and only if $h = 0$. Thus as in the proof of Theorem 2, Theorem 1 yields the following result.

THEOREM 3. *The functional $F(g)$ cannot have a local minimum at any K-contact metric.*

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REFERENCES

1. D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math. **509**, Springer, Berlin, 1976.
2. ———, *On the set of metrics associated to a symplectic or contact form*, Bull. Inst. Math. Acad. Sinica **11**(1983), 297–308.
3. ———, *The "total scalar curvature"as a symplectic invariant*, Proc. 3rd Congress of Geometry, Thessaloniki, 1991, 79–83.
4. D. E. Blair and A. J. Ledger, *Critical associated metrics on contact manifolds II*, J. Austral. Math. Soc. Ser. A **41**(1986), 404–410.
5. D. E. Blair and D. Perrone, *A variational characterization of contact metric manifolds with vanishing torsion*, Canad. Math. Bull., **35**(1992), 455–462.
6. S. S. Chern and R. S. Hamilton, *On Riemannian metrics adapted to three-dimensional contact manifolds*, Lecture Notes in Math. **1111**, Springer, Berlin, 1985, 279–308.
7. Y. Muto, *On Einstein metrics*, J. Differential Geom. **9**(1974), 521–530.
8. Z Olszak, *On contact metric manifolds*, Tôhoku Math. J. **31**(1979), 247–253.
9. D. Perrone, *Torsion and critical metrics on contact three-manifolds*, Kodai Math. J. **13**(1990), 88–100.
10. ———, *Torsion tensor and critical metrics on contact* $(2n+1)$*-manifolds*, Mh. Math. **114**(1992), 245–259.

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