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# AN OSCILLATION CRITERION FOR FIRST ORDER LINEAR DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. A new oscillation criterion is given for the delay differential equation $x'(t) + p(t)x\bigl(t - \tau(t)\bigr) = 0$, where $p, \tau \in C\bigl([0,\infty),[0,\infty)\bigr)$ and the function $T$ defined by $T(t) = t - \tau(t)$, $t \geq 0$ is increasing and such that $\lim_{t\to\infty} T(t) = \infty$. This criterion concerns the case where $\liminf_{t\to\infty} \int_{T(t)}^{t} p(s)\,ds \leq \frac{1}{e}$.

1. **Introduction and Statement of the Main Result.** The oscillatory behaviour of the solutions of first order linear delay differential equations has been extensively studied in recent years. See, for example, the book by Ladde, Lakshmikantham and Zhang [9] and the recent book by Györi and Ladas [4]. For first order linear non-autonomous delay differential equations, it is an interesting problem to obtain sufficient conditions for the oscillation of all solutions. Among numerous papers dealing with this problem we refer in particular to Elbert and Stavroulakis [1, 2], Erbe and Zhang [3], Koplatadze and Chanturija [5], Kwong [6], Ladas [7], and Ladas, Lakshmikantham and Papadakis [8]. In this paper, a new oscillation criterion is established for first order linear non-autonomous delay differential equations.

Consider the first order linear non-autonomous delay differential equation

$$x'(t) + p(t)x\bigl(t - \tau(t)\bigr) = 0, \tag{E}$$

where *p and* $\tau$ *are nonnegative continuous real-valued functions on the interval* $[0,\infty)$. It will be supposed that *the function T defined by*

$$T(t) = t - \tau(t), \quad t \geq 0$$

*is increasing and such that*

$$\lim_{t\to\infty} T(t) = \infty.$$

Let $t_0 \geq 0$. By a *solution on* $[t_0,\infty)$ of the differential equation (E) we mean a continuous real-valued function $x$ defined on the interval $[t_{-1},\infty)$, where $t_{-1} = \min_{t\geq t_0} T(t)$, which is continuously differentiable on $[t_0,\infty)$ and satisfies (E) for all $t \geq t_0$.

As usual, a solution of (E) is said to be *oscillatory* if it has arbitrarily large zeros, and otherwise the solution is said to be *nonoscillatory*. Equation (E) will be called oscillatory if all its solutions are oscillatory.

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Throughout the paper, we will use the notations

$$L = \liminf_{t \to \infty} \int_{T(t)}^{t} p(s)\, ds \quad \text{and} \quad M = \limsup_{t \to \infty} \int_{T(t)}^{t} p(s)\, ds.$$

(Clearly, $0 \leq L \leq M \leq \infty$.)

Ladas [7] and Koplatadze and Chanturija [5] established that the differential equation (E) is oscillatory if $L > 1/e$. (Note that the hypothesis that the function $T$ is increasing is not needed for this criterion to hold.) In the autonomous case, *i.e.*, in the case where $p$ and $\tau$ are positive real numbers, the condition $L > 1/e$ becomes $p\tau e > 1$ and the last condition is also necessary for the oscillation of (E) (see [7]). Moreover, Ladas, Lakshmikantham and Papadakis [8] proved that the condition $M > 1$ suffices for the oscillation of the differential equation (E).

An interesting problem is to find conditions on $L$ and/or $M$ which guarantee the oscillation of the differential equation (E), in the case where $L \leq 1/e$. Some such conditions have recently obtained by Erbe and Zhang [3], Kwong [6], and Elbert and Stavroulakis [1, 2]. In this paper, we assume that $L \leq 1/e$ and we present a new sufficient condition (depending on $L$ and $M$) for the oscillation of the differential equation (E). Our condition is not complicated.

Our main result is the following theorem.

THEOREM 1.1. *Assume that* $L \leq 1/e$. *Equation* (E) *is oscillatory if*

$$M + \frac{L^2}{2(1-L)} + \frac{L^2}{2}\lambda_0 > 1, \tag{C}$$

*where* $\lambda_0 > 0$ *is the smaller real root of the transcendental equation*

$$\lambda = e^{L\lambda}. \tag{$\star$}$$

If $L = 0$, then it is obvious that $\lambda_0 = 1$ is the unique root of ($\star$). Also, when $L = 1/e$, we can immediately see that ($\star$) has the the unique real root $\lambda_0 = e$. Moreover, if $0 < L < 1/e$, then it is not difficult to verify that ($\star$) has two real roots $\lambda_0$ and $\lambda_1$ with $0 < \lambda_0 < e < \lambda_1$.

Now, an immediate consequence of our theorem is the following result:

*Equation* (E) *is oscillatory in each of the following cases*: (i) $L = 0$ *and* $M > 1$, (ii) $L = 1/e$ *and* $M > 1 - \frac{1}{2(e-1)}$, *and* (iii) $0 < L < 1/e$ *and* $M \geq 1$.

Kwong [6] established that, if $0 < L \leq 1/e$, then for any nonoscillatory solution $x$ of the differential equation (E) it holds

$$\liminf_{t \to \infty} \frac{x\big(T(t)\big)}{x(t)} \geq \lambda_0.$$

(Note that the assumption that $T$ is increasing is not needed in this result.) This asymptotic result will be used in the proof of our theorem.

Before closing this section, we note that our theorem can be formulated in a more general form as follows:

*Assume that* $L \leq 1/e$. *If* (C) *holds, where* $\lambda_0 > 0$ *is the smaller real root of* $(\star)$, *then the differential inequality*

$$x'(t) + p(t)x\big(t - \tau(t)\big) \leq 0$$

*has no eventually positive solutions, and the differential inequality*

$$x'(t) + p(t)x\big(t - \tau(t)\big) \geq 0$$

*has no eventually negative solutions.*

2. **Proof of the Theorem.** Assume, for the sake of contradiction, that the differential equation (E) has a nonoscillatory solution $x$ on an interval $[t_0, \infty)$, $t_0 \geq 0$. Without loss of generality, we can suppose that $x(t) \neq 0$ for all $t \geq t_{-1}$, where $t_{-1} = \min_{t \geq t_0} T(t)$. Furthermore, as the negative of a solution of (E) is also a solution of the same equation, we may (and do) assume that $x$ is positive on the interval $[t_{-1}, \infty)$. Then from (E) it follows that $x'(t) \leq 0$ for every $t \geq t_0$ and so $x$ is decreasing on $[t_0, \infty)$. In what follows, the fact that $x$ is decreasing on $[t_0, \infty)$ as well as the increasing character of $T$ (on the interval $[0, \infty)$) will be used without mention.

In the sequel, for convenience, we will assume that inequalities about values of functions are satisfied eventually for all large $t$. Similarly, it will be supposed that inequalities about terms of sequences are satisfied eventually for all large $n$.

First of all, let us consider the extreme case where $L = 0$. Then condition (C) becomes $M > 1$. On the other hand, from (E) it follows that

$$x\big(T(t)\big) = x(t) + \int_{T(t)}^{t} p(s)x\big(T(s)\big)\,ds > x\big(T(t)\big)\int_{T(t)}^{t} p(s)\,ds$$

and so

$$\int_{T(t)}^{t} p(s)\,ds < 1,$$

which leads to the contradiction $M \leq 1$. Hence, in the rest of the proof, it will be assumed that $L > 0$.

Consider three arbitrary real numbers $\delta$, $\theta$ and $\gamma$ with

$$0 < \delta < L, \quad 0 < \theta < M \quad \text{and} \quad 0 < \gamma < \lambda_0.$$

Clearly, it holds

$$\int_{T(t)}^{t} p(s)\,ds \geq \delta. \tag{1}$$

Moreover, if we choose a sequence $(t_n)$ of real numbers with $\lim_{n\to\infty} t_n = \infty$ and such that

$$\lim_{n\to\infty} \int_{T(t_n)}^{t_n} p(s)\,ds = M$$

(such a sequence always exists), then we obviously have

$$\int_{T(t_n)}^{t_n} p(s)\,ds \geq \theta. \tag{2}$$

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Furthermore, by Lemma 1 in [6], $\liminf_{t\to\infty}\left[x\big(T(t)\big)/x(t)\right] \geq \lambda_0$ and consequently

$$\frac{x\big(T(t)\big)}{x(t)} \geq \gamma, \quad i.e. \quad x\big(T(t)\big) \geq \gamma x(t). \tag{3}$$

Let now $t$ be an arbitrary point which is sufficiently large. Consider an arbitrary integer $k$ with $k \geq 2$. By taking into account (1), we can choose points $\xi_0, \xi_1, \ldots, \xi_k$ with

$$T(t) = \xi_0 < \xi_1 < \cdots < \xi_{k-1} < \xi_k = t$$

and such that

$$\int_{\xi_0}^{\xi_1} p(s)\,ds = \cdots = \int_{\xi_{k-2}}^{\xi_{k-1}} p(s)\,ds = \frac{\delta}{k} \quad \text{and} \quad \int_{\xi_{k-1}}^{\xi_k} p(s)\,ds < \frac{\delta}{k}. \tag{4}$$

Then from (E) it follows that

$$x\big(T(t)\big) - x(t) = \int_{T(t)}^{t} p(s)x\big(T(s)\big)\,ds = \sum_{\nu=1}^{k}\int_{\xi_{\nu-1}}^{\xi_\nu} p(s)x\big(T(s)\big)\,ds$$

and so we have

$$x\big(T(t)\big) - x(t) \geq \sum_{\nu=1}^{k} x\big(T(\xi_\nu)\big)\int_{\xi_{\nu-1}}^{\xi_\nu} p(s)\,ds. \tag{5}$$

Furthermore, for any $\nu \in \{1, 2, \ldots, k-1\}$, from (E) we obtain

$$x\big(T(\xi_\nu)\big) - x\big(T(t)\big) = \int_{T(\xi_\nu)}^{T(t)} p(s)x\big(T(s)\big)\,ds.$$

Therefore

$$x\big(T(\xi_\nu)\big) \geq x\big(T(t)\big) + x\Big(T\big[T(t)\big]\Big)\int_{T(\xi_\nu)}^{T(t)} p(s)\,ds \quad (\nu = 1, 2, \ldots, k-1). \tag{6}$$

But, by using (1) and (4), we derive for every $\nu = 1, 2, \cdots, k-1$

$$\begin{aligned}\int_{T(\xi_\nu)}^{T(t)} p(s)\,ds &= \int_{T(\xi_\nu)}^{\xi_\nu} p(s)\,ds - \int_{T(t)}^{\xi_\nu} p(s)\,ds \\ &= \int_{T(\xi_\nu)}^{\xi_\nu} p(s)\,ds - \left[\int_{\xi_0}^{\xi_1} p(s)\,ds + \cdots + \int_{\xi_{\nu-1}}^{\xi_\nu} p(s)\,ds\right] \\ &\geq \delta - \nu\frac{\delta}{k} = (k-\nu)\frac{\delta}{k}.\end{aligned}$$

Thus, (6) gives

$$x\big(T(\xi_\nu)\big) \geq x\big(T(t)\big) + (k-\nu)\frac{\delta}{k}x\Big(T\big[T(t)\big]\Big) \quad (\nu = 1, 2, \ldots, k-1). \tag{7}$$

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Hence, by taking into account (4) and (7), from (5) we obtain

$$
\begin{aligned}
x\big(T(t)\big) - x(t) &\geq x\big(T(t)\big) \int_{\xi_{k-1}}^{t} p(s)\,ds + \sum_{\nu=1}^{k-1} x\big(T(\xi_\nu)\big) \int_{\xi_{\nu-1}}^{\xi_\nu} p(s)\,ds \\
&= x\big(T(t)\big) \left[ \int_{T(t)}^{t} p(s)\,ds - \int_{\xi_0}^{\xi_1} p(s)\,ds - \cdots - \int_{\xi_{k-2}}^{\xi_{k-1}} p(s)\,ds \right] \\
&\qquad + \sum_{\nu=1}^{k-1} x\big(T(\xi_\nu)\big) \int_{\xi_{\nu-1}}^{\xi_\nu} p(s)\,ds \\
&= x\big(T(t)\big) \left[ \int_{T(t)}^{t} p(s)\,ds - (k-1)\frac{\delta}{k} \right] + \frac{\delta}{k} \sum_{\nu=1}^{k-1} x\big(T(\xi_\nu)\big) \\
&= \left[ \int_{T(t)}^{t} p(s)\,ds \right] x\big(T(t)\big) - (k-1)\frac{\delta}{k} x\big(T(t)\big) \\
&\qquad + \frac{\delta}{k} \sum_{\nu=1}^{k-1} x\big(T(\xi_\nu)\big) \\
&\geq \left[ \int_{T(t)}^{t} p(s)\,ds \right] x\big(T(t)\big) - (k-1)\frac{\delta}{k} x\big(T(t)\big) \\
&\qquad + \frac{\delta}{k} \sum_{\nu=1}^{k-1} \left[ x\big(T(t)\big) + (k-\nu)\frac{\delta}{k} x\big(T\big[T(t)\big]\big) \right] \\
&= \left[ \int_{T(t)}^{t} p(s)\,ds \right] x\big(T(t)\big) - (k-1)\frac{\delta}{k} x\big(T(t)\big) \\
&\qquad + (k-1)\frac{\delta}{k} x\big(T(t)\big) + \Big[ \sum_{\nu=1}^{k-1} (k-\nu) \Big] \frac{\delta^2}{k^2} x\big(T\big[T(t)\big]\big)
\end{aligned}
$$

and consequently

$$
x\big(T(t)\big) - x(t) \geq \left[ \int_{T(t)}^{t} p(s)\,ds \right] x\big(T(t)\big) + \left(1 - \frac{1}{k}\right) \frac{\delta^2}{2} x\big(T\big[T(t)\big]\big).
$$

But $k$ is an arbitrary integer with $k \geq 2$. So, as $k \longrightarrow \infty$, we get

$$
x\big(T(t)\big) - x(t) \geq \left[ \int_{T(t)}^{t} p(s)\,ds \right] x\big(T(t)\big) + \frac{\delta^2}{2} x\big(T\big[T(t)\big]\big). \tag{8}
$$

This inequality holds for all sufficiently large $t$.

Next, in view of (1) and (2), from (8) it follows that

$$
x\big(T(t)\big) - x(t) \geq \delta x\big(T(t)\big) + \frac{\delta^2}{2} x\big(T\big[T(t)\big]\big) \tag{9}
$$

and

$$
x\big(T(t_n)\big) - x(t_n) \geq \theta x\big(T(t_n)\big) + \frac{\delta^2}{2} x\big(T\big[T(t_n)\big]\big). \tag{10}
$$

Because of (3), it holds

$$
x\big(T\big[T(t_n)\big]\big) \geq \gamma x\big(T(t_n)\big)
$$

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and so (10) gives

$$x\big(T(t_n)\big) \geq \theta x\big(T(t_n)\big) + x(t_n) + \frac{\delta^2}{2}\gamma x\big(T(t_n)\big). \tag{11}$$

But, from (9) we obtain

$$x\big(T(t)\big) \geq \delta x\big(T(t)\big) + \frac{\delta^2}{2} x\big(T\big[T(t)\big]\big),$$

*i.e.*,

$$x\big(T(t)\big) \geq \frac{\delta^2}{2(1-\delta)} x\big(T\big[T(t)\big]\big).$$

This yields

$$x(t) \geq \frac{\delta^2}{2(1-\delta)} x\big(T(t)\big). \tag{12}$$

(Note that $1-\delta > 0$.) By using (12), from (11) we derive

$$\begin{aligned} x\big(T(t_n)\big) &\geq \theta x\big(T(t_n)\big) + \frac{\delta^2}{2(1-\delta)} x\big(T(t_n)\big) + \frac{\delta^2}{2}\gamma x\big(T(t_n)\big) \\ &= \left[\theta + \frac{\delta^2}{2(1-\delta)} + \frac{\delta^2}{2}\gamma\right] x\big(T(t_n)\big) \end{aligned}$$

and consequently we must have

$$\theta + \frac{\delta^2}{2(1-\delta)} + \frac{\delta^2}{2}\gamma \leq 1. \tag{13}$$

We have thus proved that (13) is satisfied for all real numbers $\delta$, $\theta$ and $\gamma$ with $\delta \in (0, L)$, $\theta \in (0, M)$ and $\gamma \in (0, \lambda_0)$. As $\delta \longrightarrow L - 0$, $\theta \longrightarrow M - 0$ and $\gamma \longrightarrow \lambda_0 - 0$, (13) gives

$$M + \frac{L^2}{2(1-L)} + \frac{L^2}{2}\lambda_0 \leq 1,$$

which contradicts condition (C). (Note that $L < 1$.)

The proof of the theorem is complete.

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