Local and global structure of connections on nonarchimedean curves

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Abstract

Consider a vector bundle with connection on a $p$-adic analytic curve in the sense of Berkovich. We collect some improvements and refinements of recent results on the structure of such connections, and on the convergence of local horizontal sections. This builds on work from the author’s 2010 book and on subsequent improvements by Baldassarri and by Poineau and Pulita. One key result exclusive to this paper is that the convergence polygon of a connection is locally constant around every type 4 point.

Introduction

The theory of $p$-adic ordinary differential equations has been an active part of number theory ever since the pioneering work of Dwork, starting with his $p$-adic analytic proof of the rationality aspect of the Weil conjectures circa 1960 (predating the development of étale cohomology). The subsequent half-century saw slow but substantial progress on the question of convergence of solutions of $p$-adic differential equations; in that time, new spheres of application (rigid cohomology, $p$-adic Hodge theory, numerical computation of zeta functions, $p$-adic dynamical systems) have attracted additional attention to the area. A broad survey of the theory of $p$-adic differential equations has been given recently by the author in the book [Ked10a].

At about the time that [Ked10a] was written, it was observed by Baldassarri [Bal10, BD07] that the classical theory of $p$-adic differential equations could be rearticulated much more clearly using Berkovich’s language of analytic geometry over complete nonarchimedean fields. That is because the classical theory is heavily concerned with the convergence of local solutions of $p$-adic differential equations around certain generic points, which appear naturally in Berkovich’s framework on an equal footing with rigid analytic points. In this language, one can also naturally treat general $p$-adic curves, not just subspaces of the affine line, by using semistable models to obtain scaling parameters; Baldassarri demonstrated this in [Bal10] by establishing continuity of the radius of convergence for a differential module over a semistable $p$-adic curve.

The radius of convergence function for a differential module over a curve measures only the joint radius of convergence of all local horizontal sections around a point. A finer invariant is the convergence polygon, a Newton polygon whose slopes record the extent to which there exist subspaces of the local horizontal sections which converge on larger discs. Building on the results of [Ked10a], it has been shown recently by Poineau and Pulita [PP12a, Pul14] that...
the convergence polygon is again a continuous function which factors through the retraction onto some finite skeleton (as in the work of Payne [Pay09]). Informally, this means that the convergence of local horizontal sections is controlled by finitely many numerical invariants. Another proof is included in this paper, while a simplified version of our proof will appear in [BK]. While formally different, these proofs share many common ingredients; for instance, our key Lemma 4.3.12 is materially equivalent to [Pul14, Proposition 7.5]. In fact, the main difference between the arguments here and those in [Pul14] is that the combinatorial argument there is replaced by a compactness argument.

The purpose of this paper is to collect some results about differential modules on nonarchimedean analytic curves over fields of characteristic 0 which refine and extend the aforementioned results as well as some other results from [Ked10a]. Here is a partial list of the new results of the present paper.

- We make a finer analysis of refined differential modules over a field of analytic functions than is made in [Ked10a]; see § 2.3. This leads to results about refined differential modules on open annuli; see § 3.7.
- We provide more detailed discussion of the theory of exponents for differential modules on annuli satisfying the Robba condition (existence of horizontal sections over any open disc); see § 3.2 and § 3.4.
- We generalize the p-adic local monodromy theorem to arbitrary differential modules over an open annulus at one boundary, with no hypotheses on Frobenius structures or p-adic exponents; see § 3.8.
- We show that the convergence polygon of a differential module on a curve is constant locally around any point of type 4; see § 4.4. This strengthens the continuity theorem of [PP12a, PP12b, Pul14].
- We show that every curve admits a triangulation such that locally at any interior point, the connection decomposes into a particularly simple form; see § 5.4. Such triangulations and decompositions can be used to give a global version of the Christol–Mebkhout index formula; see [PP13a, PP13b] for some arguments along these lines.

As in [Ked10a], we have made an effort to maintain as much parity as possible between the cases of zero and positive residual characteristic. One unavoidable complication in the latter case is the existence of some pathologies in the theory of regular singularities caused by the existence of p-adic Liouville numbers (numbers which are not integers but which admit extremely good integer approximations). These complications generally emerge when considering cohomology; in this paper, we primarily limit ourselves to statements of a ‘precohomological’ nature, for which one can skirt these complications with some extra work.

Note that while many of the interesting applications of p-adic differential equations involve spaces of dimension greater than one, in this paper we follow the model of [Ked10a] and confine attention to ordinary p-adic differential equations. It is of course natural to consider also higher-dimensional spaces; in so doing, one should be able to obtain a unification of some existing work. Such work would include the study of good formal structures for formal flat meromorphic connections [Ked10b, Ked11a] in the case of zero residual characteristic, and semistable reduction for overconvergent F-isocrystals [Ked07, Ked08, Ked09, Ked11b] in the case of positive residual characteristic.
1. Preliminaries

We begin with some assorted preliminary definitions and arguments. This also provides an opportunity to set running notation for the whole paper.

**Notation 1.0.1.** Throughout the paper, let $K$ denote an analytic field (a field equipped with a nonarchimedean multiplicative norm $|\cdot|$ with respect to which it is complete) of characteristic 0. Let $\mathfrak{o}_K$, $\mathfrak{m}_K$, and $\kappa_K$ denote the valuation subring, maximal ideal, and residue field of $K$, respectively. Let $p$ denote the characteristic of $\kappa_K$; put $\omega = 1$ if $p = 0$ and $\omega = p^{-1/(p-1)}$ if $p > 0$. Let $\mathbb{C}$ denote a completed algebraic closure of $K$.

1.1 A lemma on linear groups

We need a bit of elementary analysis of linear groups in the spirit of André’s abstract analysis of filtrations [And09]. This will be used to analyze the structure of the automorphism groups of certain Tannakian categories, especially those generated by refined differential modules over fields (§2.3) and solvable differential modules over annuli (§3.8). For the formalism of Tannakian categories, including the Tannaka–Krein duality theorem, see [Saa72].

**Lemma 1.1.1.** Let $F$ be a field of characteristic 0. Fix a positive integer $n$ and let $G_0 \subseteq G_1 \subseteq \cdots$ be an increasing sequence of finite subgroups of $\text{GL}_n(F)$ such that $G_i$ is normal in $G_j$ whenever $i \leq j$.

(a) The union $G = \bigcup_{i=0}^{\infty} G_i$ contains an abelian normal subgroup $H$ of finite index.

(b) There exists an index $i$ such that $G/G_i$ is isomorphic to a subgroup of $(\mathbb{Q}/\mathbb{Z})^n$ and, in particular, is abelian.

**Proof.** By Jordan’s theorem on finite linear groups [CR06, ch. 36], there exists a constant $f(n)$ such that each $G_i$ contains an abelian normal subgroup of index at most $f(n)$. Let $S_i$ be the set of abelian normal subgroups of $G_i$ of index at most $f(n)$. For each $H_j \in S_j$ and each $i \leq j$, the map $G_i/(S_i \cap H_j) \to G_j/H_j$ is injective, so $G_i \cap H_j \in S_i$. We may thus assemble the sets $S_i$ into an inverse system via restriction, and the inverse limit is necessarily nonempty by Tikhonov’s theorem. This proves (a).

Given (a), let $\overline{F}$ be an algebraic closure of $F$. Then $H$ is an abelian torsion group which embeds into $(\overline{F}^*)^n$. This implies that $H$ is isomorphic to a subgroup of $(\mathbb{Q}/\mathbb{Z})^n$, as then is any quotient of $H$. Note also that since the group $G/H$ is finite and is the union of its subgroups $G_i/(G_i \cap H)$, there must exist an index $i$ for which the inclusion $G_i/(G_i \cap H) \to G/H$ is bijective. The group $G/G_i$ is then isomorphic to the abelian group $H/(G_i \cap H)$. This proves (b). \hfill $\Box$

**Proposition 1.1.2.** Let $F$ be a field of characteristic 0. Let $V$ be a finite-dimensional $F$-vector space. Let $G$ be an algebraic subgroup of $\text{GL}(V)$. Let $\{G^r\}_{r \in \mathbb{R}}$ be a family of normal algebraic subgroups of $G$. For $r \geq -\infty$, put $G^{r+} = \bigcup_{s \geq r} G^s$. Assume also the following conditions.

(a) For every $r, s \in \mathbb{R}$ with $r \leq s$, $G^r$ is a normal subgroup of $G^s$.

(b) For every $s \in \mathbb{R}$, there exists $r < s$ such that $G^r = G^s$.

(c) There exists $r \in \mathbb{R}$ such that $G^r$ is the trivial group.

(d) For every $r \in \mathbb{R}$ for which $G^{r+}$ is finite and all nonnegative integers $g, h$, the $G^r$-invariant subspace of $(V^\vee)^{\otimes g} \otimes V^{\otimes h}$ admits a direct sum decomposition into $G$-stable subspaces, each of which restricts to an isotypical representation of $G^r/G^{r+}$.

(e) In (d), the isotypical representations of $G^r/G^{r+}$ that occur all have finite image.
(f) For all nonnegative integers \( g, h \) and every one-dimensional \( G \)-stable subspace \( W \) of \( (V^\vee)^{\otimes g} \otimes V^{\otimes h} \), the image of \( G^{-\infty+} \) in \( \text{GL}(W) \) is finite. Then \( G^{-\infty+} \) is itself finite.

Proof. Let \( S \) be the set of \( r \in \mathbb{R} \) for which \( G^r \) is finite. By (a), the set \( S \) is up-closed. By (b), the set \( S \) does not contain its infimum. By (c), the set \( S \) is nonempty.

Put \( r = \inf S \); by the previous paragraph, \( r \notin S \). Suppose by way of contradiction that \( G^r \) is infinite. By Lemma 1.1.1, there exists \( s_0 > r \) such that \( G^{r+s_0} / G^s \) embeds into a product of finitely many copies of \( \mathbb{Q} / \mathbb{Z} \). By Tannaka–Krein duality, we can choose \( g, h \) so that \( (V^\vee)^{\otimes g} \otimes V^{\otimes h} \) contains a \( G \)-stable subspace \( X \) on which \( G^{s_0} \) acts trivially but \( G^s \) acts nontrivially for some \( s \in (r, s_0) \). By applying (d) finitely many times (with \( r \) replaced by varying choices of \( s \)), we can split \( X \) as a direct sum of \( G \)-stable summands, each of which is \( G^{r+} \)-isotypical. Since \( G^r \) is not finite, we can choose a \( G \)-stable summand \( Y \) of \( X \) such that \( G^{r+} \) has image in \( \text{GL}(Y) \) isomorphic to an infinite subgroup of \( \mathbb{Q} / \mathbb{Z} \). Put \( W = \wedge^{\dim(Y)} Y \); this space occurs as a \( G \)-invariant subspace of \( (V^\vee)^{\otimes g} \otimes V^{\otimes h} \) for some possibly different values of \( g \) and \( h \). However, the image of \( G^{r+} \) in \( \text{GL}(W) \) is again isomorphic to an infinite subgroup of \( \mathbb{Q} / \mathbb{Z} \), contradicting (f).

We conclude that \( G^r \) is finite. Suppose now that \( r \in \mathbb{R} \). By Tannaka–Krein duality, the action of \( G^r \) on the direct sum of the \( G^{r+} \)-invariant subspaces of \( (V^\vee)^{\otimes g} \otimes V^{\otimes h} \) over all nonnegative integers \( g, h \) is a faithful representation of \( G^r / G^{r+} \). However, by (e), the action on each individual summand factors through a finite group; since \( G^r \) is algebraic, this implies that \( G^r \) is finite. But then \( r \in S \), a contradiction. We must thus have \( r = -\infty \), which yields the desired result.

We will apply Proposition 1.1.2 via the following Tannakian interpretation.

Remark 1.1.3. Let \( F \) be a field of characteristic 0. Let \( C \) be a Tannakian category equipped with a fibre functor \( \omega \) to the category of finite-dimensional \( F \)-vector spaces. Assign to each nonzero element \( V \in C \) an element \( r = r(V) \in \mathbb{R} \cup \{ -\infty \} \) depending only on the isomorphism class of \( V \), subject to the following conditions.

(a) For any \( V \in C \), \( r(V^\vee) = r(V) \).

(b) For any short exact sequence \( 0 \to V_1 \to V \to V_2 \to 0 \) in \( C \), \( r(V) = \max\{ r(V_1), r(V_2) \} \).

(c) For any \( V_1, V_2 \in C \), \( r(V_1 \otimes V_2) \leq \max\{ r(V_1), r(V_2) \} \).

For \( V \in C \), let \([V]\) denote the Tannakian subcategory of \( C \) generated by \( V \); note that \( r(W) \leq r(V) \) for all \( W \in [V] \). Let \( G(V) \subseteq \text{GL}(\omega(V)) \) denote the automorphism group of the restriction of \( \omega \) to \([V]\); this is an algebraic group over \( F \), so all of its pro-algebraic quotients are also algebraic. For \( r \in \mathbb{R} \), let \( G^r(V) \) be the subgroup of \( G(V) \) acting trivially on \( \omega(W) \) for all \( W \in [V] \) with \( r(W) < r \). For \( r \in \mathbb{R} \cup \{ -\infty \} \), put \( G^{r+}(V) = \bigcup_{s > r} G^s(V) \); if this group is finite, then it equals the subgroup of \( G(V) \) acting trivially on \( \omega(W) \) for all \( W \in [V] \) with \( r(W) \leq r \) (because there exists \( s > r \) for which \( G^s(V) = G^{r+}(V) \) and hence \( r(W) \notin (r, s) \) for all \( W \in [V] \)).

The groups \( G^r(V) \) then satisfy conditions (a), (b), (c) of Proposition 1.1.2. This is evident for (a) and (c). For (b), note that the objects \( W \in [V] \) for which \( G^s(V) \) acts trivially on \( \omega(W) \) form a Tannakian category which is finitely generated (because restricting \( \omega \) gives a fibre functor whose automorphism group \( G(V) / G^s(V) \) is algebraic, not just pro-algebraic).

To enforce conditions (d), (e), (f) of Proposition 1.1.2, it would suffice to have the following additional information about \( C \).

(i) Every \( V \in C \) with \( r(V) > -\infty \) admits a direct sum decomposition \( V = \bigoplus_i V_i \) in which each summand \( V_i \) satisfies \( r(V_i^\vee \otimes V_i) < r(V) \). (This implies (d).)
By Remark 1.1.3, \( p \) is an integer and \( n \) is the smallest positive integer such that \( r(V^\otimes n) < r(V) \). (Given (i), this implies (e).)

(iii) For every \( V \in \mathcal{C} \) with \( \dim_F \omega(V) = 1 \), there exists a positive integer \( n \) such that \( r(V^\otimes n) = -\infty \). (This implies (f).)

Note also that if in (ii) and (iii) the integer \( n \) can always be taken to be a power of a fixed prime \( p \), then the group \( G^{-\infty+}(V) \) is forced to be not only finite but also a \( p \)-group.

Lemma 1.1.4. Suppose that the conditions of Remark 1.1.3 hold and that in (ii) and (iii), the integer \( n \) can always be taken to be a power of a fixed prime \( p \). Then for any \( V \in \mathcal{C} \) with \( r(V) > -\infty \), there exists \( W \in \mathcal{C} \) such that the action of \( G^{-\infty+}(V) \) on \( W \) is \( \tau \)-isotypical for some character \( \tau : G^{-\infty+}(V) \to GL_1(F) \) of order \( p \).

Proof. By Remark 1.1.3, \( G^{-\infty+}(V) \) is a finite \( p \)-group, which must be nontrivial since \( r(V) > -\infty \). The group \( G^{-\infty+}(V) \) thus admits a character \( \tau : G^{-\infty+}(V) \to GL_1(F) \) of order \( p \). Let \( r(\tau) > -\infty \) be the smallest value of \( r \) for which \( G^{r+}(V) \subseteq \ker(\tau) \), and choose \( \tau \) to minimize \( r(\tau) \).

By Tannaka–Krein duality for \( G^{-\infty+}(V) \), we may choose nonnegative integers \( g, h \) such that \( \tau \) occurs in the action of \( G^{-\infty+}(V) \) on \( (V^\vee)^{\otimes g} \otimes V^{\otimes h} \). Then \( \tau \) also occurs in the action of \( G^{-\infty+}(V) \) on some irreducible subquotient \( W \) of \( (V^\vee)^{\otimes g} \otimes V^{\otimes h} \).

Since \( G^{-\infty+}(V) \) is a finite group, its action on \( W \) is completely reducible and thus admits an isotypical decomposition. Since \( W \) is irreducible, all of its isotypical components must correspond to conjugates of \( \tau \) by the action of \( G(V) \) on its normal subgroup \( G^{-\infty+}(V) \). In particular, each of these conjugates \( \tau' \) is a character of order \( p \) with \( r(\tau') = r(\tau) \).

It follows that \( r(W) = r(\tau') \). By property (i) of Remark 1.1.3, we have \( r(W^{\vee} \otimes W) < r(W) \); however, the irreducible representations of \( G^{-\infty+}(V) \) appearing in \( W^{\vee} \otimes W \) are characters of order dividing \( p \), so by our minimization of \( r(\tau) \) these characters must be trivial. That is, \( r(W^{\vee} \otimes W) = -\infty \), which implies that \( W \) is \( \tau \)-isotypical.

### 1.2 A lemma on local fields

We introduce an auxiliary calculation concerning local fields in positive characteristic. This is needed for the study of solvable differential modules at type 4 points (§4.4). We use without comment some basic facts about higher ramification of local fields, for which see [Ked10a, ch. 3] for a brief summary or [Ser79] for a complete treatment.

Hypothesis 1.2.1. Throughout this subsection, assume that \( p > 0 \) and let \( k \) be an algebraically closed field of characteristic \( p \).

Definition 1.2.2. Let \( N \) be the pro-unipotent pro-algebraic group over \( k \) whose \( p \)-points are identified with the \( t \)-adically continuous \( k \)-linear automorphisms \( \psi \) of \( k((t)) \) fixing \( t \) modulo \( t^2 \).

The group \( N \) is filtered by the pro-algebraic subgroups

\[
N_m = \ker(N \to \text{Aut}(k[[t]]/t^{m+1})) \quad (m = 1, 2, \ldots)
\]

for which \( N_1 = N \) and each successive quotient \( N_m/N_{m+1} \) is isomorphic to the additive group (though not canonically). We will write \( N_t \) and \( N_{m,t} \) instead of \( N \) and \( N_m \) when it is necessary to specify the series variable \( t \) in the notation. (The analogous construction with \( k = \mathbb{F}_p \) is sometimes called the Nottingham group.)

Hypothesis 1.2.3. For the remainder of this subsection, let \( m \) be a positive integer, and let \( \Gamma_m \) be a copy of the additive group over \( k \) equipped with a homomorphism \( \Gamma_m \to N_m \) of pro-algebraic groups over \( k \) such that the composition \( \Gamma_m \to N_m \to N_m/N_{m+1} \) is surjective and separable.
Example 1.2.4. The key case of Hypothesis 1.2.3 for our intended applications is the one in which \( m = 1 \) and \( \Gamma_m \) is the group of translations \( t^{-1} \mapsto t^{-1} + c \). However, we will need the full generality of Hypothesis 1.2.3 in order to make certain inductive arguments in towers of field extensions.

Lemma 1.2.5. Let \( E \) be a \( \mathbb{Z}/p\mathbb{Z} \)-extension of \( k((t)) \) equipped with an extension of the action of \( \Gamma_m \).

(a) The ramification number \( e \) of \( E \) is a positive integer no greater than \( m \) and not divisible by \( p \).

(b) Put \( m' = (m - e)p + e \). For any \( k \)-linear homeomorphism \( E \cong k((u)) \), the action of \( \Gamma_m \) on \( E \) induces a homomorphism \( \Gamma_m \to N_{m',u} \) of pro-algebraic groups such that the composition \( \Gamma_m \to N_{m',u} \to N_{m',u}/N_{m'+1,u} \) is surjective and separable.

Proof. Let \( \varphi \) denote the \( p \)-power Frobenius endomorphism of \( k((t)) \). Write \( E \) as an Artin–Schreier extension \( k((t))[z]/(z^p - z - x) \) with the \( t \)-adic valuation of \( x \) as large as possible. We then have \( x = at^{-e} + \cdots \) for some nonzero \( a \in k \), where \( e \) is the ramification number of \( E \). In particular, \( e \) is a positive integer not divisible by \( p \) (it cannot be 0 because \( k \) has been assumed to be algebraically closed).

For each \( c \in k \), the element \( \psi_c \in \Gamma_m \) corresponding to \( c \) has the property that \( \psi_c(x) \) defines the same Artin–Schreier extension of \( k((t)) \) as does \( x \), and so the elements \( x \) and \( \psi_c(x) \) must generate the same \( \mathbb{F}_p \)-subspace of \( \text{coker}(\varphi - 1, k((t))) \). Since \( x \) and \( \psi_c(x) \) both have the form \( at^{-e} + \cdots \) and \( e \) is not divisible by \( p \), the images of \( x \) and \( \psi_c(x) \) in \( \text{coker}(\varphi - 1, k((t))) \) must in fact coincide.

Write \( \psi_c(t) = t + \sum_{i=m+1}^{\infty} P_i(c)t^i \) for certain polynomials \( P_i(T) \in k[T] \). Because \( \Gamma_m \to N_m/N_{m+1} \) is separable, \( P_{m+1} \) is not a \( p \)th power. Moreover, the map \( c \mapsto P_{m+1}(c) \) must be additive in order to come from a group action.

Suppose that \( e > m \), and write \( x = \sum_{j \geq -e} a_j t^j \) with \( a_{-e} = a \). We then have

\[
\psi_c(x) - x \equiv \sum_{j=m-e}^{-1} Q_j(c)t^j \pmod{k[t]}
\]

for certain polynomials \( Q_j(T) \in k[T] \), and in particular \( Q_{m-e}(T) = -ea_{m+1}(T) \). Since \( \psi_c(x) - x \in \text{coker}(\varphi - 1, k((t))) \), we must have

\[
\sum_{i=0}^{\infty} Q_{(m-e)/p^i}(c)p^i = 0 \quad (c \in k). \quad (1.2.5.1)
\]

However, in the sum \( \sum_{i=0}^{\infty} Q_{(m-e)/p^i}(T)p^i \), the \( i = 0 \) term is not a \( p \)th power whereas all of the other terms are. Consequently, (1.2.5.1) asserts that a nonzero polynomial over \( k \) vanishes at all \( c \in k \), a contradiction. We conclude that \( e \leq m \), proving (a).

To prove (b), note that it is sufficient to check the claim for a single \( k \)-linear homeomorphism \( E \cong k((u)) \). We will check the claim with \( u = z^r t^s \) for an arbitrary pair of integers \( r, s \) satisfying \(-re + ps = 1 \) (which exist because \( e \) is not divisible by \( p \)). To begin with, we have \( z = Au^{-e} + \cdots \), \( t = Bu^p + \cdots \) for some \( A, B \in k \); using the equalities

\[
u = z^r t^s, \quad z^p = at^{-e} + \cdots, \quad u = z^r t^s, \quad z^p = au^{-e} + \cdots,
\]

we can solve for \( A \) and \( B \) to obtain

\[
z = a^su^{-e} + \cdots, \quad t = a^{-r} u^p + \cdots.
\]
By (a), we have $m - e \geq 0$. For $c \in k$, we thus have
\[
(\psi_c(z) - z)^p - (\psi_c(z) - z) = \psi_c(x) - x
= (\psi_c - 1)(at^{-e} + \cdots)
= -eaP_{m+1}(c)t^{m-e} + \cdots \in k[t].
\]
If $e < m$, this implies that $\psi_c(z) = z + eaP_{m+1}(c)t^{m-e} + \cdots$. Since the $u$-adic valuation of $(\psi_c(z) - z)/z$ is $(m - e)p + e = m'$ while the valuation of $(\psi_c(t) - t)/t$ is the strictly larger value $mp$, we obtain
\[
\psi_c(u) = \psi_c(z)^r\psi_c(t)^s
= z^r t^s + reP_{m+1}(c)t^{m-e+s}z^{r-1} + \cdots
= u + reP_{m+1}(c)a^{1-r(m-e)-s}u^{m'+1} + \cdots.
\]
If $e = m$, we instead have $\psi_c(z) = z + d + \cdots$ for some $d \in k$ satisfying $d - dp = eaP_{m+1}(c)$. Computing as before, we obtain
\[
\psi_c(u) = u + rda^{-s}u^{m'+1} + \cdots.
\]
In both cases, we obtain (b). \qed

**Proposition 1.2.6.** Let $E$ be a finite Galois extension of $k((t))$ equipped with an extension of the action of $\Gamma_m$. Then the ramification breaks of $E/k((t))$ in the upper numbering are all less than or equal to $m$.

**Proof.** Note that $E$ is totally ramified because we assumed that $k$ is algebraically closed. Also, by replacing $m$ by a multiple, we may reduce to the case where $E$ is totally wildly ramified. In this case, we induct on the degree of $E$; the case $E = k((t))$ serving as a trivial base case.

Suppose that $E \neq k((t))$. Let $e$ be the least ramification break of $E$ in the upper numbering, and let $F_e$ be the corresponding subfield of $E$. Since the definition of the ramification filtration is invariant under automorphisms of $k((t))$, we obtain an action of $\Gamma_m$ on $F_e$. Moreover, $\Gamma_m$ acts on $H = \text{Gal}(F_e/k((t)))$ via a discrete quotient, but the additive group has no nontrivial discrete quotients. Consequently, if we pick any $\mathbb{Z}/p\mathbb{Z}$-subextension $F$ of $F_e$, then $\Gamma_m$ acts on $F$.

By Lemma 1.2.5(a), we have $e \leq m$. In addition, if we put $m' = (m - e)p + e$ and choose a homeomorphism $F \cong k((u))$, then by Lemma 1.2.5(b), we obtain a homomorphism $\Gamma_m \to N_{m',u}$ such that the composition $\Gamma_m \to N_{m',u} \to N_{m',u}/N_{m'+1,u}$ is surjective and separable. This last fact allows us to invoke the induction hypothesis, which implies that the ramification breaks of $E/F$ in the upper numbering are all less than or equal to $m'$. By Herbrand’s rule for transferring ramification breaks from a group to a subgroup [Ser79, §IV.3], this in turn implies that the ramification breaks of $E/k((t))$ for the upper numbering are all less than or equal to $m$, as desired. \qed

**2. Differential modules over complete fields**

We next recall some definitions and results from [Ked10a] concerning the spectral behavior of differential modules over complete fields. We then make a few additional calculations leading to a finiteness result concerning the Tannakian automorphism group of a differential module.

**Convention 2.0.1.** For a matrix over a ring equipped with a norm, we will always interpret the norm of the matrix to be the supremum norm over entries of the matrix.
2.1 Differential rings and modules

We need some general terminology concerning differential rings and modules.

**Definition 2.1.1.** By a differential ring, we will mean a pair \((R, d)\) in which \(R\) is a commutative unital ring and \(d\) is a derivation on \(R\). By a differential module over \((R, d)\), we will mean a pair \((M, D)\) in which \(M\) is a finite projective \(R\)-module and \(D\) is a differential operator on \(M\) with respect to \(d\). For example, for each nonnegative integer \(n\), \(R^{\otimes n}\) may be viewed as a differential module by setting \(D(r_1, \ldots, r_n) = (d(r_1), \ldots, d(r_n))\); any differential module isomorphic to one of this form is said to be trivial. We will often omit mention of \(d\) and/or \(D\) when they may be inferred from context.

**Remark 2.1.2.** Let \((M, D)\) be a differential module over a differential ring \(R\) which is freely generated by the basis \(e_1, \ldots, e_n\). Then the action of \(D\) on \(M\) can be reconstructed from the matrix \(N\) defined by \(D(e_j) = \sum_i N_{ij} e_i\) (the matrix of action of \(D\) on the basis). Any other basis \(e'_1, \ldots, e'_n\) is uniquely determined by the invertible matrix \(U\) over \(R\) defined by \(e'_j = \sum_i U_{ij} e_i\) (the change-of-basis matrix from \(e_i\) to \(e'_j\)); the matrix of action of \(D\) on this new basis has the form \(U^{-1}NU + U^{-1}d(U)\).

**Definition 2.1.3.** The differential modules over a given differential ring form a tensor category. For \(M\) a differential module, we write \(\text{End}(M)\) as shorthand for \(M^\vee \otimes M\); there is a natural composition morphism \(- \circ - : \text{End}(M) \otimes \text{End}(M) \to \text{End}(M)\).

**Definition 2.1.4.** Let \((M, D)\) be a differential module of rank \(n\) over a differential ring \((R, d)\). A cyclic vector for \(M\) is an element \(v \in M\) such that \(D(v), \ldots, D^{n-1}(v)\) form a basis of \(M\) as an \(R\)-module.

**Lemma 2.1.5 (Cyclic vector theorem).** Let \((R, d)\) be a differential ring such that \(R\) is a field of characteristic 0 and \(d\) is nonzero. Then every differential module over \(R\) admits a cyclic vector.

**Proof.** See, for instance, [Ked10a, Theorem 5.4.2].

**Corollary 2.1.6.** Let \((R, d)\) be a differential ring such that \(R\) is a domain of characteristic 0 and \(d\) is nonzero. Then every differential module \(M\) over \((R, d)\) contains a cyclic vector for \(M \otimes_R \text{Frac}(R)\).

**Definition 2.1.7.** For \((M, D)\) a differential module, write \(H^0(M)\) and \(H^1(M)\) for \(\ker(D)\) and \(\text{coker}(D)\), respectively. Note that \(H^1(M)\) may be interpreted as a Yoneda extension group.

2.2 Differential modules over fields

We next review some of the theory of differential modules over completed rational function fields (also known as fields of analytic elements) as presented in [Ked10a, chs. 9–10].

**Hypothesis 2.2.1.** Throughout this subsection, choose \(\rho > 0\), let \(F_\rho\) be the completion of \(K(t)\) for the \(\rho\)-Gauss norm, and let \(E\) be a finite tamely ramified extension of \(F_\rho\). View \(F_\rho\) as a differential field for the derivation \(d = dt/dt\), which extends uniquely to \(E\).

**Definition 2.2.2.** Let \((V, D)\) be a differential module over \(E\). For \(V\) nonzero, let \(IR(V)\) denote the intrinsic radius of \(V\) in the sense of [Ked10a, Definition 9.4.7]. That is, \(\omega/(\rho IR(V))\) equals the spectral radius of \(D\) as a \(K\)-linear endomorphism of \(V\) for any \(E\)-Banach norm on \(V\). The following properties are easily derived (see [Ked10a, Lemma 6.2.8]).
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(a) For any $V$, $IR(V^\vee) = IR(V)$.

(b) For any short exact sequence $0 \rightarrow V_1 \rightarrow V \rightarrow V_2 \rightarrow 0$, $IR(V) = \min\{IR(V_1), IR(V_2)\}$.

(c) For any $V_1, V_2$, $IR(V_1 \otimes V_2) \geq \min\{IR(V_1), IR(V_2)\}$, with equality if $IR(V_1) = IR(V_2)$.

As in [Ked10a, Definition 9.8.1], the multiset of intrinsic subsidiary radii of $V$ is constructed as follows: for each Jordan–Hölder constituent $W$ of $V$, include $IR(W)$ with multiplicity \(\dim_{F_p}(W)\). This multiset is invariant under arbitrary extensions of the constant field and under finite tamely ramified extensions of $E$ [Ked10a, Proposition 10.6.6], and its maximum element equals $IR(V)$.

Let $s_1 \leq \cdots \leq s_n$ be the intrinsic subsidiary radii of $V$. The spectral polygon of $V$, denoted $P(V)$, is then defined to be the convex polygonal curve starting at $(-n, 0)$ and consisting of segments of width one and slopes $\log s_1, \ldots, \log s_n$ in that order.

**Definition 2.2.3.** Let $V$ be a nonzero differential module over $E$. We say that $V$ is pure if its intrinsic subsidiary radii are all equal. We say that $V$ is refined if $IR(\text{End}(V)) > IR(V)$; this condition implies that $IR(V) < 1$, and also that $V$ is pure (using [Ked10a, Lemma 9.3.4]). Consequently, this definition of refinedness agrees with that of [Ked10a, Definition 6.2.12].

We say that two refined differential modules $V_1, V_2$ over $E$ are equivalent if $IR(V_1) = IR(V_2) < IR(V_1^\vee \otimes V_2)$. As the terminology suggests, this is an equivalence relation [Ked10a, Lemma 6.2.14].

**Lemma 2.2.4.** Let $V_1, V_2$ be nonzero differential modules over $F_p$ such that $IR(V_1), IR(V_2) < IR(V_1^\vee \otimes V_2)$. Then $IR(\text{End}(V_1)) = IR(\text{End}(V_2)) \leq IR(V_1^\vee \otimes V_2)$; consequently, $V_1$ and $V_2$ are both refined of the same intrinsic radius.

**Proof.** The first claim holds because $V_2$ is a direct summand of $V_1 \otimes (V_1^\vee \otimes V_2)$ and $V_1$ is a direct summand of $V_2 \otimes (V_1^\vee \otimes V_2)^\vee$. The second claim holds because $V_1^\vee \otimes V_1$ is a direct summand of $V_1^\vee \otimes V_2 \otimes V_2^\vee \cong (V_1^\vee \otimes V_2)^\vee \otimes (V_1^\vee \otimes V_2)$. \hfill \(\square\)

**Remark 2.2.5.** The intrinsic subsidiary radii of $V$ behave for many purposes like the reciprocal norms of the eigenvalues of some linear transformation associated to $V$. In this model, a refined differential module (respectively two equivalent refined modules) over $E$ corresponds to a linear transformation (respectively two linear transformations) whose eigenvalues all have a single image in the graded ring associated to an algebraic closure of $F_p$.

For radii in the range $(0, \omega)$ (called the visible range in [Ked10a]), this intuition can be made precise using cyclic vectors; see Proposition 2.2.6 below. When $p > 1$, one must use pullback and pushforward along Frobenius to access radii in the range $[\omega, 1)$, as described in [Ked10a, ch. 10]. We will see these techniques in action in §2.3.

**Proposition 2.2.6 (Christol–Dwork).** Let $V$ be a differential module over $E$ of rank $n$, let $v$ be a cyclic vector of $V$, and write $D^n(v) = a_0v + \cdots + a_{n-1}D^{n-1}(v)$ with $a_0, \ldots, a_{n-1} \in E$. Then the multiset of slopes of the spectral polygon of $V$ less than $\log \omega$ consists of $\log \omega - \log \rho + s$ for $s$ running over the multiset of slopes of the Newton polygon of the polynomial $T^n - a_{n-1}T^{n-1} - \cdots - a_0 \in E[T]$ less than $\log \rho$.

**Proof.** See [Ked10a, Corollary 6.5.4]. \hfill \(\square\)

**Corollary 2.2.7.** For any $s < \omega$ and any positive integers $n_1, n_2, m$, there exists $\delta \in (s, \omega)$ for which the following statements hold. For $i = 1, 2$, let $V_i$ be a differential module over
Proof. We describe only (a) in detail, as the proof of (b) is similar. Equip \( V_i \) with a norm as in the proof of [Ked10a, Theorem 6.5.3]; by enlarging \( K \) if necessary, we may ensure that this norm is the supremum norm defined by a basis. Equip \( V_i^\vee \) with the dual basis, then equip \( V_i^\vee \otimes V_2 \) with the product basis and the resulting supremum norm. The claim then follows by applying [Ked10a, Theorem 6.7.4].

DEFINITION 2.2.8. Let \( V \) be a differential module over \( E \). A spectral decomposition of \( V \) is a direct sum decomposition \( V = \bigoplus_{s \in (0,1]} V_s \) such that the intrinsic subsidiary radii of \( V_s \) are all equal to \( s \). A refined decomposition of \( V \) is a direct sum decomposition of \( V \) refining a spectral decomposition in which \( V_1 \) remains whole, but each \( V_s \) with \( s < 1 \) is split into inequivalent refined summands.

PROPOSITION 2.2.9. Let \( V \) be a differential module over \( E \).

(a) There exists a unique spectral decomposition of \( V \).

(b) A refined decomposition of \( V \) is unique if it exists. Moreover, there exists a finite tamely ramified extension \( E' \) of \( E \) such that \( V \otimes_E E' \) admits a refined decomposition.

Proof. By restriction of scalars, we may reduce to the case \( E = F_\rho \). In this case, see [Ked10a, Theorems 10.6.2 and 10.6.7] for (a) and (b), respectively. \( \square \)

COROLLARY 2.2.10. Let \( V \) be a differential module over \( E \) such that \( IR(V) < 1 \).

(a) If \( V \) is indecomposable, then \( V \) is pure.

(b) If \( V \otimes_E E' \) is indecomposable for every finite tamely ramified extension \( E' \) of \( E \), then \( V \) is refined.

For \( p = 0 \), one can state an even stronger version of Corollary 2.2.10, closely related to the classical Turrittin–Levelt–Hukuhara decomposition theorem for formal meromorphic connections (see, for instance, [Ked10a, ch. 7]).

PROPOSITION 2.2.11. Assume that \( p = 0 \). Let \( V \) be a differential module over \( E \) such that \( IR(V) < 1 \). If \( V \otimes_E E' \) is indecomposable for every finite tamely ramified extension \( E' \) of \( E \), then there exists a differential module \( W \) over \( E \) of dimension 1 such that \( IR(W^\vee \otimes V) = 1 \).

Proof. Put \( n = \dim_E(V) \). Let \( v \) be a generator of \( \wedge^n V \) and define \( f \in E \) by the formula \( D(v) = f v \). Let \( W \) be the differential module of dimension 1 over \( E \) on the generator \( w \) for which \( D(w) = (f/n)w \); then \( W^\otimes n \cong \wedge^n V \), so \( \wedge^n(W^\vee \otimes V) \) is trivial. If \( IR(W^\vee \otimes V) < 1 \), then by
Corollary 2.2.10, $W^V \otimes V$ would be refined; however, [Ked10a, Proposition 6.8.4] would then imply that $IR(W^V \otimes V) = IR(\wedge^n(W^V \otimes V)) = 1$, a contradiction. Hence $IR(W^V \otimes V) = 1$ as desired.

2.3 More on refined modules

Proposition 2.2.11 gives a fairly precise description of the indecomposable differential modules over finite tamely ramified extensions of $F_p$ in the case $p = 0$. We next turn to the situation where $p > 0$, in which case things are more complicated.

**Hypothesis 2.3.1.** Throughout this subsection retain Hypothesis 2.2.1, but assume in addition that $p > 0$. Let $\mu_p$ denote the group of $p$th roots of unity in some algebraic closure of $K$.

We recall the basic formalism of Frobenius pullback and pushforward, as in [Ked10a, ch. 10].

**Definition 2.3.2.** For each $\zeta \in \mu_p$, the $K$-linear substitution $t \mapsto \zeta t$ induces a continuous automorphism $\zeta^* \in \text{Gal}(E(\mu_p)/E)$ and the automorphisms $\zeta^*$ for $\zeta \in \mu_p$; we may then view $E'$ as a differential field for the derivation $d = d/dt^p$, and thus define the intrinsic radius of a nonzero differential module $(V', D')$ over $E'$ so that $\rho^p/(\omega IR(V'))$ equals the spectral radius of $D'$.

For $m = 0, \ldots, p - 1$, let $(W_m, D')$ denote the differential module over $E'$ on the single generator $v$ given by $D'(v) = (m/p)t^{-p}v$. By Proposition 2.2.6, $IR(W_m) = \omega^p$ for $m \neq 0$ (see also [Ked10a, Definition 10.3.3]).

For $(V', D')$ a differential module over $E'$, define the differential module $\varphi^*V'$ over $E$ to have underlying module $V' \otimes_{E'} E$ and derivation given by $D = D' \otimes pt^{p-1}$.

For $(V, D)$ a differential module over $E$, define the differential module $\varphi_*V$ over $E'$ to have underlying module $V$ and derivation given by $D' = p^{-1}t^{1-p}D$.

**Lemma 2.3.3.** For any nonzero differential module $V'$ over $E'$, 

$$IR(\varphi^*V') \geq \min\{IR(V')^{1/p}, pIR(V')\}.$$ 

**Proof.** See [Ked10a, Lemma 10.3.2].

**Proposition 2.3.4.** Let $V$ be a nonzero differential module over $E$ such that $IR(V) > \omega$. Then there exists a unique differential module $V'$ over $E$ (called the Frobenius antecedent of $V$) such that $IR(V') > \omega^p$ and $\varphi^*V' \cong V'$; moreover, this module satisfies $IR(V') = IR(V)^p$.

**Proof.** See [Ked10a, Theorem 10.4.2].

**Proposition 2.3.5.** Let $V$ be a differential module over $E$ of rank $n$ with intrinsic subsidiary radii $s_1, \ldots, s_n$. Then the intrinsic subsidiary radii of $\varphi_*V$ (called the Frobenius descendant of $V$) comprise the multiset

$$\bigcup_{i=1}^n \begin{cases} \{s_i^p\} \cup \{\omega^p \text{ (p - 1 times)}\} & \text{if } s_i > \omega, \\ \{p^{-1}s_i \text{ (p times)}\} & \text{if } s_i \leq \omega. \end{cases}$$

**Proof.** See [Ked10a, Theorem 10.5.1].

**Lemma 2.3.6.** (a) For $V$ a differential module over $E$, there are canonical isomorphisms

$$\iota_m : (\varphi_*V) \otimes W_m \cong \varphi_*V \quad (m = 0, \ldots, p - 1).$$
(b) For $V$ a differential module over $E$, a submodule $U$ of $\varphi_*V$ has the form $\varphi_*X$ for some differential submodule $X$ of $V$ if and only if $\iota_m(U \otimes W_m) = U$ for $m = 0, \ldots, p - 1$.

(c) For $V'$ a differential module over $E'$, there is a canonical isomorphism

$$\varphi_*\varphi^*V' \cong \bigoplus_{m=0}^{p-1} (V' \otimes W_m).$$

Proof. See [Ked10a, Lemma 10.3.6(a, b, c)]. \hfill $\square$

**Lemma 2.3.7.** Let $V'$ be an indecomposable differential module over $E'$ of intrinsic radius $\omega^p$ such that $IR(\varphi^*V') > \omega$. Then there exists a unique $m \in \{0, \ldots, p - 1\}$ such that $IR(V' \otimes W_m) > \omega^p$.

Proof. By Proposition 2.3.5, at least one of the intrinsic subsidiary radii of $\varphi_*\varphi^*V'$ is greater than $\omega^p$. By Lemma 2.3.6(c), we have $\varphi_*\varphi^*V' \cong \bigoplus_{m=0}^{p-1} (V' \otimes W_m)$, so for some $m$, at least one of the intrinsic subsidiary radii of $V' \otimes W_m$ is greater than $\omega^p$. Since $V' \otimes W_m$ is indecomposable, this implies $IR(V' \otimes W_m) > \omega^p$ by Corollary 2.2.10. This proves the existence of $m$; uniqueness holds because $IR(W_m) = \omega^p$ for $m \neq 0$. \hfill $\square$

**Corollary 2.3.8.** Let $V'$ be a nonzero differential module over $E'$ of intrinsic radius $\omega^p$ such that $IR(\varphi^*V') > \omega$. Then there exists a unique direct sum decomposition $V' = \bigoplus_{m=0}^{p-1} V_m'$ such that $IR(V_m' \otimes W_m) > \omega^p$ for $m = 0, \ldots, p - 1$.

**Lemma 2.3.9.** Let $V'_1, V'_2$ be nonzero differential modules over $E'$ of intrinsic radius $\omega^p$ such that $V'_1$ is refined, $V'_2$ is indecomposable, and $IR(\varphi^*((V'_1)^\vee \otimes V'_2)) > \omega$. Then there exists a unique $m \in \{0, \ldots, p - 1\}$ such that $IR((V'_1)^\vee \otimes V'_2 \otimes W_m) > \omega^p$.

Proof. By Corollary 2.3.8, we have a decomposition $(V'_1)^\vee \otimes V'_2 = \bigoplus_{m=0}^{p-1} X_m$ such that $IR(X_m \otimes W_m) > \omega^p$ for $m = 0, \ldots, p - 1$. Contracting with $V'_1$ produces an inclusion $V'_2 \to \bigoplus_{m=0}^{p-1} (V'_1 \otimes X_m)$; since $V'_2$ is indecomposable, we have $V'_2 \subseteq V'_1 \otimes X_m$ for some $m$. Therefore

$$IR((V'_1)^\vee \otimes V'_2 \otimes W_m) \geq IR((V'_1)^\vee \otimes V'_1 \otimes X_m \otimes W_m)$$

$$\geq \min\{IR((V'_1)^\vee \otimes V'_1), IR(X_m \otimes W_m)\}$$

$$> \omega^p.$$ Again, $m$ is unique because $IR(W_m) = \omega^p$ for $m \neq 0$. \hfill $\square$

**Remark 2.3.10.** Let $V$ be a refined differential module over $E$ of intrinsic radius $\omega$ such that $\varphi_*V$ admits a refined decomposition $\bigoplus X_i$. By Lemma 2.3.6(a), there are canonical isomorphisms $\psi_m : (\varphi_*V) \otimes W_m \cong \varphi_*V$ for $m = 0, \ldots, p - 1$; we may view these as an action of $\mathbb{Z}/p\mathbb{Z}$ on $\varphi_*V$, which induces an action on the collection of the $X_i$. Since $IR(W_m) = \omega^p$ for $m \neq 0$, any two distinct $X_i$ in the same orbit are refined and pairwise inequivalent. By Lemma 2.3.6(b), the $X_i$ in a single orbit constitute the pushforward of a direct summand of $V$.

**Lemma 2.3.11.** Let $V$ be a refined differential module over $E$ of intrinsic radius $s \geq \omega$. Then for some finite unramified extension $E'_1$ of $E'$, there exists a refined differential module $V'$ over $E'_1$ of intrinsic radius $s^p$ such that $\varphi^*V' \cong V \otimes E'/E'_1$.

Proof. In the case $s > \omega$, we may take $V'$ to be the Frobenius antecedent of $V$ (Proposition 2.3.4); we thus assume that $s = \omega$ hereafter. Suppose first that $V$ is indecomposable. By Proposition 2.3.5, the intrinsic subsidiary radii of $\varphi_*V$ are all equal to $\omega^p$. We may thus
apply Proposition 2.2.9 to produce a finite Galois tamely ramified extension $E_1'$ of $E'$ such that $\varphi_\star V \otimes_{E'} E_1'$ admits a refined decomposition $\bigoplus_i X_i$.

Define an action of $\mathbb{Z}/p\mathbb{Z}$ on the collection of the $X_i$ as in Remark 2.3.10. Since we assumed that $V$ is indecomposable, it follows that the $X_i$ form a single orbit under $\mathbb{Z}/p\mathbb{Z}$.

The group $G = \text{Gal}(E_1'/E')$ also acts on the set of the $X_i$; since $W_m$ is defined over $E'$, this action defines a homomorphism $G \to \mathbb{Z}/p\mathbb{Z}$. We may replace $E_1'$ with the fixed field of the kernel of this homomorphism; this field has tame degree over $\varphi$ a map $Z$ inequivalent, the map $\varphi$ since distinct terms $i$ independently, we can find $m_i \in \{0, \ldots, p - 1\}$ such that $\text{IR}((V_i')^\vee \otimes V_i' \otimes W_m) > \omega^p$. We may thus take $V' = \bigoplus_{i=0}^r V_i' \otimes W_m$.

**Lemma 2.3.12.** Let $V$ be a pure differential module over $E$ of intrinsic radius $s \geq \omega$ such that $\varphi_\star V$ admits a refined decomposition. Group summands in this decomposition according to their $\mathbb{Z}/p\mathbb{Z}$-orbit as per Remark 2.3.10. Then the resulting decomposition descends to a refined decomposition of $V$.

**Proof.** We may use Proposition 2.3.4 to check the claim when $s > \omega$, so we may assume that $s = \omega$ hereafter. The claim may be checked after enlarging $E$, so by Proposition 2.2.9 we may ensure that $V$ itself admits a refined decomposition $\bigoplus_i V_i$. After enlarging $E$ again, by Lemma 2.3.11 we may ensure that each $V_i$ can be written as $\varphi_\star V_i'$ for some refined differential module $V_i'$ over $E'$. By Lemma 2.3.6(c), we then have $\varphi_\star V_i \cong \bigoplus_{m=0}^{p-1} (V_i' \otimes W_m)$. For $i, j$ distinct and $m \in \{0, \ldots, p - 1\}$, we cannot have $\text{IR}((V_i')^\vee \otimes V_j' \otimes W_m) > \omega^p$ or else Proposition 2.3.5 would imply $\text{IR}(V_i' \otimes V_j) > \omega$. It follows that the $V_i' \otimes W_m$ are refined and pairwise inequivalent, so they form the refined decomposition of $\varphi_\star V$. This proves the claim.

**Proposition 2.3.13.** Let $V$ be a refined differential module over $E$. Then $\text{IR}(V^{\otimes p}) > \text{IR}(V)$.

**Proof.** It is sufficient to prove that for each nonnegative integer $h$, the claim holds when $\text{IR}(V) < \omega^{p-h}$. For $h = 0$, this follows by Corollary 2.2.7(a, b) with the parameter $m$ in (b) taken to be $p$. Given this assertion for some $h$, we may check it for $h + 1$ by forming a module $V''$ as in Lemma 2.3.11, applying the known case to deduce that $\text{IR}(V^{\otimes p}) > \text{IR}(V) = \text{IR}(V')^p$, then observing that $\varphi_\star (V')^{\otimes p} = V^{\otimes p}$ and invoking Lemma 2.3.3 to deduce that $\text{IR}(V^{\otimes p}) > \text{IR}(V)$.

When $V$ has dimension 1, we can prove an even stronger assertion.

**Lemma 2.3.14.** Let $V$ be a differential module over $E$ of dimension 1. Then

$$\min\{\omega, \text{IR}(V^{\otimes p})\} = \min\{\omega, p\text{IR}(V)\}.$$  

**Proof.** This is immediate from Proposition 2.2.6.

**Lemma 2.3.15.** Let $V$ be a differential module over $E$ of dimension 1 such that $\omega^p \leq IR(V) \leq \omega$. Then $IR(V^{\otimes p}) \geq IR(V)^{1/p}$.

**Proof.** By enlarging $K$ and rescaling, we may reduce to the case $\rho = 1$. Put $d = d/dt$ and $s = IR(V)$. Choose a generator $v$ of $V$ and write $D(v) = nv$ with $n \in E$. By Proposition 2.2.6, $|n| = \omega/s$. The differential module $V^{\otimes p}$ is generated by $v^{\otimes p}$ and $D(v^{\otimes p}) = pn v^{\otimes p}$. Since $|d| = 1$ and $|p| = p^{-1}\omega/s = \omega^p/s \leq 1$, for any $a \in E$ we have $D^p(a v^{\otimes p}) = b v^{\otimes p}$ for some $b \in E$ with $|b - p^d(a)| \leq p^{-1}(\omega/s)|a|$. Since $|p^d| = p^{-1} \leq p^{-1}\omega/s$, we conclude that the operator norm of $D^p$ on $V$ is at most $p^{-1}\omega/s = \omega^p/s$, so the spectral norm of $D$ on $V$ is at most $\omega/s^{1/p}$. This implies the desired inequality. \hfill $\square$

**Proposition 2.3.16.** Let $V$ be a differential module over $E$ of dimension 1 such that $IR(V) < 1$. Then $IR(V^{\otimes p}) \geq \min\{IR(V)^{1/p}, pIR(V)\}$.

**Proof.** The claim is trivial if $IR(V) = 1$, so we may assume that $IR(V) < 1$. If $IR(V) \leq \omega^p$, then $\min\{IR(V)^{1/p}, pIR(V)\} = pIR(V)$, and in this case the claim follows from Lemma 2.3.14. To complete the proof, it suffices to check the claim when $\omega^{p-h+1} \leq IR(V) < \omega^{p-h}$ for some nonnegative integer $h$. We prove this by induction on $h$, with the base case $h = 0$ following from Lemma 2.3.15. Given the claim for $h - 1$, we may deduce the claim for $h$ by forming $V'$ as in Lemma 2.3.11 (after enlarging $E$ if necessary), applying the induction hypothesis to $V'$, and then applying Lemma 2.3.3. \hfill $\square$

We are now ready to deduce a finiteness theorem for Tannakian automorphism groups.

**Theorem 2.3.17.** Let $V$ be a differential module over $E$. Let $[V]$ be the Tannakian category of differential modules over $E$ generated by $V$. Let $\omega$ be the fibre functor on $[V]$ which extracts underlying $E$-vector spaces. Let $G$ be the automorphism group of $\omega$. For $s < 1$, let $G^s$ be the subgroup of $G$ acting trivially on $\omega(W)$ for every $W \in [V]$ with $IR(W) > s$. Then $G^s$ is a finite $p$-group.

**Proof.** Instead of working with differential modules over $E$, we work with the direct limit of the categories of differential modules over all finite tamely ramified extensions of $E$: this does not change the groups $G^s$ except for a base extension. In this larger category, we may apply Proposition 1.1.2 using Remark 1.1.3: conditions (i), (ii), (iii) of the remark may be verified using Propositions 2.2.9(b), 2.3.13, 2.3.16, respectively. \hfill $\square$

**Remark 2.3.18.** The group $\bigcup_{s < 1} G^s$ need not be finite in general. For example, if $V$ is free on one generator $v$ and $D(v) = \lambda t^{-1}v$ for some $\lambda \in K \setminus \mathbb{Q}_p$, then $IR(V^{\otimes n}) < 1$ for all positive integers $n$ [Ked10a, Example 9.5.2] and so $\bigcup_{s < 1} G^s \cong \mathbb{Q}_p/\mathbb{Z}_p$.

In order to obtain finiteness for some class of differential modules, one must impose additional hypotheses to ensure that when $V$ is of dimension 1, there exists a nonnegative integer $m$ for which $IR(V^{\otimes pm}) = 1$. For an example of such hypotheses, see Theorem 3.8.16.

**Remark 2.3.19.** If we assume that $p = 0$ but otherwise set notation as in Theorem 2.3.17, then the group $\bigcup_{s < 1} G^s$ becomes a torus, as one may deduce easily from Proposition 2.2.11.
3. Differential modules over discs and annuli

We next continue in the vein of [Ked10a], treating differential modules on discs and annuli. In this section, we maintain continuity with [Ked10a] by phrasing everything in the language of modules over rings of convergent power series. Starting in § 4, we will switch to the language of Berkovich spaces in order to articulate more precise and general results.

3.1 Rings of convergent power series

We first introduce the relevant rings of convergent power series on a disc or annulus, modifying the notation somewhat from that used in [Ked10a, ch. 8]. We next continue in the vein of [Ked10a], treating differential modules on discs and annuli. In § 4, we will switch to the language of Berkovich spaces in order to articulate more precise and general results.

Definition 3.1.1. For $\rho \in [0, +\infty)$, let $|\cdot|_\rho$ denote the $\rho$-Gauss seminorm on $K[t]$, defined by the formula $|\sum_n c_n t^n|_\rho = \max\{|c_n| |\rho^n\}$. For $I$ a subinterval of $[0, +\infty)$, let $R_I$ denote the Fréchet completion of $K[t]$ (if $0 \in I$) or $K[t, t^{-1}]$ (if $0 \notin I$) for the seminorms $|\cdot|_\rho$ for $\rho \in I$. View $R_I$ as a differential ring for the derivation $d/dt$. We will occasionally write $R_{I,K}$ instead of $R_I$ when it is necessary to specify $K$.

Remark 3.1.2. Let us briefly recall how the rings $R_I$ appear in the notation of [Ked10a].

- If $I = [0, \beta]$, then $R_I$ appears as $K\langle t/\beta \rangle$, the ring of analytic functions on the closed disc $|t| \leq \beta$.
- If $I = [0, \beta)$, then $R_I$ appears as $K\{t/\beta\}$, the ring of analytic functions on the open disc $|t| < \beta$.
- If $I = [\alpha, \beta]$ with $\alpha > 0$, then $R_I$ appears as $K\langle t/\alpha, t/\beta \rangle$, the ring of analytic functions on the closed annulus $\alpha \leq |t| \leq \beta$.
- If $I = (\alpha, \beta)$ with $\alpha > 0$, then $R_I$ appears as $K\{t/\alpha, t/\beta\}$, the ring of analytic functions on the open annulus $\alpha < |t| < \beta$.

Remark 3.1.3. Suppose that $I$ is a closed interval. Then $R_I$ is an affinoid algebra for the norm $|\cdot| = \sup\{ |\cdot|_\rho : \rho \in I \}$. By the log-convexity of $|\cdot|_\rho$ [Ked10a, Proposition 8.2.3] (see also Lemma 3.1.5), one has $|\cdot|_{[0,\beta]} = |\cdot|_\beta$ and $|\cdot|_{[\alpha,\beta]} = \max\{|\cdot|_{[\alpha,\beta]} | \alpha, | \cdot|_{[\alpha,\beta]} \}$. By the log-convexity of $|\cdot|_\rho$ [Ked10a, Proposition 8.2.3] (see also Lemma 3.1.5), one has $|\cdot|_{[0,\beta]} = |\cdot|_\beta$ and $|\cdot|_{[\alpha,\beta]} = \max\{|\cdot|_{[\alpha,\beta]} | \alpha, | \cdot|_{[\alpha,\beta]} \}$. By the log-convexity of $|\cdot|_\rho$ [Ked10a, Proposition 8.2.3] (see also Lemma 3.1.5), one has $|\cdot|_{[0,\beta]} = |\cdot|_\beta$ and $|\cdot|_{[\alpha,\beta]} = \max\{|\cdot|_{[\alpha,\beta]} | \alpha, | \cdot|_{[\alpha,\beta]} \}$.

Now let $I$ be arbitrary. In this case, $R_I$ is a Fréchet–Stein algebra in the sense of [ST03, §3]; this means that every coherent sheaf on the associated analytic space is generated by its module of global sections. Moreover, any coherent locally free sheaf of rank $n$ is uniformly finitely generated (because exactly $n$ generators are needed over any closed disc or annulus), and so corresponds to a finite projective module over $R_I$ by [KPX14, Proposition 2.1.15] or [Bel13, Corollary 2.2.5].

Definition 3.1.4. For $x \in \mathbb{R}$, let $\langle x \rangle$ denote the distance from $x$ to the nearest integer, that is, $\langle x \rangle = \min\{ x - |x|, -x - |x| \}$. We will frequently use the fact that for $m$ a positive integer, $m\langle x/m \rangle$ is the distance from $x$ to the nearest multiple of $m$.

Lemma 3.1.5. Choose $\eta > 1$ and $\alpha, \alpha', \beta, \beta' \in [0, +\infty)$ such that $\alpha' < \beta'$, $\alpha' = \alpha \eta$, $\beta' = \beta/\eta$.

Choose a positive integer $m$, an element $h \in \mathbb{Z}$, and an element $f \in R_{[\alpha,\beta]}$ whose terms all have exponents congruent to $h$ modulo $m$.

(a) Put $h' = m(h/m)$. Then $|f|_{[\alpha',\beta']} \leq \eta^{-h'} |f|_{[\alpha,\beta]}$. 

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(b) Assume that \( h = 0 \). Let \( f_0 \) be the constant coefficient of \( f \). Then
\[
|f - f_0|_{[\alpha', \beta']} \leq \eta^{-m}|f|_{[\alpha, \beta]}.
\]

Proof. Both assertions reduce at once to the case \( f = t^n \) for some \( n \in h + m\mathbb{Z} \), for which the claim is evident. \( \square \)

We will also need the construction of rings of analytic elements.

**Definition 3.1.6.** Let \( J \) be the closure of \( I \). Let \( R_J^\text{an} \) be the Fréchet completion of the ring of rational functions in \( K(t) \) with no poles in the region \( |t| \in I \) for the norms \( |\cdot|_\rho \) for \( \rho \in J \). This is called the ring of *analytic elements* in the region \( |t| \in I \); it is a principal ideal domain [Ked10a, Proposition 8.5.2].

- If \( I \) is closed, then \( R_J^\text{an} = R_J \).
- If \( I = (0, \beta) \), then \( R_J^\text{an} \) appears in [Ked10a] as \( K[[t/\beta]] \).
- If \( I = (\alpha, \beta) \) with \( \alpha > 0 \), then \( R_J \) appears in [Ked10a] as \( K[[\alpha/t, t/\beta]] \).

### 3.2 The Robba condition

We now introduce a special class of differential modules over annuli; this class is closely related to the class of *regular* meromorphic differential modules on a Riemann surface.

**Hypothesis 3.2.1.** Throughout this subsection, let \( I \) be an open subinterval of \( [0, +\infty) \) and let \( M \) be a differential module of rank \( n \) over \( R_I \) for the derivation \( t(d/dt) \). For \( \rho \in I \setminus \{0\} \), put \( M_\rho = M \otimes_{R_I} F_\rho \); for \( J \) a closed subinterval of \( I \) of positive length, put \( M_J = M \otimes_{R_I} R_J \).

**Definition 3.2.2.** We say that \( M \) satisfies the *Robba condition* if \( IR(M_\rho) = 1 \) for all \( \rho \in I \setminus \{0\} \). In this case, we may define an action of the multiplicative group \( 1 + m_K \) on \( M \) by the formula
\[
\lambda(v) = \sum_{i=0}^{\infty} (\lambda - 1)^i \left( \frac{D}{i} \right)(v) \quad (\lambda \in 1 + m_K, v \in M),
\]

since the Taylor series on the right is guaranteed to converge. (Note that this formula is given incorrectly in [Ked10a, Definition 13.5.2]; it differs from the analogous formula in [Ked10a, Definition 5.8.1] because the latter is adapted to differential modules for the derivation \( d/dt \).)

We may also interpret the action of \( \lambda \in 1 + m_K \) as an isomorphism \( \lambda^*(M) \cong M \), where \( \lambda^* \) is the pullback along the substitution \( t \mapsto \lambda t \).

**Example 3.2.3.** For \( \lambda \in K \), let \( M_\lambda \) denote the differential module over \( R_I \) on a single generator \( v \) satisfying \( D (v) = \lambda dv \). If \( p = 0 \), then \( M_\lambda \) satisfies the Robba condition whenever \( |\lambda| \leq 1 \), and is trivial if and only if \( \lambda \in \mathbb{Z} \). By contrast, if \( p > 0 \), then \( M_\lambda \) satisfies the Robba condition if and only if \( \lambda \in \mathbb{Z}_p \) [Ked10a, Example 9.5.2], and is again trivial if and only if \( \lambda \in \mathbb{Z} \) [Ked10a, Proposition 9.5.3].

**Definition 3.2.4.** For \( A \) a finite multisubset of \( \mathfrak{o}_{K_{alg}} \), we say that \( A \) is *prepared* if no two elements \( a_1, a_2 \) of \( A \) have the property that \( |a_1 - a_2 - m| < 1 \) for some nonzero integer \( m \). For \( A, B \) two finite multisubsets of \( \mathfrak{o}_{K_{alg}} \) of the same cardinality \( n \), we say that \( A \) and \( B \) are *equivalent* if there exist orderings \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \) of \( A \) and \( B \) respectively, such that \( a_i - b_i \in \mathbb{Z} \) for \( i = 1, \ldots, n \); this indeed defines an equivalence relation.

**Definition 3.2.5.** We say that \( M \) is of *cyclic type* if \( \text{End}(M) \) satisfies the Robba condition. For example, if there exists a differential module \( N \) over \( R_I \) of positive rank such that \( N^\vee \otimes M \) satisfies the Robba condition, then \( M \) is of cyclic type by Lemma 2.2.4. Note that the tensor product of modules of cyclic type is again of cyclic type.
Lemma 3.2.6. Suppose that $M$ is of cyclic type. For each $\lambda \in 1 + \mathfrak{m}_K$, view the Taylor isomorphism $T_\lambda : \lambda^*(\text{End}(M)) \cong \text{End}(M)$ as a horizontal element of
\[
\lambda^*(M^\vee \otimes M) \otimes (M^\vee \otimes M) \cong \lambda^*(M^\vee) \otimes \lambda^*(M) \otimes M^\vee \otimes M \\
\cong \lambda^*(M) \otimes M^\vee \otimes \lambda^*(M^\vee) \otimes M \\
\cong (\lambda^*(M^\vee) \otimes M)^\vee \otimes \lambda^*(M^\vee) \otimes M \\
\cong \text{End}(\lambda^*(M^\vee) \otimes M).
\]
Then the corresponding endomorphism of $\lambda^*(M^\vee) \otimes M$ is a projector of rank 1.

Proof. The construction of the Taylor isomorphism on modules satisfying the Robba condition is functorial, so the diagram
\[
\begin{array}{ccc}
\lambda^*(\text{End}(M)) & \otimes & \lambda^*(\text{End}(M)) \\
\downarrow T_\lambda \otimes T_\lambda & & \downarrow T_\lambda \\
\text{End}(M) \otimes \text{End}(M) & \to & \text{End}(M)
\end{array}
\]
commutes. From this, it follows formally that the endomorphism of $\lambda^*(M^\vee) \otimes M$ is a projector. The trace of this projector is an analytic function of $\lambda$, but is also equal to the rank of the projector and so always belongs to $\{0, \ldots, \text{rank}(M)\}$. It is thus a constant function; moreover, the constant value must equal 1 because for $\lambda = 1$, the endomorphism of $\lambda^*(M^\vee) \otimes M \cong \text{End}(M)$ in question is the projector onto the trace component. This proves the claim. \qed

3.3 The Robba condition: residue characteristic 0

We continue to study the Robba condition in the case of residue characteristic 0. The methods used are familiar, but the exact result seems to be inexplicably missing from the literature.

Hypothesis 3.3.1. Throughout this subsection, retain Hypothesis 3.2.1, but also assume that $p = 0$ and that $M$ satisfies the Robba condition.

Definition 3.3.2. An exponent for $M$ is a finite multiset of $\mathfrak{o}_{K^{\text{alg}}}$ such that $M[t^{-1}] \otimes_K K^{\text{alg}}$ admits a basis on which $D$ acts via a matrix over $\mathfrak{o}_{K^{\text{alg}}}$ with multiset of eigenvalues equal to $A$.

Lemma 3.3.3. Assume that $0 \in I$ and that the eigenvalues of $D$ on $M/tM$ belong to $\mathfrak{o}_{K^{\text{alg}}}$. Then there exists a differential module $M'$ over $R_I$ with $M[t^{-1}] \cong M'[t^{-1}]$ such that the eigenvalues of $D$ on $M'/tM'$ belong to $\mathfrak{o}_{K^{\text{alg}}}$ and are prepared.

Proof. This is an example of the use of shearing transformations [Ked10a, Proposition 7.3.10]. Split $M/tM$ as a direct sum in which each summand consists of the generalized eigenspaces for a single Galois orbit of eigenvalues for the action of $D$. If we consider the differential submodule $M'$ of $M$ consisting of those elements whose images in $M/tM$ project to zero in a particular summand, the eigenvalues of $D$ on $M'/tM'$ are the same as on $M/tM$ except that one Galois orbit has been shifted by one.

It thus suffices to establish the existence of a sequence of shifts having the desired property. This follows from the following two observations (both of which require $p = 0$).

(a) If $\lambda_1, \lambda'_1 \in \kappa_K^{\text{alg}}$ are Galois conjugate, $\lambda_2, \lambda'_2 \in \kappa_K^{\text{alg}}$ are Galois conjugate, and $\lambda_1 - \lambda_2, \lambda'_1 - \lambda'_2 \in \mathbb{Z}$, then $\lambda_1 - \lambda_2 = \lambda'_1 - \lambda'_2$. (This follows by taking traces from some finite extension of $K$ containing $\lambda_1, \lambda'_1, \lambda_2, \lambda'_2$.)

(b) If $\lambda, \lambda' \in \kappa_K^{\text{alg}}$ are Galois conjugate and differ by an integer, then they are equal. (This follows from (a) by taking $\lambda_1 = \lambda'_1 = \lambda_2 = \lambda, \lambda'_2 = \lambda'$.) \qed
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**Lemma 3.3.4.** Assume that $0 \in I$ and the eigenvalues of $D$ on $M/tM$ belong to $\mathfrak{o}_K^\times$ and are prepared. Then there exists a basis of $M$ on which $D$ acts via a matrix over $\mathfrak{o}_K$.

**Proof.** Let $P(T) \in \mathfrak{o}_K[T]$ be the characteristic polynomial of the action of $D$ on $M/tM$. Since the roots of $P$ are prepared, for each positive integer $j$ there exists a unique polynomial $Q_j(T) \in \mathfrak{o}_K[T]$ of degree at most $n-1$ such that $P(T-j)Q_j(T) \equiv 1 \pmod{P(T)}$.

It is straightforward to check (see, for example, [Ked10a, Proposition 7.3.6]) that there exists a basis of $M \otimes_{R_0} K[T]$ on which $D$ acts via a matrix over $\mathfrak{o}_K$. We may reconstruct this basis by starting with any elements $e_1, \ldots, e_n \in M$ which lift a basis of $M/tM$ and forming the $t$-adic limits of the sequences

$$e_{i,m} = \left( \prod_{j=1}^{m} P(D-j)Q_j(D) \right) e_i \quad (i = 1, \ldots, n; m = 1, 2, \ldots).$$

For any given $\rho \in I - \{0\}$, these sequences are bounded for the norm induced by $|\cdot|_\rho$ using a basis of $M_{\rho,0}$ (because $M$ satisfies the Robba condition); since these sequences also converge $t$-adically, they converge under $|\cdot|_{\rho'}$ for any $\rho' \in (0, \rho)$ by Lemma 3.1.5(b) (with $m = 1$). This proves the existence of the desired basis.

**Lemma 3.3.5.** Assume that for some $\rho \in I$, $M$ admits a basis $e_1, \ldots, e_n$ on which $D$ acts via a matrix $N = \sum_{i \in \mathbb{Z}} N_i t^i$ with $|N_0| \leq 1$ and $|N - N_0|_\rho < 1$ for all $\rho \in I$. Then there exists a basis of $M$ on which $D$ acts via a matrix over $\mathfrak{o}_K$.

**Proof.** By applying Lemma 3.3.3 with $K$ replaced by $\kappa_K$ (equipped with the trivial norm) and using the fact that $\kappa_K[t^\pm]$ is a principal ideal domain (so every invertible square matrix over it factors as a product of elementary matrices), we may ensure that the eigenvalues of $N_0$ are prepared. In this case, for any nonzero $i \in \mathbb{Z}$, the eigenvalues of the linear operator $X \mapsto N_0 X - X N_0 + iX$ on $n \times n$ matrices over $\kappa_K$ are all nonzero (because each of them has the form $\lambda - \lambda' + i$ for some eigenvalues $\lambda, \lambda'$ of $N_0$). Consequently, this linear operator is invertible; it follows that for any $n \times n$ matrix $X$ over $K$ and any nonzero $i \in \mathbb{Z}$, $|N_0 X - X N_0 + iX| = |X|$.

We next produce a sequence $U_0, U_1, \ldots$ of invertible matrices over $R$ such that $|U_i - I_n|_\rho < 1$ for all $i \in \{0, 1, \ldots\}$ and $\rho \in I$. Start with $U_0 = I_n$. Given $U_l$ for some $l$, put $N_l = U_l^{-1} N U_l + U_l^{-1} D(U_l)$. Write $N_l = \sum_{i \in \mathbb{Z}} N_{l,i} t^i$ and apply the previous paragraph to construct $X_l$ so that $|X_l|_\rho = |N_l - N_{l,0}|_\rho$ for all $\rho \in I$ and $N_l = X_l N_0 - N_0 X_l + D(X_l)$. Then put $V_l = I_n + X_l$ and $U_{l+1} = U_l V_l$; note that $N_{l+1} = V_l^{-1} N V_l + V_l^{-1} D(V_l)$.

Suppose that for $\rho \in I$ and $\epsilon > 0$, we have $|N - N_0|_\rho \leq \epsilon$ and $|N_l - N_{l,0}|_\rho \leq \epsilon^{l+1}$. Then we have $|V_l - I_n|_\rho \leq \epsilon^{l+1}$, so

$$|N_{l+1} - N_l + X_l N_0 - N_0 X_l - D(X_l)|_\rho \leq \epsilon^{l+2}.$$ 

However, the matrix on the left-hand side is exactly $N_{l+1} - N_{l,0}$, so we must have $|N_{l+1} - N_{l+1,0}|_\rho \leq \epsilon^{l+2}$.

From the previous paragraph, it follows that the $U_l$ converge to an invertible matrix $U$ over $R_I$. The elements $e'_1, \ldots, e'_n$ of $M$ given by $e'_j = \sum_i U_{ij} e_i$ then form a basis with the desired property.

**Theorem 3.3.6.** Assume that $p = 0$ and that $M$ satisfies the Robba condition.

(a) There exists a Galois-invariant exponent for $M$.

(b) Any two exponents of $M$ are equivalent.

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Proof. Apply Corollary 2.1.6 to choose $v \in M$ which is a cyclic vector for $M \otimes_{R_I} \text{Frac}(R_I)$. For any closed subinterval $J$ of $I$ having positive length, the quotient of $M_J$ by the span of $v, D(v), \ldots, D^{n-1}(v)$ is killed by some nonzero element of $R_I$; since the slopes of the Newton polygon of this element form a discrete subset of $J$, we can shrink $J$ so as to force this element to become a unit. That is, we may choose $J$ so that $v, D(v), \ldots, D^{n-1}(v)$ form a basis of $M_J$.

Let $N$ be the matrix of action of $D$ on the basis $v, D(v), \ldots, D^{n-1}(v)$ of $M_J$. By Proposition 2.2.6, we have $|N|_J \leq 1$. In particular, if we write $N = \sum_{t \in \mathbb{Z}} N_t t^i$, then $|N| = \sum_{t \in \mathbb{Z}} |N_t|$. Since $J$ has positive length and $|N - N_0|_J \leq 1$, by shrinking $J$ and applying Lemma 3.1.5(b) (with $m = 1$) we may ensure that $|N - N_0|_J < 1$. We may then apply Lemma 3.3.5 to obtain the conclusion of (a) for $M_J$. We may then use Lemmas 3.3.3 and 3.3.4 to extend the convergence from $J$ to $I$. This yields (a). Given (a), (b) follows from the fact that $M_\lambda$ is trivial if and only if $\lambda \in \mathbb{Z}$.

\[\square\]

3.4 The Robba condition: residue characteristic $p > 0$

When $p > 0$, the structure of modules satisfying the Robba condition is more complicated; it is best understood using the Christol–Mebkhout theory of $p$-adic exponents. Here we follow and refine the exposition in [Ked10a, ch. 13].

Hypothesis 3.4.1. Throughout this subsection, retain Hypothesis 3.2.1, but also assume that $p > 0$ and that $M$ satisfies the Robba condition.

Definition 3.4.2. We say that $a \in \mathbb{Z}_p$ is a $p$-adic Liouville number if $a \notin \mathbb{Z}$ and

$$\liminf_{m \to \infty} \frac{p^m}{m} \left\langle \frac{a}{p^m} \right\rangle < +\infty. \quad (3.4.2.1)$$

Otherwise, we say that $a$ is a $p$-adic non-Liouville number.

For $A$ a multisubset of $\mathbb{Z}_p$, we say that $A$ is $p$-adic non-Liouville if it contains no $p$-adic non-Liouville number. We say that $A$ has $p$-adic non-Liouville differences if the difference multiset of $A$, defined as

$$A - A = \{a_1 - a_2 : a_1, a_2 \in A\},$$

is $p$-adic non-Liouville.

Definition 3.4.3. Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ be two finite multisubsets of $\mathbb{Z}_p$ of the same cardinality $n$. We say that $A$ and $B$ are weakly equivalent if there exist a constant $c > 0$ and a sequence $\sigma_1, \sigma_2, \ldots$ of permutations of $\{1, \ldots, n\}$ such that

$$p^m \left\langle \frac{a_{\sigma_m(i)} - b_i}{p^m} \right\rangle \leq cm \quad (m = 1, 2, \ldots; i = 1, \ldots, n).$$

This is evidently an equivalence relation. Note that $A, B$ are weakly equivalent if they are equivalent in the sense of Definition 3.2.4; the converse is false in general (see [Ked10a, Example 13.4.6]) but is true for $n = 1$ (see Corollary 3.4.7 below).

All of the key properties of weak equivalence can be expressed in terms of the following construction.

Definition 3.4.4. Let $A, A_1, \ldots, A_k$ be multisubsets of $\mathbb{Z}_p$ such that $A$ is the multiset union of $A_1, \ldots, A_k$. We say that $A_1, \ldots, A_k$ form an integer partition (respectively a Liouville partition) of $A$ if there do not exist distinct values $g, h \in \{1, \ldots, k\}$ and elements $a_g \in A_g, a_h \in A_h$ such...
that $a_g - a_h$ is an integer (respectively an integer or a $p$-adic Liouville number). This implies in particular that $A_g$ and $A_h$ are disjoint, so $A_1, \ldots, A_k$ form a partition of $A$.

Note that $A$ always admits a maximal integer partition, namely the partition into $\mathbb{Z}$-cosets. This partition is a Liouville partition if and only if $A$ has $p$-adic non-Liouville differences.

**Proposition 3.4.5.** Let $A$ be a finite multisubset of $\mathbb{Z}_p$ and let $A_1, \ldots, A_k$ be a Liouville partition of $A$.

(a) Let $B_1, \ldots, B_k$ be multisubsets of $\mathbb{Z}_p$ such that $B_g$ is weakly equivalent to $A_g$ for $g = 1, \ldots, k$. Then $B_1, \ldots, B_k$ form a Liouville partition of $B = B_1 \cup \cdots \cup B_k$; in particular, $B_1, \ldots, B_k$ are pairwise disjoint.

(b) Suppose that $B$ is a multisubset of $\mathbb{Z}_p$ weakly equivalent to $A$. Then $B$ admits a Liouville partition $B_1, \ldots, B_k$ such that $B_g$ is weakly equivalent to $A_g$ for $g = 1, \ldots, k$.

**Proof.** By the conditions on $A$, for each $c > 0$, there exists $m_0 = m_0(c)$ such that for all $m \geq m_0$, $g, h \in \{1, \ldots, k\}$ with $g \neq h$, $a_g \in A_g$, $a_h \in A_h$,

$$p^m \left( \frac{a_g - a_h}{p^m} \right) > (3c + 1)m. \tag{3.4.5.1}$$

Assume now the hypotheses of (a). Suppose by way of contradiction that there exist $g, h \in \{1, \ldots, k\}$ with $g \neq h$, $b_g \in B_g$, $b_h \in B_h$ such that $b_g - b_h$ is an integer or a $p$-adic Liouville number. Then there exists $c > 0$ such that for each $m$, on the one hand

$$p^m \left( \frac{b_g - b_h}{p^m} \right) \leq cm$$

and on the other hand there exist $a_g \in A_g$, $a_h \in A_h$ such that

$$p^m \left( \frac{a_g - b_g}{p^m} \right), p^m \left( \frac{a_h - b_h}{p^m} \right) \leq cm.$$ 

But then

$$p^m \left( \frac{a_g - a_h}{p^m} \right) \leq 3cm,$$

which combined with (3.4.5.1) yields the desired contradiction.

Assume now the hypotheses of (b). Label the elements of $A$ and $B$ as $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, respectively. Then there exists $c > 0$ such that for each $m$, there exists a permutation $\sigma_m$ of $\{1, \ldots, n\}$ such that

$$p^m \left( \frac{a_{\sigma_m(i)} - b_i}{p^m} \right) \leq cm \quad (i = 1, \ldots, n).$$

In particular,

$$p^m \left( \frac{a_{\sigma_m(i)} - a_{\sigma_{m+1}(i)}}{p^m} \right) \leq (2c + 1)m \quad (i = 1, \ldots, n),$$

which by (3.4.5.1) yields that for $m \geq m_0(c)$, $\sigma_m^{-1} \circ \sigma_{m+1}$ must respect the partition of $A$. Define $B_1, \ldots, B_k$ so that $B_g$ consists of those $b_i$ for which $a_{\sigma_m(i)} \in A_g$ for $m \geq m_0(c)$; by the above argument, $B_g$ is weakly equivalent to $A_g$. By (a), $B_1, \ldots, B_k$ is a Liouville partition of $B$, as desired.

**Corollary 3.4.6.** Let $A, B$ be two finite multisubsets of $\mathbb{Z}_p$ which are weakly equivalent. Then $A$ contains an integer or a $p$-adic non-Liouville number if and only if $B$ does.
Proof. Note that $A$ contains an integer or $p$-adic non-Liouville number if and only if $\{0\}$ and $A$ fail to form a Liouville partition of $\{0\} \cup A$. The claim thus follows by applying Proposition 3.4.5(a) to $\{0\} \cup A$ and $\{0\} \cup B$.

The following corollary reproduces [Ked10a, Lemma 13.4.3].

**Corollary 3.4.7.** For $a, b \in \mathbb{Z}_p$, the singleton multisets $\{a\}, \{b\}$ are weakly equivalent if and only if $a - b \in \mathbb{Z}$.

*Proof.* By translating both $a$ and $b$, we may assume that $b = 0$. If $a \in \mathbb{Z}$, then $\{a\}$ and $\{0\}$ are equivalent and hence weakly equivalent. Conversely, if $\{a\}$ and $\{0\}$ are weakly equivalent, then $a$ satisfies (3.4.2.1) and so must be either an integer or a $p$-adic Liouville number, but the latter case is ruled out by Corollary 3.4.6.

The following corollary reproduces [Ked10a, Proposition 13.4.5].

**Corollary 3.4.8.** Let $A, B$ be two finite multisubsets of $\mathbb{Z}_p$ which are weakly equivalent. Suppose that $A$ has $p$-adic non-Liouville differences. Then $A$ and $B$ are equivalent.

*Proof.* By partitioning $A$ into $\mathbb{Z}$-cosets and applying Proposition 3.4.5(b), we may reduce to the case where $A$ is a multisubset of $\mathbb{Z}$. In this case, for each $b \in B$, the singleton multisets $\{0\}$ and $\{b\}$ are weakly equivalent, so Corollary 3.4.7 implies that $b \in \mathbb{Z}$. This proves the claim.

**Corollary 3.4.9.** Let $A, B$ be two finite multisubsets of $\mathbb{Z}_p$ which are weakly equivalent. Suppose that $A$ is $p$-adic non-Liouville. Then there exist Liouville partitions $A_1, A_2$ of $A$ and $B_1, B_2$ of $B$ satisfying the following conditions.

(a) The multisets $A_1, B_1$ consist entirely of integers.

(b) The multisets $A_2, B_2$ are weakly equivalent and contain no integers or $p$-adic Liouville numbers.

In particular, $B$ is also $p$-adic non-Liouville.

*Proof.* Partition $A$ into two parts $A_1, A_2$ so that $A_1$ consists precisely of the integers appearing in $A$; by hypothesis, this is a Liouville partition of $A$. By Proposition 3.4.5(b), $B$ admits a Liouville partition $B_1, B_2$ in which $B_1$ is weakly equivalent to $A_1$ for $i = 1, 2$. Since $A_1$ consists only of integers, by Corollary 3.4.8, $B_1$ also consists only of integers. Since $A_2$ does not contain any integer or $p$-adic Liouville number, neither does $B_2$ by Corollary 3.4.6. This proves the desired results.

**Corollary 3.4.10.** Let $A$ be a finite multisubset of $\mathbb{Z}_p$ such that $A - A$ is weakly equivalent to a $p$-adic non-Liouville multiset. Then $A$ has $p$-adic non-Liouville differences.

*Proof.* By Corollary 3.4.9, $A - A$ is $p$-adic non-Liouville.

**Definition 3.4.11.** Recall that we are assuming that $M$ satisfies the Robba condition. Let $J$ be a closed subinterval of $I$ of positive length. We say that the multisubset $A = \{a_1, \ldots, a_n\}$ of $\mathbb{Z}_p$ is an *exponent* for $M$ over $J$ if there exist elements $v_{m,A,j} \in M_J[t^{-1}]$ for $m = 1, 2, \ldots$ and $j = 1, \ldots, n$ satisfying the following conditions. (For this definition, we fix an ordering of $A$, but this choice is manifestly immaterial.)

(a) For all $m, j$, for all $\zeta \in K^{\text{alg}}$ with $\zeta^{p^m} = 1$, we have $\zeta^*(v_{m,A,j}) = \zeta^{a_j}v_{m,A,j}$ as an equality in $M_J[t^{-1}] \otimes_K K(\overline{\zeta})$.
(b) For some (and hence any) basis $e_1, \ldots, e_n$ of $M_J$, there exists $k > 0$ such that the $n \times n$ matrices $S_{m,A}$ over $R_I$ defined by $v_{m,A,j} = \sum_i (S_{m,A})_{ij} e_i$ are invertible and satisfy

$$|S_{m,A}|, |^{-1}_{m,A}| \leq p^{km} \quad (m = 1, 2, \ldots).$$

Note that if $A$ is an exponent for $M$ over $J$, then so is any multiset weakly equivalent to $A$ (but not necessarily any multiset weakly equivalent to $A$).

Remark 3.4.12. In [Ked10a, Definition 13.5.2], the hypotheses on the matrix $S_{m,A}$ are slightly different: it is assumed that $S_{m,A}$ is invertible and satisfies $|S_{m,A}| \leq p^{km}$ and $|\det(S_{m,A})| \geq 1$. It is easy to see that this hypothesis is equivalent to the one given in Definition 3.4.11 modulo rescaling the vectors $v_{m,A,j}$ and rechoosing the constant $k$; we may thus safely quote results from [Ked10a] in what follows.

Example 3.4.13. As noted in Example 3.2.3, for any $\lambda \in \mathbb{Z}_p$, the differential module $M_\lambda$ generated by a single element $v$ satisfying $D(v) = \lambda v$ satisfies the Robba condition [Ked10a, Example 9.5.2]. This module admits the singleton multiset $\{\lambda\}$ as an exponent.

Remark 3.4.14. If $M_1, M_2$ are two differential modules over $R_I$ for the derivation $t \frac{d}{dt}$ admitting respective exponents $A_1, A_2$ over some $J$, we then have the following:

(a) if there exists an exact sequence $0 \to M_1 \to M \to M_2 \to 0$ of differential modules over $R_I$, then $M$ admits the multiset union $A_1 \cup A_2$ as an exponent over $J$;

(b) the differential module $M_1 \otimes M_2$ admits the multiset $A_1 + A_2 = \{a_i + a_j : a_i \in A_1, a_j \in A_2\}$ as an exponent over $J$;

(c) the differential module $M_1^\vee$ admits the multiset $-A_1 = \{-a : a \in A_1\}$ as an exponent over $J$.

Remark 3.4.15. If $0 \in I$, then it is straightforward to check the following by imitating the proof of [Ked10a, Theorem 13.2.2].

(a) Let $A$ be the set of eigenvalues of $D$ on $M/tM$. Then $A$ belongs to $\mathbb{Z}_p^n$ and is an exponent for $M$.

(b) Any Liouville partition of $A$ corresponds to a unique direct sum decomposition of $M$.

(c) If $A$ is a multiset of $\lambda + \mathbb{Z}$, then there exists another differential module $M'$ over $R_I$ with $M'[t^{-1}] \cong M[t^{-1}]$ such that $D$ acts on $M'/tM'$ via a matrix with all eigenvalues equal to $\lambda$. This again follows from the use of shearing transformations as in Lemma 3.3.3

(d) If $A$ is a multiset of $\{\lambda\}$, then there exists a basis of $M$ on which $D$ acts via a matrix over $K$ with all eigenvalues equal to $\lambda$.

We may thus safely assume that $0 \notin I$ in what follows.

Theorem 3.4.16. (a) For any closed subinterval $J$ of $I$ of positive length not containing $0$, there exists an exponent for $M$ over $J$.

(b) Any two exponents for $M$ (possibly over different intervals) are weakly equivalent.

Proof. For (a), see [Ked10a, Theorem 13.5.5]. For (b), let $J_1, J_2$ be two closed subintervals of $I$ of positive length not containing $0$. Let $A_1, A_2$ be exponents for $M$ over $J_1, J_2$. If $J_1 = J_2$, we may apply [Ked10a, Theorem 13.5.6] to deduce that $A_1$ is weakly equivalent to $A_2$. Otherwise, let $J$ be a third such interval containing both $J_1$ and $J_2$. By (a), there exists an exponent $A$ for $M$ over $J$, which then restricts to an exponent for $M$ over $J_1$ and over $J_2$. By [Ked10a, Theorem 13.5.6] again, $A$ is weakly equivalent to both $A_1$ and $A_2$, so $A_1$ and $A_2$ are weakly equivalent to each other. ☐
Remark 3.4.17. If $M$ admits a basis (e.g., if $K$ is spherically complete), the proofs of [Ked10a, Theorems 13.5.5, 13.5.6] show that the exponent $A$ and the elements $v_{m,A,J}$ can be chosen uniformly in $J$. We will not need to use this fact in this paper.

Definition 3.4.18. We say that $M$ has $p$-adic non-Liouville exponents if for some closed subinterval $J$ of $I$ of positive length not containing 0, $M$ admits an exponent $A$ over $J$ which is $p$-adic non-Liouville. By Theorem 3.4.16(b) and Corollary 3.4.9, this implies that every exponent of $M$ (over every $J$) is $p$-adic non-Liouville.

We say that $M$ has $p$-adic non-Liouville exponent differences if $\text{End}(M)$ has $p$-adic non-Liouville exponents. For alternate characterizations, see Lemma 3.4.19 below.

Lemma 3.4.19. The following conditions are equivalent.

(a) The module $M$ has $p$-adic non-Liouville exponent differences.
(b) Some exponent of $M$ has $p$-adic non-Liouville differences.
(c) Every exponent of $M$ has $p$-adic non-Liouville differences.

Moreover, when these conditions hold, then any two exponents of $M$ are equivalent (not just weakly equivalent).

Proof. By Theorem 3.4.16(a), (c) implies (b). By Remark 3.4.14, (b) implies (a).

Suppose now that (a) holds. Let $A$ be any exponent for $M$, and let $B$ be an exponent for $\text{End}(M)$ which is $p$-adic non-Liouville. By Remark 3.4.14, $A - A$ is an exponent for $\text{End}(M)$, so by Theorem 3.4.16(b), $A - A$ and $B$ are weakly equivalent. By Corollary 3.4.10, $A$ has $p$-adic non-Liouville differences, yielding (c). Moreover, if $A'$ is another exponent for $M$, then $A$ and $A'$ are weakly equivalent by Theorem 3.4.16(b), so $A$ and $A'$ are equivalent by Corollary 3.4.8. □

The primary structure theorem for differential modules satisfying the Robba condition is the Christol–Mebkhout decomposition theorem; see, for instance, [Ked10a, Theorem 13.6.1] and the errata to [Ked10a]. Here, we divide the statement into two parts in order to clarify the exposition and strengthen one of the two parts. One of the two parts, which by itself is sufficient for many applications, is the following structure theorem for modules admitting a singleton exponent.

Theorem 3.4.20. Suppose that $M$ admits an exponent identically equal to some $\lambda \in \mathbb{Z}_p$. Then for any closed subinterval $J$ of $I$ of positive length, $M_J$ admits a basis on which $D$ acts via a matrix over $K$ whose eigenvalues are all equal to $\lambda$.

Proof. We may assume that $0 \notin I$ thanks to Remark 3.4.15. By replacing $M$ with its twist $M_\lambda \otimes M$, we may reduce the theorem to the special case $\lambda = 0$. Let $J$ be any closed subinterval of $I$ of positive length; by Lemma 3.4.19, the zero $n$-tuple is an exponent for $M$ over $J$. Choose $\eta > 0$ and $\alpha, \alpha', \beta, \beta' \in I$ such that $\alpha' < \beta'$, $\alpha' = \alpha \eta$, $\beta' = \beta / \eta$, and $J \subseteq [\alpha', \beta']$. Fix a basis of $M_J$ and define the matrices $S_{m,A}$ as in Definition 3.4.11 for $A = \{0, \ldots, 0\}$. Choose $\lambda \in (0, 1)$ and $c > 0$ so that $p^{10k} \eta^{-c} \leq \lambda$, then choose $m_0 > 0$ so that $p^m > cm$ for all $m \geq m_0$. We will construct invertible matrices $R_m$ over $K$ for $m \geq m_0$ such that $R_{m_0} = I_n$ and

$$|I_n - R_m S_{m,A} S_{m+1,A} R^{-1}_{m+1}|_\rho \leq \lambda^m \quad (\rho \in [\alpha', \beta'], m \geq m_0).$$

This will imply that for $m \geq m_0$ and $\rho \in [\alpha', \beta']$,

$$|I_n - S_{m_0,A} S_{m,A} R^{-1}_m|_\rho < 1, \quad |S_{m_0,A} S_{m,A} R^{-1}_m - S_{m_0,A} S_{m+1,A} R^{-1}_{m+1}|_\rho < \lambda^m.$$
Consequently, the sequence $S_{m_0,A}^{-1}S_m,A R_m^{-1}$ for $m = m_0, m_0 + 1, \ldots$ will converge to an invertible matrix $U$ over $R_{\alpha',\beta'}$ such that $S_{m_0,A} U$ is the change-of-basis matrix to a basis of $M_{\alpha',\beta'}$ of the desired form. This will complete the proof.

The construction of the $R_m$ proceeds recursively as follows. Given $R_{m_0}, \ldots, R_m$, we first verify that

$$|R_m|, |R_m^{-1}| \leq p^{2km}.$$

This is clear for $m = m_0$, so we may assume that $m > m_0$. Choose any $\rho \in [\alpha', \beta']$. As noted above, we have $|I_n - S_{m_0,A}^{-1}S_m,A R_m^{-1}| \rho < 1$, so $|S_{m_0,A}^{-1}S_m,A R_m^{-1}| = |R_m S_{m_0,A}^{-1}S_m,A| \rho = 1$. We then deduce the claim from the bound $|S_{m,A}|, |S_{m,A}^{-1}| \rho \leq p^{km}$.

Next, put $T_m = R_m S_{m,A}^{-1} S_{m+1,A}$; we then have

$$|T_m|, |T_m^{-1}| \leq p^{4km+k}.$$

Let $T_{m,0}$ be the constant coefficient of $T_m$. Since $T_m$ is a series in $\theta^m$, Lemma 3.1.5(b) implies

$$|T_m - T_{m,0}|, |T_m - T_{m,0}|^{-1} \leq p^{4km+k} \eta^{-p^m}.$$

We may now take $R_{m+1} = T_{m,0}$, because

$$|I_n - R_{m+1} T_m^{-1}|, |T_m - T_{m,0}|, |I_n - R_{m+1} T_m^{-1}|, |T_m - T_{m,0}| \leq p^{8km+2k} \eta^{-p^m}$$

and so $|I_n - T_m R_{m+1}|, |I_n - T_m R_{m+1}|^{-1} \leq \lambda^m$. This completes the construction of the $R_m$ and thus the proof.

Remark 3.4.21. Theorem 3.4.20 is sufficient to recover the full Christol–Mebkhout decomposition theorem in the case of a differential module admitting an exponent contained in $\mathbb{Z}_p \cap \mathbb{Q}$, by pulling back along the map $t \mapsto t^m$ for a suitably divisible integer $m \in \mathbb{Z}$.

The second part is a splitting theorem for modules admitting an exponent with $p$-adic non-Liouville differences. This may be generalized as follows.

Theorem 3.4.22. Suppose that $M$ admits an exponent $A$ admitting the Liouville partition $A_1, \ldots, A_k$. Then for any closed subinterval $J$ of $I$ of positive length, there exists a unique direct sum decomposition $M_J = M_1 \oplus \cdots \oplus M_k$ such that for $g = 1, \ldots, k$, $M_g$ admits an exponent over $J$ weakly equivalent to $A_g$.

Proof. We may assume that $0 \notin I$ thanks to Remark 3.4.15. We first verify uniqueness. Suppose to the contrary that there is a second decomposition $M_J = M'_1 \oplus \cdots \oplus M'_k$ of the desired form for which there exist $g \neq h$ such that $M_{gh} = M'_g \cap M'_h$ is nonzero. Apply Theorem 3.4.16(a) to produce exponents $B_1, B_2, B_3$ of $M_{gh}, M_g/M_{gh}, M_h/M_{gh}$. By Remark 3.4.14 and Theorem 3.4.16(b), $B_1 \cup B_2$ is weakly equivalent to $A_g$ and $B_1 \cup B_3$ is weakly equivalent to $A_h$. We then obtain the desired contradiction by applying Proposition 3.4.5(a).
We next verify existence. To simplify notation, we may reduce to the case \( k = 2 \). Let \( J \) be any closed subinterval of \( I \) of positive length; by Theorem 3.4.16(a, b) and Proposition 3.4.5(b), \( M \) admits an exponent \( A \) over \( J \) of the specified form. Choose an ordering \( A = \{ a_1, \ldots, a_n \} \).

Choose \( \eta > 1 \) and \( \alpha, \alpha', \beta, \beta' \in I \) such that \( \alpha' < \beta' \), \( \alpha' = \alpha \eta \), \( \beta' = \beta / \eta \), and \( J \subseteq [\alpha', \beta'] \). Fix a basis of \( M \) and define the matrices \( S_{m, A} \) as in Definition 3.4.11. Choose \( \lambda \in (0, 1) \) and \( c > 0 \) so that \( \lambda^{k_0} \eta^{-c} \leq \lambda \). By hypothesis, there exists \( m_0 > 0 \) such that for \( m \geq m_0 \), \( b_1 \in A_1 \), \( b_2 \in A_2 \), the congruence \( h \equiv b_1 - b_2 \mod p^m \) forces \( |b| \geq cm \).

Let \( \Pi_m \) be the projector onto the submodule of \( M \) generated by \( v_{m, A, i} \) for those \( i \) for which \( a_i \in A_1 \); then

\[
|\Pi_m|_{[\alpha, \beta]} \leq p^{2km}.
\]

For those \( j \) for which \( a_j \in A_1 \), write \( (\Pi_m - \Pi_{m+1})(v_{m, A, j}) = \sum_i a_{m, i} v_{m+1, A, i} \), so that

\[
|a_{m, i}|_{[\alpha', \beta']} \leq p^{4km+2k}.
\]

Since \( (\Pi_m - \Pi_{m+1})(v_{m, A, j}) = (1 - \Pi_{m+1})(v_{m, A, j}) \), we have \( a_{m, i} = 0 \) when \( a_i \in A_1 \). On the other hand, when \( a_i \in A_2 \), the coefficient of \( t^b \) in \( a_{m, i} \) can only be nonzero if \( h \equiv a_i - a_j \mod p^m \); this implies

\[
|a_{m, i}|_{[\alpha', \beta']} \leq p^{4km+2k} \eta^{-cm} \quad (m \geq m_0)
\]

by Lemma 3.1.5(a), and so

\[
|(\Pi_m - \Pi_{m+1})(v_{m, A, j})|_{[\alpha', \beta']} \leq p^{5km+2k} \eta^{-cm} \quad (m \geq m_0).
\]

Similarly, for those \( j \) for which \( a_j \in A_2 \),

\[
|(\Pi_m - \Pi_{m+1})(v_{m, A, j})|_{[\alpha', \beta']} \leq p^{5km+3k} \eta^{-cm} \quad (m \geq m_0)
\]

and so

\[
|(\Pi_m - \Pi_{m+1})|_{[\alpha', \beta']} \leq p^{6km+3k} \eta^{-cm} \leq \lambda^m \quad (m \geq m_0).
\]

Therefore the \( \Pi_m \) converge to an endomorphism of \( M_J \), which is forced to be a projector defining the desired splitting. \( \square \)

We may put Theorems 3.4.20 and 3.4.22 together to separate integer exponents from \( p \)-adic non-Liouville exponents.

**Corollary 3.4.23.** Suppose that \( M \) has \( p \)-adic non-Liouville exponents. Then there exists a unique direct sum decomposition \( M \cong M_1 \oplus M_2 \) with the following properties.

(a) The module \( M_1[t^{-1}] \) admits a basis on which \( D \) acts via a nilpotent matrix over \( K \). In particular, \( M_1[t^{-1}] \) is unipotent (i.e., it is a successive extension of trivial differential modules over \( R_I[t^{-1}] \)).

(b) No exponent of \( M_2 \) contains an integer or a \( p \)-adic Liouville number.

**Proof.** We obtain the splitting \( M \cong M_1 \oplus M_2 \) using Theorem 3.4.22. We then obtain (a) using Theorem 3.4.22 and (b) using Corollary 3.4.6. \( \square \)

We recover as a corollary the original decomposition theorem of Christol and Mebkhout, as stated in [Ked10a, Theorem 13.6.1].

**Corollary 3.4.24.** Fix a set \( S \) of coset representatives of \( \mathbb{Z} \) in \( \mathbb{Z}_p \). Suppose that \( M \) has \( p \)-adic non-Liouville exponent differences. Then there exists a unique direct sum decomposition \( M \cong \bigoplus_{\lambda \in S} N_\lambda \) in which \( N_\lambda \) admits a basis on which \( D \) acts via a matrix over \( K \) with all eigenvalues equal to \( \lambda \). In particular, \( N_\lambda \) is isomorphic to a successive extension of copies of \( M_\lambda \).
3.5 Frobenius antecedents and descendants

The construction of Frobenius antecedents and descendants can be generalized to differential modules over power series. We record here some key facts from [Ked10a, ch. 10] which we will use.

Hypothesis 3.5.1. Throughout §3.5, assume that $p > 0$, fix a subinterval $I$ of $[0, +\infty)$, and take $R = R_I$ or $R = R_I^\infty$. Let $(M, D)$ be a differential module of rank $n$ over $(R, d/dt)$.

Definition 3.5.2. Let $I^p$ be the subinterval of $[0, +\infty)$ consisting of $\gamma^p$ for all $\gamma \in I$. For $R = R_I$ (respectively $R = R_I^\infty$), let $R'$ be a copy of $R_{I^p}$ (respectively $R_{I^p}^\infty$) in the variable $\iota^p$, identified with a subring of $R$. We may then view $(R', d/d\iota^p)$ as a differential ring.

If $0 \notin I$, we may form the Frobenius descendant $\varphi_* M$ as in [Ked10a, Definition 10.3.4]; that is, $\varphi_* M$ is a copy of $M$ viewed as an $R'$-module equipped with the derivation $D' = p^{-1}t^{1-p}D$. For any $\rho \in I$, $(\varphi_* M) \otimes_R F'_\rho$ may be naturally identified with the Frobenius descendant of $M_\rho$.

Proposition 3.5.3. Suppose that $f_1(M, r) < r - \log \omega$ for all $r \in -\log I$. Then there exists a unique (up to unique isomorphism) differential module $M'$ over $(R', d/d\iota^p)$ such that for $\rho \in I \backslash \{0\}$, $M' \otimes_R F'_\rho$ is the Frobenius antecedent of $M_\rho$.

Proof. See [Ked10a, Theorem 10.4.4].

We will also need to consider ‘off-center Frobenius descendants’ as in [Ked10a, §10.8].

Definition 3.5.4. Suppose that $R = R_{[0,1]}^\infty$. Let $R''$ be a copy of $R_{[0,1]}^\infty$ in the variable $u$. For $\rho \in (0, 1]$, let $F''_\rho$ be a copy of $F_\rho$ in the variable $u$, so that $R''$ maps to $F''_\rho$. Choose $\lambda \in K$ with $|\lambda| = 1$, then identify $R''$ with a subring of $R$ by identifying $u$ with $(t + \lambda)^p - \lambda^p$. 


Proposition 3.5.5. Suppose that \( R = R^n_{[0,1]} \). Let \( \psi_*M \) be a copy of \( M \) viewed as a differential module over \( (R^n, d/du) \). Then for \( \rho \in (0,1] \), the multiset consisting of the intrinsic subsidiary radii of \( (\psi_*M) \otimes R^n F^\rho_{\psi_*} \) is the union of the multisets
\[
\begin{cases}
\{s^p\} \cup \{\omega^p \rho^{-p} (p-1 \text{ times})\} & \text{if } s > \omega \rho, \\
\{p^{-1} s \rho^{1-p} (p \text{ times})\} & \text{if } s \leq \omega / \rho,
\end{cases}
\]
for \( s \) running over the intrinsic subsidiary radii of \( M_\rho \).

Proof. By rescaling \( t \), we may reduce to the case where \( \lambda = 1 \). In this case, see [Ked10a, Theorem 10.8.3].

3.6 Variation of intrinsic radii

We now consider differential modules not necessarily satisfying the Robba condition, with an eye toward the variation of the intrinsic subsidiary radii. The results we report here are taken from [Ked10a, chs. 11–12]; starting in §4, we will see how to make more definitive statements in the language of Berkovich spaces. To facilitate this transition, we record a couple of important direct corollaries of the results of [Ked10a].

Hypothesis 3.6.1. Throughout this subsection, fix a subinterval \( I \) of \([0, +\infty)\), and let \( M \) be a differential module of rank \( n \) over \((R^n, d/dt)\).

Definition 3.6.2. For \( \rho \in I \setminus \{0\} \), put \( M_\rho = M \otimes R^n F^\rho \). For \( r \in -\log I \) and \( i = 1, \ldots, n \), define \( f_i(M, r) \) so that the list of intrinsic subsidiary radii of \( M_{e^{-r}} \) in increasing order is
\[
\exp(r - f_1(M, r)), \ldots, \exp(r - f_n(M, r)).
\]
Put \( F_i(M, r) = f_1(M, r) + \cdots + f_n(M, r) \). As observed in Definition 2.2.2, the functions \( f_i \) and \( F_i \) are invariant under enlargement of the constant field \( K \).

Proposition 3.6.3. For \( i = 1, \ldots, n \), we have the following.

(a) (Linearity) The functions \( f_i(M, r) \) and \( F_i(M, r) \) are continuous and piecewise affine. Moreover, these functions assume only finitely many different slopes over any closed subinterval of \(-\log I\) (even if \( 0 \in I \)).

(b) (Integrality) If \( i = n \) or \( f_i(M, r_0) > f_{i+1}(M, r_0) \), then the slopes of \( F_i(M, r) \) in some neighborhood of \( r_0 \) belong to \( \frac{1}{\log \mu} \mathbb{Z} \). Consequently, the slopes of each \( f_i(M, r) \) and \( F_i(M, r) \) belong to \( \frac{1}{\log \mu} \mathbb{Z} \).

(c) (Subharmonicity) Suppose that \( K \) is algebraically closed, \( \alpha < 1 < \beta \), and \( f_i(M, 0) > 0 \). Let \( s_{\infty,i}(M) \) and \( s_{0,i}(M) \) be the left and right slopes of \( F_i(M, r) \) at \( r = 0 \). For \( \bar{\mu} \in \kappa_K^\times \), choose any \( \mu \in \sigma_K \) lifting \( \bar{\mu} \), let \( T_\mu \) denote the substitution \( t \mapsto t + \mu \), and let \( s_{\bar{\mu},i}(M) \) be the right slope of \( F_i(T_\mu^*(M), r) \) at \( r = 0 \). Then
\[
s_{\infty,i}(M) \leq \sum_{\bar{\mu} \in \kappa_K} s_{\bar{\mu},i}(M),
\]
with equality if either \( i = n \) and \( f_n(M, 0) > 0 \), or \( i < n \) and \( f_i(M, 0) > f_{i+1}(M, 0) \).

(d) (Monotonicity) Suppose that \( 0 \in I \). Then for any point \( r_0 \) where \( f_i(M, r_0) > r_0 \), the slopes of \( F_i(M, r) \) are nonpositive in some neighborhood of \( r_0 \).

(e) (Convexity) The function \( F_i(M, r) \) is convex.

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Proof. See [Ked10a, Theorem 11.3.2].

COROLLARY 3.6.4. Suppose that $0 \in I$ and $f_1(M,r_0) = r_0$ for some $r_0 \in -\log I$. Then $f_1(M,r) = r$ for all $r \geq r_0$.

Proof. By Proposition 3.6.3(a, d, e), the function $f_1(M,r)$ is piecewise affine and convex everywhere, and nonincreasing wherever it is greater than $r$. Since $f_1(M,r_0) = r_0$ and $f_1(M,r) \geq r$ everywhere, all of the slopes of $f_1(M,r)$ for $r \geq r_0$ must be at least one. However, none of them can be strictly greater than one because this would force $f_1(M,r) > r$ for some $r$, and then $f_1(M,r)$ would be forced to be nonincreasing. This proves the claim.

COROLLARY 3.6.5. Suppose that $0 \in I$ and for some $r_0 \in -\log I$, some $r_1 > r_0$, and some $j \in \{0, \ldots, n\}$, the functions $f_1(M,r), \ldots, f_j(M,r)$ are equal to some constant value $c$ for $r \in (r_0,r_1)$. Then

$$f_i(M,r) = \max\{r,c\} \quad (r > r_0; \ i = 1, \ldots, j).$$

Proof. We prove that the claim holds for $f_1, \ldots, f_i$ by induction on $i$, with base case $i = 0$. Given the induction hypothesis for $i - 1$, note that since $f_i(M,r) \geq r$ for all $r \geq r_0$, we must have $c > r_0$. By Proposition 3.6.3(d, e) and the induction hypothesis, the function $f_i(M,r)$ is convex everywhere and nonincreasing wherever it is greater than $r$. It follows that $f_i(M,r) = c$ for $r \in (r_0,c]$. By Corollary 3.6.4, we then have $f_i(M,r) = r$ for $r \geq c$.

PROPOSITION 3.6.6. For $i = 1, \ldots, n$, on any interval where $f_i(M,r)$ is affine, it has the form $ar + b$ for some $a \in \mathbb{Q}$ and some $b$ in the divisible closure of $\log |K^x|$.

Proof. See [Ked10a, Corollary 11.8.2].

PROPOSITION 3.6.7. Suppose that $0 \in I$ and that for some $i \in \{1, \ldots, n - 1\}$ and some $\gamma \in I \setminus \{0\}$, the following conditions hold.

(a) The function $F_i(M,r)$ is constant for $r < -\log \gamma$.

(b) We have $f_i(M,r) > f_{i+1}(M,r)$ for $r < -\log \gamma$.

Then $M$ admits a unique direct sum decomposition separating the first $i$ intrinsic subsidiary radii of $M_\rho$ for all $\rho > \gamma$.

Proof. See [Ked10a, Theorem 12.5.1].

COROLLARY 3.6.8. Suppose that $I = [0, \beta]$ or $I = [0, \beta)$ for some $\beta > 0$ and put $r_0 = -\log \beta$. Suppose that for some $i \in \{0, \ldots, n\}$ and some $r_1 > r_0$, the following conditions hold.

(a) For $j = 1, \ldots, i$, the function $f_j(M,r)$ is constant for $r \in (r_0,r_1)$.

(b) If $i < n$, then $\liminf_{r \to r_1^+} f_{i+1}(M,r) = r_0$.

Then $M$ admits a direct sum decomposition $M_0 \oplus M_1 \oplus \cdots$ such that

$$f_1(M_0,r) = \cdots = f_{\text{rank}(M_0)}(M_0,r) = r \quad (r > r_0)$$

and for each $k > 0$, there is a constant $c_k > r_0$ such that

$$f_1(M_k,r) = \cdots = f_{\text{rank}(M_k)}(M_k,r) = \max\{r,c_k\} \quad (r > r_0).$$

In particular, for $j = i + 1, \ldots, n$, $f_j(M,r) = r$ for $r > r_0$.  

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Proof. We induct on $i$. Suppose first that $i = 0$. In this case, we cannot have \( f_1(M, r_1) > r_1 \) for any \( r_1 > r_0 \), as then Proposition 3.6.3(d) would imply \( f_1(M, r) > r_1 \) for all \( r \in (r_0, r_1) \) and hence \( \liminf_{r \to r_1^+} f_1(M, r) \geq r_1 \), violating condition (b). Hence for all \( r > r_0 \) and all \( j \), we have \( r = f_1(M, r) \geq f_j(M, r) \geq r \), proving the claim with \( M = M_0 \).

Suppose next that \( i > 0 \). Let \( c_1 \) be the constant value of \( f_i(M, r) \) for \( r \in (r_0, r_1) \). By Corollary 3.6.5, we have \( f_1(M, r) = \max\{r, c_1\} \) for \( r > r_0 \). Let \( m \) be the largest value for which \( f_1(M, r) = f_m(M, r) \) for \( r \) in some right neighborhood \( (r_0, r_1) \) of \( r_0 \). Split \( M \) as \( M_1 \oplus M_2 \) as per Proposition 3.6.7 so that \( M_1 \) accounts for the first \( m \) intrinsic subsidiary radii of \( M_\rho \) for \( \rho > e^{-r_1} \). For \( r > c_1 \), for all \( j \) we have \( r \leq f_j(M_1, r) \leq f_1(M_1, r) \leq f_1(M, r) = r \) and so \( f_j(M_1, r) = r \). By Proposition 3.6.3(d), \( TM_\rho(M_1, r) \) is convex; since it agrees with the constant function \( \text{rank}(M_1)c_1 \) for \( r \in (r_0, r_1) \) and for \( r = c_1 \), we must have \( TM_\rho(M_1, r) = \text{rank}(M_1)c_1 \) for \( r \in (r_0, c_1) \). Since in addition \( f_j(M_1, r) \leq f_1(M_1, r) \leq f_1(M, r) = \max\{r, c_1\} \), we must have \( f_j(M_1, r) = c_1 \) for \( r \in (r_0, c_1) \). Thus \( M_1 \) has all of the desired properties, so we may apply the induction hypothesis to \( M_2 \) to prove the claim.

**Proposition 3.6.9.** Suppose that \( I = (\alpha, \beta) \) for some \( \alpha, \beta > 0 \) and that for some \( i \in \{1, \ldots, n-1\} \), the following conditions hold.

(a) The function \( f_i(M, r) \) is affine for \( r \in -\log I \).
(b) We have \( f_i(M, r) > f_{i+1}(M, r) \) for \( r \in -\log I \).

Then \( M \) admits a unique direct sum decomposition separating the first \( i \) intrinsic subsidiary radii of \( M_\rho \) for every \( \rho \in I \).

**Proof.** See [Ked10a, Theorem 12.4.2].

### 3.7 Decompositions over open annuli

We now embark on a deeper analysis of differential modules over open annuli than is found in [Ked10a], concentrating on spectral decompositions and on properties of refined modules. For the latter, we incorporate some ideas of Xiao [Xia09, Xia12].

**Hypothesis 3.7.1.** Throughout this subsection, continue to retain Hypothesis 3.6.1, but assume further that \( n > 0 \) and \( I = (\alpha, \beta) \) for some \( \alpha, \beta > 0 \).

**Definition 3.7.2.** We say that \( M \) is pure if the functions \( f_1(M, r), \ldots, f_n(M, r) \) for \( r \in -\log I \) are all equal to a single affine function. A spectral decomposition of \( M \) is a direct sum decomposition \( M = \bigoplus_i M_i \) in which each summand \( M_i \) is pure and the values \( f_i(M_i, r) \) are all distinct for each \( r \in -\log I \). If such a decomposition exists, it specializes to the spectral decomposition of \( M_\rho \) for all \( \rho \in I \); in particular, a spectral decomposition is unique if it exists.

**Lemma 3.7.3.** Consider the following conditions.

(a) The module \( M \) admits a spectral decomposition.
(b) For \( i = 1, \ldots, n \), the function \( f_i(M, r) \) is affine for \( r \in -\log I \).
(c) The functions \( F_n(M, r) \) and \( F_n(\text{End}(M), r) \) are affine for \( r \in -\log I \). (That is, \( M \) is clean in the sense of [Ked10a, Definition 12.8.2].)

Then (a) and (b) are equivalent, and (c) implies both of them.

**Proof.** It is clear that (a) implies (b), and [Ked10a, Theorem 12.8.3] shows that (c) implies (b), so it remains to check that (b) implies (a). Given (b), for \( i = 1, \ldots, n-1 \), if there exists \( r_0 \in -\log I \) for which \( f_i(M, r_0) = f_i(M, r_0) \), then we must have \( f_i(M, r) = f_{i+1}(M, r) \) identically; otherwise,
since \( f_i(M, r) \) and \( f_i(M, r) \) are both affine, the inequality \( f_i(M, r) \geq f_{i+1}(M, r) \) would have to be violated on one side of \( r_0 \). In other words, for \( i = 1, \ldots, n - 1 \), either \( f_i(M, r) = f_{i+1}(M, r) \) for \( r = -\log I \) or \( f_i(M, r) > f_{i+1}(M, r) \) for \( r = -\log I \). This allows us to apply Proposition 3.6.9 to obtain a spectral decomposition, yielding (a).

**Definition 3.7.4.** Suppose that \( M \) admits a spectral decomposition. By the *Robba component* of \( M \), we mean the summand \( M_1 \) in the spectral decomposition of \( M \) for which \( f_1(M_1, r) = r \) for each \( r \in -\log I \), or the zero submodule if no such summand exists.

**Lemma 3.7.5.** Suppose that \( M \) admits a spectral decomposition. Let \( M_1 \) be the Robba component of \( M \). Then the natural maps \( H^i(M_1) \to H^i(M) \) are bijections for \( i = 0, 1 \).

**Proof.** Let \( M_2 \) be the summand complementary to \( M_1 \) in the spectral decomposition of \( M \). It is clear that \( H^0((M_2)_\rho) = 0 \) for \( \rho \in I \), proving the desired bijectivity for \( i = 0 \). For \( i = 1 \), note that \( f_1(M_2, r) > r \) for each \( r \in -\log I \), so any extension \( 0 \to R_I \to N \to M_2 \to 0 \) splits by Lemma 3.7.3.

**Lemma 3.7.6.** Suppose that \( M \) admits a spectral decomposition. Let \( M_1 \) be the Robba component of \( M \). Assume either that \( p = 0 \) or that \( p > 0 \) and \( M_1 \) has \( p \)-adic non-Liouville exponents.

(a) Let \( M_2 \) be the maximal unipotent submodule of \( M_1 \). Then the natural maps \( H^i(M_2) \to H^i(M) \) are bijections for \( i = 0, 1 \).

(b) The composition \( H^0(M) \times H^1(M^\vee) \to H^1(R_I) \to K \) in which the first map is induced by the natural pairing \( M \times M^\vee \to R_I \) and the second map is the residue map is a perfect pairing of finite-dimensional \( K \)-vector spaces.

(c) For any open subinterval \( J \) of \( I \), the map
\[
H^i(M) \to H^i(M_J)
\]
is a bijection for \( i = 0, 1 \).

**Proof.** To prove (a), we may replace \( K \) by a finite extension \( K' \), since \( M \) may be viewed as a direct summand of \( M \otimes_K K' \). After a suitable such extension, by Theorem 3.3.6 we may decompose \( M_1 = M_2 \oplus M_3 \) in such a way that \( M_3 \) becomes a successive extension of copies of \( M_1 \) for various \( \lambda \in o_K \setminus \mathbb{Z} \). To see that \( H^i(M_3) = 0 \) for \( i = 0, 1 \), we may use the snake lemma to reduce to the case \( M = M_\lambda \) for some \( \lambda \in o_K \setminus \mathbb{Z} \). In this case, vanishing of \( H^0 \) follows from the nontriviality of \( M_\lambda \), while vanishing of \( H^1 \) follows from Theorem 3.3.6 applied to an extension \( 0 \to R_I \to N \to M_\lambda \to 0 \).

To prove (a) for \( p > 0 \), apply Corollary 3.4.23 to decompose \( M_3 = M_2 \oplus M_3 \) where \( M_3 \) has an exponent containing no integer or \( p \)-adic Liouville number. On one hand, \( H^0(M_3) = 0 \) because otherwise Remark 3.4.14 would force \( M \) to have an exponent containing 0. On the other hand, \( H^1(M_3) = 0 \) because we may split any extension \( 0 \to R_I \to N \to M_3 \to 0 \) using Remark 3.4.14 and Theorem 3.4.22.

To prove (b) and (c), we may use (a) to reduce to the case \( M = M_2 \). We may then use the snake lemma to reduce to the case \( M = R_I \), for which both claims are easily verified.

**Lemma 3.7.7.** Suppose that \( M \) admits a spectral decomposition. Assume either that \( p = 0 \) or that \( p > 0 \) and the Robba component of \( M \) has \( p \)-adic non-Liouville exponent differences. Then for any \( \rho \in I \), the map \( H^0(M) \to H^0(M_\rho) \) is a bijection.
Proof. For $p = 0$, this is immediate from Lemma 3.7.6(a). For $p > 0$, apply Corollary 3.4.24 to reduce to the case $M = M_\lambda$ for some $\lambda \in \mathbb{Z}_p$. The claim then holds because by [Ked10a, Proposition 9.5.3], $H^p(M_{\lambda,\rho}) = 0$ whenever $\lambda \notin \mathbb{Z}$. \hfill \Box

Definition 3.7.8. We say that $M$ is refined if $M$ is pure and, moreover, $f_1(M, r) > f_1(M^\vee \otimes M, r)$ for all $r \in -\log I$ (that is, $M$ is pure and $M_{\rho}$ is refined for all $\rho \in I$). If $M_1, M_2$ are refined, we say that they are equivalent if $f_1(M_1^\vee \otimes M_2, r) < f_1(M_1, r)$ or $f_1(M_2, r)$ for all $r \in -\log I$. Note that if $M_1$ and $M_2$ are inequivalent, then by convexity (Proposition 3.6.3(e)) we must have $f_1(M_1^\vee \otimes M_2, r) = \max\{f_1(M_1, r), f_1(M_2, r)\}$ for all $r \in -\log I$.

A refined decomposition of $M$ is a direct sum decomposition in which each summand $M_i$ is either refined or satisfies the Robba condition, at most one summand satisfies the Robba condition, and any two distinct refined summands $M_i, M_j$ are inequivalent. Such a decomposition specializes to a refined decomposition of $M_{\rho}$ for each $\rho \in I$, and hence is unique if it exists.

It is easiest to obtain refined decompositions using the following construction of test modules (compare [Xia12, Example 1.3.20]).

Definition 3.7.9. For any finite tamely ramified extension $K'$ of $K$, any $\lambda \in K'$, any positive integer $m$ not divisible by $p$, any positive integer $e$ which is a power of $p$ (which must be 1 if $p = 0$), and any integer $h$ coprime to $em$, let $N_{\lambda,h,e,m}$ be the differential module over $(R_t \otimes_{K[t]} K'[t^{1/m}], d/dt^{1/m})$ on the generators $v_1, \ldots, v_e$ given by

$$D(v_1) = t^{-1/m}v_2, \ldots, D(v_{e-1}) = t^{-1/m}v_e, D(v_e) = \lambda t^{-1/m+h/m}v_1.$$ 

Lemma 3.7.10. With notation as in Definition 3.7.9, for $\rho > 0$ we have

$$\min\{\omega, \rho - 1/e, -h/(em)\}.$$ 

Proof. This is immediate from Proposition 2.2.6. \hfill \Box

Lemma 3.7.11. Suppose that $M$ is pure and $f_1(M, r) > r - \log \omega$ for $r \in -\log I$. Then for any $\rho \in I$, there exist a finite tamely ramified extension $K'$ of $K$ and a positive integer $m$ not divisible by $p$ such that $M_{\rho} \otimes_{K[t]} K'[t^{1/m}]$ admits a refined decomposition in which for each summand $V$, there exist a scalar $\lambda \in K'$, a positive integer $e$ which is a power of $p$, and an integer $h$ coprime to $em$ such that $IR(M_{\rho}) = IR((N_{\lambda,h,e,m})_{\sigma})$ for $\sigma$ in some neighborhood of $\rho$ and $IR(V^\vee \otimes (N_{\lambda,h,e,m})_{\rho}) > IR(V)$. 

Proof. We imitate the proof of [Ked10a, Lemma 6.8.1]. Apply Corollary 2.1.6 to produce $v \in M$ which is a cyclic vector in $M \otimes_{R_t} \text{Frac}(R_t)$. Write $D^n(v) = a_0v + \cdots + a_{n-1}D^{n-1}(v)$ with $a_0, \ldots, a_{n-1} \in \text{Frac}(R_t)$. Factor the polynomial $P(T) = T^m - a_{n-1}T^{m-1} - \cdots - a_0$ over an algebraic closure of $\text{Frac}(R_t)$ within an algebraic closure of $F_{\rho}$. For each root $\alpha$, we can find $\lambda, h, e, m$ such that $|\alpha - m^{-1}1/\lambda\epsilon_{t^{1/m+h}/(em)}|_\rho < |\alpha|_\rho$; by Corollary 2.2.7, $IR(M_{\rho}) = IR((N_{\lambda,h,e,m})_{\sigma})$ for $\sigma$ in a neighborhood of $\rho$ and one of the intrinsic subsidiary radii of $M_{\rho} \otimes (N_{\lambda,h,e,m})_{\rho}$ is greater than $IR(M_{\rho})$. Apply Proposition 2.2.9 to construct a refined decomposition of $M_{\rho} \otimes_{F_{\rho}} E$ for some finite tamely ramified extension $E$ of $F_{\rho}$; then each summand is equivalent to $(N_{\lambda,h,e,m})_{\rho}$ for some $\lambda, h, e, m$, and in particular is stable under $\text{Gal}(E'/F')$ for $F' = F_{\rho} \otimes_{K[t]} K'[t^{1/m}]$ and $E'$ a compositum of $E$, $F'$, and $K(\mu_m)$. We thus obtain a refined decomposition of $M_{\rho} \otimes_{K(t)} K'[t^{1/m}]$ with the desired property. \hfill \Box

Theorem 3.7.12. Suppose that $M$ is pure. Then there exist a finite tamely ramified extension $K'$ of $K$ and a positive integer $m$ not divisible by $p$ such that $M \otimes_{K[t]} K'[t^{1/m}]$ admits a refined decomposition. 

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Proof. By virtue of the uniqueness of refined decompositions, we may work locally in a neighborhood of some $\rho \in (\alpha, \beta)$. Suppose first that $IR(M_\rho) < \omega$. To simplify notation, we may assume that the conclusion of Lemma 3.7.11 holds with $K' = K$ and $m = 1$, so that $M_\rho$ admits a refined decomposition. In addition, for each summand $V$ in the refined decomposition of $M_\rho$, we can find a differential module $N$ over $R_I$ such that $IR(V \otimes N_\rho) > IR(V)$. By continuity (Proposition 3.6.3(a)), for $\sigma$ in a neighborhood of $\rho$, $M_\rho^\sigma \otimes N_\sigma$ has an intrinsic subsidiary radius strictly greater than $IR(M_\rho) = IR(N_\rho)$. Apply Proposition 3.6.9 to $N^\sigma \otimes M$ to pull off a summand corresponding to the intrinsic subsidiary radii of $N_\rho^\sigma \otimes M_\rho$ less than $IR(V)$, then tensor with $N$ and project the decomposition from $N \otimes N^\sigma \otimes M$ to $M$. Repeating this process gives the desired decomposition.

Suppose next that $p > 0$ and $IR(M_\rho) = \omega$. Let $M'$ be the global Frobenius descendant of $M$ (Definition 3.5.2). By Proposition 2.3.5, $IR(\varphi_* M_\rho) = \omega^p$, so we may apply the previous paragraph to exhibit a finite tamely ramified extension $K'$ of $K$ and a positive integer $m$ not divisible by $p$ such that $M' \otimes_{K[p^m]} K'[p^m]$ admits a refined decomposition. To simplify notation, we may assume that $K' = K$ and $m = 1$, i.e., that $M'$ itself admits a refined decomposition. In particular, $\varphi_* M_\rho$ admits a refined decomposition. By Remark 2.3.10, if we group summands of $\varphi_* M$ into $\mathbb{Z}/p\mathbb{Z}$-orbits, the resulting decomposition descends to a decomposition specializing to a refined decomposition of $M$.

Suppose, finally, that $p > 0$ and $IR(M_\rho) > \omega$. Using Frobenius antecedents (Proposition 3.5.3), we may reduce to one of the previous cases. $\square$

Theorem 3.7.13. Suppose that either:

(a) $M$ is refined and $\text{rank}(M)$ is not divisible by $p$; or
(b) $p > 0$ and $M$ is of cyclic type.

Then the slopes of $f_1(M, r)$ are in $\mathbb{Z}$.

Proof. Using Proposition 3.6.3(a), $f_1(M, r)$ is piecewise affine. It thus suffices to compute its slope on a closed subinterval $J$ of $I$ on which $f_1(M, r)$ is affine. We may assume that this slope is not equal to zero or one, as otherwise there is nothing left to check.

Suppose first that we are in case (a) with $p = 0$. Choose a generator $\mathbf{v}$ of $\wedge^n M_J$, define $c \in R_J$ by the formula $D(\mathbf{v}) = cv$, and let $N$ be the differential module over $R_J$ on a single generator $\mathbf{v}$ given by $D(\mathbf{w}) = (c/n)\mathbf{v}$. We then have $N^{\wedge^n} \cong \wedge^n M$ and so $f_1(N^\sigma \otimes M, r) < f_1(M, r)$ for $r \in I$ by [Ked10a, Proposition 6.8.4]. In particular, in some range we have $f_1(M, r) = f_1(N, r)$, whereas $f_1(N, r)$ has integer slopes by Proposition 3.6.3(b). This proves the claim in this case.

Suppose next that we are in case (a) with $p > 0$. Since we assumed that the slope of $f_1(M, r)$ is neither zero nor one, we may shrink $J$ to ensure that $f_1(M, r) \neq r - p^{-j} \log \omega$ for all $r \in J$ and all nonnegative integers $j$. We may then use Frobenius antecedents (Proposition 3.5.3) to reduce to the case where $f_1(M, r) > r - \log \omega$ for all $r \in J$, and then argue as in (a).

Suppose finally that we are in case (b). We may again assume that $f_1(M, r) > r - \log \omega$ for all $r \in J$; we may also assume that $K$ is algebraically closed. Pick any $r_0 \in J$ and apply Lemma 3.7.11 to construct $\lambda, h, e, m$ for which $IR(M_\rho^\sigma \otimes (N_{\lambda, h, e, m})_\rho) > IR(M_\rho)$ for $\rho = e^{-r_0}$; by continuity (Proposition 3.6.3(a)), the same inequality holds for $\rho$ in a neighborhood of $e^{-r_0}$. For $\mu \in 1 + m_K$, apply Corollary 2.2.7(a) to $\mu^* N_{\lambda, h, e, m}^\sigma \otimes N_{\lambda, h, e, m}$; it implies that there exists $a > 0$ for which $f_1(\mu^* M^\sigma \otimes M, r) = f_1(M, r) + a \log |\mu - 1|$ for $|\mu - 1|$ sufficiently close to 1 and $r$ sufficiently close to $r_0$. By Lemma 3.2.6, there exists a rank 1 submodule $Q_\mu$ of $\mu^* M^\sigma \otimes M$. Since $\mu^* M^\sigma \otimes M$ is of cyclic type, we have $f_1(Q_\mu, r) = f_1(\mu^* M^\sigma \otimes M, r) = f_1(M, r) + a \log |\mu - 1|$. 

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for suitable \( \mu, r \). Since \( f_1(Q_\mu, r) \) has integer slopes by Proposition 3.6.3(b) again, so then does \( M \) in a neighborhood of \( r_0 \); this proves the claim in this case. \( \square \)

Remark 3.7.14. Theorem 3.7.13(b) is new to this paper. It was known previously that if \( p > 0 \), \( M \) is of cyclic type, and \( \text{End}(M) \) has \( p \)-adic non-Liouville exponent differences, then \( M \) is a successive extension of differential modules of rank 1 over \( R_I \); namely, this is an easy consequence of Corollary 3.4.24. That previous result figures in the proofs of the \( p \)-adic local monodromy theorem given by Andr\'e [And02] and Mebkhout [Meb02]; see Remark 3.8.26.

The following refinement of Lemma 3.7.11 will be used in the study of solvable modules in \( \S \) 3.8.

Lemma 3.7.15. Choose \( \gamma, \delta \) with \( \alpha < \gamma < \delta < \beta \). Suppose that \( p > 0 \), \( K \) is algebraically closed, \( M \) is refined, and there exists a nonnegative integer \( b \) such that \( IR(M_\rho) = (\alpha/\rho)^b < \omega \) for \( \rho \in [\gamma, \delta] \). Then there exists a differential module \( N \) over \( R_\gamma \) which is free of rank 1 with \( IR(N_\rho) = (\alpha/\rho)^b \) for \( \rho \in (\alpha, \delta) \) and \( IR((N^\vee \otimes M)_\rho) < IR(M_\rho) \) for \( \rho \in [\gamma, \delta] \).

Proof. We may rescale to reduce to the case \( \rho = \alpha = 1 \). Using Lemma 3.7.11, we may replace \( M \) with \( N_{\lambda, h, e, m} \); note that the fact that \( b \in \mathbb{Z} \) forces \( e = m = 1 \). After making the substitution \( t \mapsto t^{-1} \), we may perform the construction from the proof of [Ked10a, Theorem 12.7.2] to obtain the desired \( N \). \( \square \)

3.8 Solvable modules

We continue in the vein of [Ked10a], next treating differential modules over rings of convergent power series on an open annulus which are solvable at a boundary. This gives a uniform statement of the classical Turrittin–Levelt–Hukuhara decomposition as well as a strong \( p \)-adic analogue.

Note that for differential modules on an open annulus, one can equally well discuss solvability at the inner boundary or the outer boundary. In [Ked10a] and other literature, it is typical to consider outer boundaries because one has in mind the boundary of a residue disc. However, in this paper we will mostly need to consider inner boundaries (see \( \S \) 4.4), so we will set notation to address that case.

Hypothesis 3.8.1. Throughout this subsection, fix \( \alpha > 0 \) and put

\[
\mathcal{R}_\alpha = \bigcup_{\beta > \alpha} R_{(\alpha, \beta)},
\]

viewed as a differential ring for the derivation \( d = d/dt \). Let \( M \) be a differential module over \( \mathcal{R}_\alpha \) which is solvable at \( \alpha \) in the sense of Definition 3.8.3 below.

Convention 3.8.2. The functions \( f_i(M, r) \) and \( F_i(M, r) \) are not well defined for any particular \( r < -\log \alpha \); however, the germs of these functions in left neighborhoods of \( -\log \alpha \) may be interpreted unambiguously. We will use these germs frequently in what follows.

Definition 3.8.3. The module \( M \) is solvable at \( \alpha \) if

\[
\lim_{r \to (-\log \alpha)^-} f_1(M, r) = -\log \alpha.
\]

By Proposition 3.6.3 plus an extra argument (see [Ked10a, Lemma 12.6.2]), this implies that there exist nonnegative rational numbers \( b_1(M) \geq \cdots \geq b_n(M) \) such that at the level of germs, we have

\[
f_i(M, r) = r + b_i(M)(-\log \alpha - r) \quad (i = 1, \ldots, n). \tag{3.8.3.1}
\]

We call the \( b_i(M) \) the slopes of \( M \).
Definition 3.8.4. Suppose that $M \neq 0$. We say that $M$ satisfies the Robba condition if $b_1(M) = 0$. We say that $M$ is pure if $b_1(M) = \cdots = b_{\text{rank}(M)}(M)$. We say that $M$ is refined if $b_1(M) > b_1(\text{End}(M))$; this implies that $M$ is pure. We say that $M$ is of cyclic type if $b_1(\text{End}(M)) = 0$; this implies that $M$ either is refined or satisfies the Robba condition.

By (3.8.3.1) plus Lemma 3.7.3, $M$ admits a unique decomposition $\bigoplus \_j M_j$ into pure summands of distinct slopes; we call this the spectral decomposition of $M$. By the Robba component of $M$, we mean the summand of the spectral decomposition of slope 0, or the zero submodule of $M$ if no such summand exists.

Definition 3.8.5. Define a binary relation on irreducible solvable differential modules over $\mathcal{R}_\alpha$ by declaring that $M \sim N$ if at least one slope of $M^\vee \otimes N$ is nonzero. This relation is evidently reflexive and symmetric; it is also transitive by Lemma 3.8.6 below.

Lemma 3.8.6. With notation as in Definition 3.8.5, the relation $\sim$ is transitive.

Proof. Suppose $M \sim N$ and $N \sim P$. Let $S,T$ be the Robba components of $M^\vee \otimes N$, $N^\vee \otimes P$. Since $N$ is irreducible and $S,T \neq 0$, the images of the elements of $S \subseteq \text{Hom}_{\mathcal{R}_\alpha}(M,N)$ span $N$ and the kernels of the elements of $T \subseteq \text{Hom}_{\mathcal{R}_\alpha}(N,P)$ have zero intersection in $N$. It follows that the image of $S \otimes T$ under the contraction map $M^\vee \otimes N \otimes N^\vee \otimes P \to M^\vee \otimes P$ is nonzero; this image satisfies the Robba condition. This proves the claim.

Lemma 3.8.7. If $b_{\text{rank}(M)}(M) > 0$, then $H^1(M) = 0$.

Proof. Each element of $H^1(M)$ corresponds to an extension $0 \to M \to N \to \mathcal{R}_\alpha \to 0$ of differential modules, but any such extension is split by the spectral decomposition of $N$.

Proposition 3.8.8. There exists a unique direct sum decomposition $M = \bigoplus \_i M_i$ such that for any irreducible subquotients $P,Q$ of $M_i, M_j$, we have $P \sim Q$ in the sense of Definition 3.8.5 if and only if $i = j$.

Proof. It suffices to check that if $P,Q$ are inequivalent irreducible solvable differential modules over $\mathcal{R}_\alpha$, then $H^1(P^\vee \otimes Q) = 0$. But this is immediate from Lemma 3.8.7.

Lemma 3.8.9. Suppose that either:

(a) $p = 0$ and $M$ is refined; or
(b) $p > 0$, $K$ is algebraically closed, $M$ is refined, and $\dim(M)$ is not divisible by $p$; or
(c) $p > 0$, $K$ is algebraically closed, $M$ is of cyclic type, and $b_1(M) > 0$.

Then there exists a differential module $N$ over $\mathcal{R}_\alpha$ which is free of rank 1, is solvable at $\alpha$, and satisfies $b_1(N^\vee \otimes M) < b_1(M)$.

Proof. Realize $M$ as a refined differential module over $R_{(\alpha,\beta)}$ for some $\beta > \alpha$. By Theorem 3.7.13, $b_1(M)$ is a positive integer. We may thus imitate the proof of [Ked10a, Theorem 12.7.2] as follows.

In case (a), we may apply Lemma 3.7.11 to construct $N_{\lambda,h,e,m}$ with $\text{IR}((N_{\lambda,h,e,m}^\vee \otimes M)_\rho) < \text{IR}(M_\rho)$ for $\rho$ in some interval; because $b_1(M) \in \mathbb{Z}$, we are forced to take $e = m = 1$. By Lemma 3.7.10, $N_{\lambda,\beta,1,1}$ is solvable at $\alpha$. It remains to check that we may take $\lambda$ in $K$, not just in a finite extension of $K$; for this, we argue as in Proposition 2.2.11. Put $n = \text{rank}(M)$. Choose a generator $v$ of the restriction of $\wedge^n M$ to $R_I$ for some closed interval $I$, and write $D(v) = \alpha v$ with $\alpha \in R_I$. Let $M'$ be the differential module over $R_I$ on the single generator $w$.
with \( D(\mathbf{w}) = (a/n)\mathbf{w} \); then \((M')^{\otimes n}\) is isomorphic to the restriction of \(\wedge^n M\) to \(R_I\). It follows that 
\[
|a/n - \lambda^h|_\rho < |a/n|_\rho = |\lambda^h|_\rho \text{ for } \rho \in I, \text{ so there must exist } \lambda' \in K \text{ with } |\lambda - \lambda'| < |\lambda| = |\lambda'|.
\]
We may thus replace \(N_{\lambda,h,1,1}\) with \(N_{\lambda',h,1,1}\) without affecting the preceding arguments.

In cases (b) and (c), by taking global Frobenius antecedents (Proposition 3.5.3) as needed, we can ensure that there exist \(\gamma, \delta\) with \(\alpha < \gamma < \delta < \beta\) such that \(\text{IR}(M_\rho) > \omega\) for \(\rho \in [\gamma, \delta]\). By Lemma 3.7.15, we obtain the desired module \(N\).

**Corollary 3.8.10.** Suppose that either:

(a) \(p = 0\) and \(M\) is indecomposable and refined; or
(b) \(p = 0\) and \(M\) is of cyclic type; or
(c) \(p > 0\), \(K\) is algebraically closed, \(M\) is indecomposable and refined, and \(\dim(M)\) is not divisible by \(p\); or
(d) \(p > 0\), \(K\) is algebraically closed, \(M\) is of cyclic type, and \(b_1(M) > 0\).

Then there exists a factorization \(M \cong N \otimes P\) in which \(N\) is free of rank 1 and \(b_1(P) = 0\). In particular, \(M\) is of cyclic type.

**Proof.** This follows by repeated application of Lemma 3.8.9. Note that since \(b_1(M) \in \mathbb{Z}\) by Theorem 3.7.13, only finitely many iterations are needed before \(b_1(M)\) is reduced to zero. \(\square\)

When \(p = 0\), the structure of solvable modules is relatively simple.

**Theorem 3.8.11.** Assume that \(p = 0\). Then there exist a finite extension \(K'\) of \(K\) and a positive integer \(m\) such that \(M \otimes_{K[t]} K'[t^{1/m}]\) admits a direct sum decomposition in which each summand is of cyclic type.

**Proof.** This follows from Theorem 3.7.12 and Corollary 3.8.10. \(\square\)

**Remark 3.8.12.** By taking \(K = \mathbb{C}\) with the trivial norm, we may deduce from Theorem 3.8.11 the usual Turrittin–Levelt–Hukuhara decomposition theorem for differential modules over \(\mathbb{C}(t)\) [Ked10a, Theorem 7.5.1].

**Definition 3.8.13.** Put \(F = \text{Frac}(\mathcal{R}_\alpha)\). Let \([M]\) denote the Tannakian subcategory generated by \(M\) within the category of differential modules over \(\mathcal{R}_\alpha\), equipped with the fibre functor \(\omega\) taking each \(N \in [M]\) to the \(F\)-vector space \(N \otimes_{\mathcal{R}_\alpha} F\). Note that the objects of \([M]\) are all solvable at \(\alpha\).

Let \(G(M)\) be the automorphism group of \(\omega\). For \(r \geq 0\), let \(G^r(M)\) denote the subgroup of \(G(M)\) which acts trivially on \(\omega(N)\) for each nonzero \(N \in [M]\) for which \(b_1(N) < r\). Also put \(G^+(M) = \bigcup_{s \geq r} G^s(M)\).

**Remark 3.8.14.** As in Remark 2.3.19, we may use Theorem 3.8.11 to deduce that when \(p = 0\), the group \(G^{0+}(M)\) is a torus. The structure of \(G^{0+}(M)\) for \(p > 0\) will be clarified by Theorem 3.8.16 below; this will imply that for any \(p\) and any \(r \geq 0\), \(G^+(M)\) equals the subgroup of \(G(M)\) which acts trivially on \(\omega(N)\) for each nonzero \(N \in [M]\) for which \(b_1(N) \leq r\).

**Lemma 3.8.15.** If \(p > 0\) and \(M\) is of cyclic type, then there exists a nonnegative integer \(h\) such that \(b_1(M^{\otimes p}) = 0\).

**Proof.** If \(b_1(M) > 0\), then by Proposition 2.3.13, we have \(b_1(M^{\otimes p}) < b_1(M)\). Since \(b_1(M)\) and \(b_1(M^{\otimes p})\) are nonnegative integers by Theorem 3.7.13, this proves the claim. \(\square\)
Theorem 3.8.16. If \( p > 0 \), then \( G^{0+}(M) \) is a finite \( p \)-group.

Proof. This follows from Proposition 1.1.2 using Remark 1.1.3 as follows. Replace the category of differential modules over \( \mathcal{R}_\alpha \) with the direct limit of the categories of differential modules over \( \mathcal{R}_\alpha \otimes_{K[t]} K'[t^{1/m}] \) over all finite extensions \( K' \) of \( K \) and all positive integers \( m \) not divisible by \( p \); this does not change the groups \( G^r(M) \) except for a base extension. We may then deduce conditions (i), (ii), (iii) of Remark 1.1.3 using Theorem 3.7.12, Proposition 2.3.13, Lemma 3.8.15, respectively.

Corollary 3.8.17. There exist a finite extension \( K' \) of \( K \) and a positive integer \( m \) such that for all nonnegative integers \( g, h \), \((M^\forall)^{\otimes g} \otimes M^{\otimes h} \otimes_{K[t]} K'[t^{1/m}]\) admits a refined decomposition.

Proof. This is apparent from Theorem 3.8.11 if \( p = 0 \). If \( p > 0 \), for each pair \((g, h)\) we may choose a suitable \( m \) by Theorem 3.7.12, so we need only check that \( m \) may be chosen uniformly. But this follows from Theorem 3.8.16: it is enough to list each of the finitely many isomorphism classes of irreducible representations \( \tau \) of \( G^{0+}(M) \) and, for each \( \tau \), ensure that \( m \) works for one pair \( g, h \) such that \( \tau \) appears in \((M^\forall)^{\otimes g} \otimes M^{\otimes h}\).

Corollary 3.8.18. If \( p > 0 \) and \( b_1(M) > 0 \), then there exist a finite extension \( K' \) of \( K \), a positive integer \( m \), and an object \( N \in [M \otimes_{K[t]} K'[t^{1/m}]] \) of cyclic type such that \( b_1(N) > 0 \) but \( b_1(N^\otimes p) = 0 \).

Proof. This follows from Remark 1.1.3 plus the proof of Theorem 3.8.16 (in which it is shown that the conditions of Remark 1.1.3 are satisfied).

Lemma 3.8.19. Suppose that \( p > 0 \), \( K \) contains a primitive \( p \)th root of unity, \( M \) is free of rank 1, and \( b_1(M^\otimes p) = 0 \). Then there exists another differential module \( N \) over \( \mathcal{R}_\alpha \) which is solvable on \( \alpha \), is free on a single generator \( \mathbf{v} \) such that \( D(\mathbf{v}) = P(t) \) for some \( P(t) \in K[t] \) with \( |P(t)|_\alpha = \omega \), and satisfies \( b_1(N^\otimes \otimes M) = 0 \).

Proof. This follows from [Ked10a, Theorem 17.1.6, Remark 17.1.7].

Definition 3.8.20. Let \( \mathcal{R}_\alpha^{bid} \) be the subring of \( \mathcal{R}_\alpha \) consisting of germs of bounded analytic functions. This ring is henselian but not complete for the \( \alpha \)-Gauss norm; let \( \mathcal{R}_\alpha^{int} \) denote the valuation subring.

If \( S \) is a connected finite étale cover, it makes sense to impose the Robba condition on \( M \otimes_{\mathcal{R}_\alpha^{int}} S \) provided that \( S \) can be identified with a ring of the form \( \mathcal{R}_\alpha^{int} \) in a suitable power series coordinate; the resulting condition will not depend on the choice of this identification. Such an identification can always be made if \( \kappa_K \) is algebraically closed.

Theorem 3.8.21. If \( p > 0 \), then there exists a connected finite étale cover \( S \) of \( \mathcal{R}_\alpha^{int} \) such that \( MS = M \otimes_{\mathcal{R}_\alpha^{int}} S \) satisfies the Robba condition in the sense of Definition 3.8.20.

Proof. Since \( G^{0+}(M) \) is finite by Theorem 3.8.16 and is trivial if and only if \( M \) satisfies the Robba condition, it suffices to produce a cover that decreases \( G^{0+}(M) \). This may be achieved as follows. We may assume from the outset that \( K \) contains an element \( \pi \) with \( \pi^{p-1} = -p \); this also forces \( K \) to contain a primitive \( p \)th root of unity. Pick out an object \( N \in [M \otimes_{K[t]} K'[t^{1/m}]] \) for some \( K', m \) as in Corollary 3.8.18. Apply Corollary 3.8.10 to produce a free rank 1 object \( N' \in [M \otimes_{K[t]} K'[t^{1/m}]] \) for some \( K', m \) such that \( N^\forall \otimes N' \) satisfies the Robba condition. By Lemma 3.8.19, we may choose \( N' \) to be free on one generator \( \mathbf{v} \) satisfying \( D(\mathbf{v}) = P(t) \) for
some \( P \in K[t] \) with \( |P(t)|_\alpha = \omega \). We may then trivialize \( N' \) by extending scalars from \( \mathcal{R}_\alpha^\int \) to \( \mathcal{R}_\alpha^\int[z]/(z^p - z - \pi^{-1}P(t)) \) and recalling that the power series \( \exp(\pi(z^p - z)) \) in \( z \) has radius of convergence strictly greater than 1 (see, for example, [Ked10a, Example 9.9.3]). \( \square \)

**Corollary 3.8.22.** Assume that \( p > 0 \), \( \kappa_K \) is algebraically closed, and \( \alpha = 1 \).

(a) There is a unique minimal choice of \( S \) satisfying the conclusion of Theorem 3.8.21.

(b) The residue field of \( S \) is a finite Galois extension of \( \kappa_K((t)) \) whose highest ramification break is equal to \( b_1(M) \).

**Proof.** This follows from Theorem 3.8.21 as in the proof of [Ked05, Theorem 5.23] (see also [Ked10a, Theorem 19.4.1]). \( \square \)

**Corollary 3.8.23.** Assume that \( p > 0 \), and decompose \( M = \bigoplus M_i \) as in Proposition 3.8.8. Then for each \( i \), there exists an isomorphism \( M_i \cong N \otimes P \) for some solvable differential modules \( N, P \) over \( \mathcal{R}_\alpha^\int \) such that \( N \) is irreducible, \( N_S \) is trivial for some connected finite étale cover \( S \) of \( \mathcal{R}_\alpha^\int \), and \( P \) satisfies the Robba condition.

**Proof.** Let \( Q \) be an irreducible subquotient of \( M_i \). By Theorem 3.8.21, we may choose \( S \) so that \( M_i \otimes \mathcal{R}_\alpha^\int S \) satisfies the Robba condition, as then does \( Q \otimes \mathcal{R}_\alpha^\int S \). Let \( T \) be the restriction of scalars of \( \mathcal{R} \otimes \mathcal{R}_\alpha^\int S \) to \( \mathcal{R} \), viewed as a solvable differential module; then \( Q \otimes T \) has a nontrivial Robba component, so \( Q \) is equivalent to some irreducible subquotient \( N \) of \( T \). Let \( P \) be the Robba component of \( N^\vee \otimes M_i \); by construction, there is a natural map \( N \otimes P \to M_i \) factoring through the contraction \( N^\vee \otimes N \otimes M_i \to M_i \). We may check that this map is an isomorphism by induction on the length of a Jordan–Hölder filtration of \( M_i \). \( \square \)

**Corollary 3.8.24.** Suppose that \( p > 0 \) and that the Robba component of \( \text{End}(M) \) has \( p \)-adic non-Liouville exponents. Then for \( S \) as in Theorem 3.8.21, \( M_S \) splits as a direct sum, each summand of which is a successive extension of copies of \( M_\lambda \) for some \( \lambda \in \mathbb{Z}_p \).

**Proof.** We may assume that \( M \) is indecomposable. By Corollary 3.8.23, we may write \( M \cong N \otimes P \) where \( N \) is irreducible, \( N_S \) is trivial, and \( P \) satisfies the Robba condition. We then have \( \text{End}(M) \cong \text{End}(N) \otimes \text{End}(P) \) and hence \( \text{End}(M_S) \cong \text{End}(N_S) \otimes \text{End}(P_S) \). Choose an exponent \( A \) of \( P \); then \( A - A \) is an exponent of \( \text{End}(P) \), and the multiset obtained from \( A - A \) by multiplying each multiplicity by \( \text{rank}(N)^2 \) is an exponent of \( \text{End}(M_S) \). On the other hand, since \( \text{End}(N) \) contains a nontrivial Robba component (namely the trace component), \( \text{End}(P) \) is isomorphic to a submodule of the Robba component of \( \text{End}(M) \). Therefore \( \text{End}(P) \) has \( p \)-adic non-Liouville exponents, as then does \( \text{End}(M_S) \). By Corollary 3.4.24, \( M \) has the desired form. \( \square \)

**Remark 3.8.25.** In Corollary 3.8.24, it is not true in general that the differences between the different values of \( \lambda \) are \( p \)-adic Liouville numbers. That is because if \( M \) splits nontrivially as in Proposition 3.8.8, then \( M_i^\vee \otimes M_j \) has no Robba component and thus imposes no restriction on the exponents of \( (M_i^\vee \otimes M_j)_S \). For instance, choose inequivalent irreducible solvable differential modules \( N_i, N_j \) over \( \mathcal{R}_\alpha \) with \( N_i,S, N_j,S \) trivial, and choose \( \lambda, \mu \in \mathbb{Z}_p \) which differ by a \( p \)-adic Liouville number. Then

\[
M = (N_i \otimes M_\lambda) \oplus (N_j \otimes M_\mu)
\]

satisfies the hypothesis of Corollary 3.8.24 but \( \text{End}(M) \) admits an exponent containing \( \lambda - \mu \).
Remark 3.8.26. Theorem 3.8.21 includes a result variously known as the $p$-adic Turrittin theorem (the implicit analogy being perhaps clearest from Corollary 3.8.23) and the $p$-adic local monodromy theorem. That result, due to André [And02], Mebkhout [Meb02], and the author [Ked04], assumes the existence of a Frobenius structure on $M$ (see [Ked10a, ch. 17]); in addition, $K$ must be discretely valued and $p$ must equal one.

The methods of André and Mebkhout can be used to derive Theorem 3.8.21 also in the case where all of the objects in $\mathcal{M}$ have $p$-adic non-Liouville exponent differences. In these arguments, the non-Liouville condition is needed to ensure that irreducible objects satisfying the Robba condition are all of rank 1. The proof of Theorem 3.8.21 provides a workaround in cases where advance information about exponents is not available.

4. Berkovich discs

We are at last ready to shift language and perspective toward Berkovich’s nonarchimedean analytic spaces. In this section, we introduce the topological spaces which play the role of discs in Berkovich’s theory, and consider radii of convergence of local horizontal sections of differential analytic spaces. In this section, we introduce the topological spaces which play the role of discs in Berkovich’s theory, and consider radii of convergence of local horizontal sections of differential modules on such spaces. This draws heavily on the results of §3, but some additional maneuvering is needed. In addition, the behavior of differential modules around points of type 4 requires some extra work.

4.1 Underlying topological spaces

We begin by defining the Gel’fand spectrum of a Banach ring. For now, we just consider the resulting topological space; we postpone discussion of the analytic space structure to §5.

Definition 4.1.1. For $R$ a ring equipped with a submultiplicative norm (e.g., a commutative Banach algebra over $K$), the Gel’fand spectrum $\mathcal{M}(R)$ is defined as the set of bounded (by the given norm) multiplicative seminorms on $R$, topologized as a subset of the product $\mathbb{R}^R$. Note that $\mathcal{M}(R)$ may also be viewed as a closed subset of a product of bounded closed intervals, and hence is compact; it is also nonempty provided that $R \neq 0$ [Ber90, Theorem 1.2.1]. For $x \in \mathcal{M}(R)$, let $\mathcal{H}(x)$ denote the completion of $\text{Frac}(R/\ker(x))$ for the multiplicative norm induced by $x$.

Remark 4.1.2. Any bounded homomorphism $R \to S$ of commutative Banach algebras over $K$ defines a continuous restriction map $\mathcal{M}(S) \to \mathcal{M}(R)$. If this map is surjective, then it is a quotient map because the source and target are compact: the induced map from the quotient space is a continuous bijection from a quasicompact space to a Hausdorff space, hence a bijection [Bou71, §9, No. 4, Corollaire 2].

For example, suppose that $R$ is a commutative Banach algebra over $K$ and that $K'$ is a complete field extension of $K$. Then the completed tensor product $R' = R \widehat{\otimes}_K K'$ is a Banach algebra over $K'$ and the restriction map $\mathcal{M}(R') \to \mathcal{M}(R)$ is always surjective [Ked13, Lemma 1.20].

In the previous paragraph, if $K'$ is the completion of an algebraic Galois extension of $K$ (such as $\mathbb{C}$), we can say more: not only is the restriction map $\mathcal{M}(R') \to \mathcal{M}(R)$ surjective, but the group of continuous automorphisms of $K'$ over $K$ acts transitively on the fibres of the restriction map. See [Ber90, Corollary 1.3.6].

Definition 4.1.3. Let $R$ be a commutative Banach algebra over $K$, and put $R' = R \widehat{\otimes}_K \mathbb{C}$. For $x \in \mathcal{M}(R)$, choose any lift $\tilde{x} \in \mathcal{M}(R')$ of $x$, and define the signature of $x$ as the triple

$$(\dim(\ker(\tilde{x})), \text{rank}(|\mathcal{H}(x)^x/|K^x|), \text{trdeg}(\kappa_{\mathcal{H}(x)}/\kappa_K)).$$

Note that one can have $\dim(\ker(\tilde{x})) > \dim(\ker(x))$. 

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4.2 Discs
We now specialize the previous discussion to rings of convergent power series on discs. Due to the increasing prevalence of such rings and their associated Gel'fand spectra in various branches of mathematics, numerous expositions of this material can be found in the literature; among these, perhaps the most comprehensive is the book of Baker and Rumely [BR10, ch. 1]. However, that treatment assumes that the ground field $K$ is algebraically closed, which we prefer not to do here; to avoid imposing this condition, we refer also to [Ked13, §2].

**Definition 4.2.1.** For $\beta > 0$, the space $M(R_{[0,\beta]}')$ is called the Berkovich closed disc of radius $\beta$ with coordinate $t$ over $K$, and also denoted $D_{\beta,K}$. For $z \in \mathbb{C}$ with $|z| \leq \beta$ and $\rho \in [0,\beta]$, the restriction to $R_t \cong K(t/\beta)$ of the $\rho$-Gauss norm on $\mathbb{C}(\langle(t-z)/\beta\rangle)$ defines a point $\zeta_{z,\rho} \in D_{\beta,K}$; the point $\zeta_{0,\beta}$ is called the Gauss point of $D_{\beta,K}$. For $\beta' > \beta$, the natural map $R_{[0,\beta']} \to R_{[0,\beta]}$ induces an inclusion $D_{\beta,K} \to D_{\beta',K}$; the direct limit of the $D_{\beta,K}$ along these maps is called the Berkovich affine line over $K$.

**Lemma 4.2.2.** The restriction map $D_{\beta,C} \to D_{\beta,K}$ identifies $D_{\beta,K}$ with the quotient of $D_{\beta,C}$ by the action of the group of continuous automorphisms of $\mathbb{C}$ over $K$.

**Proof.** See [Ber90, Proposition 1.3.5].

**Proposition 4.2.3.** For $\beta > 0$, $x \in D_{\beta,K}$ and $\rho \in [0,\beta]$, define

$$H(x,\rho)(f) = \max\left\{\rho^i x \left(\frac{1}{i!} \frac{d^i}{dt^i}(f)\right) : i = 0,1,\ldots\right\}$$

(4.2.3.1)

with the interpretation that $\rho^0 = 1$ even for $\rho = 0$.

(a) Formula (4.2.3.1) defines a continuous map

$$H : D_{\beta,K} \times [0,\beta] \to D_{\beta,K}.$$

(b) For $x \in D_{\beta,K}$, $H(x,0) = x$ and $H(x,\beta) = \zeta_{0,\beta}$.

(c) For $x \in D_{\beta,K}$ and $\rho,\sigma \in [0,\beta]$, $H(H(x,\rho),\sigma) = H(x,\rho,\sigma)$. For $z \in \mathbb{C}$ with $|z| \leq \beta$ and $\rho \in [0,\beta]$, $H(\zeta_{z,\rho},\rho) = \zeta_{z,\rho}$.

(e) For $x,y \in D_{\beta,K}$, $y$ dominates $x$ (that is, $y(f) \geq x(f)$ for all $f \in R_t$) if and only if $y = H(x,\rho)$ for some $\rho \in [0,\beta]$.

**Proof.** See [Ber90, Remark 6.1.3(ii)] or [Ked13, Lemma 2.3] for (a)–(d) and [Ked13, Theorem 2.11] for (e).

**Definition 4.2.4.** For $\beta > 0$ and $x \in D_{\beta,K}$, define the diameter of $x$, denoted $\rho(x)$, to be the maximum $\rho \in [0,\beta]$ for which $H(x,\rho) = x$. Beware that the diameter is stable under base extension from $K$ to $\mathbb{C}$ (see Proposition 4.2.7), but not under general base extensions (see Remark 4.2.5). It is also stable under increasing $\beta$.

**Remark 4.2.5.** For $\beta > 0$ and $x \in D_{\beta,K}$, let $t_x \in \mathcal{H}(x)$ be the image of $t$ under the natural map $R_{[0,\beta]} \to \mathcal{H}(x)$. We may then realize $x$ as the restriction of the seminorm $\zeta_{t_x,0} \in M(R_t,\mathcal{H}(x))$ of radius zero.

At the other extreme, we have the following.

**Lemma 4.2.6.** For $\beta > 0$, $x \in D_{\beta,K}$, and $K'$ an analytic field containing $K$, there exists $y \in D_{\beta,K'}$ lifting $x$ with $\rho(y) = \rho(x)$.
**Definition 4.2.9** For some nonempty finite subset \( \{x_1, \ldots, x_m\} \subseteq \mathbb{D}_{\beta,K} \); we sometimes say that this skeleton is generated by \( x_1, \ldots, x_m \). A strict rooted skeleton is a rooted skeleton generated by a set of points of type 2.

For any rooted skeleton \( S \) of \( \mathbb{D}_{\beta,K} \), define the map \( \pi_S : \mathbb{D}_{\beta,K} \to S \) taking each \( x \in \mathbb{D}_{\beta,K} \) to \( H(x, \rho) \) for \( \rho \) the least value in \([0, \beta]\) for which \( H(x, \rho) \in S \). By Proposition 4.2.3, \( \pi_S \) is a deformation retract.
Proposition 4.2.10. Form the inverse system consisting of the rooted skeleta of $\mathbb{D}_{\beta,K}$ with morphisms given as follows: for every pair of rooted skeleta $S, S'$ with $S \subseteq S'$, include a morphism $S' \to S$ given by the restriction of $\pi_S$. Define a map from $\mathbb{D}_{\beta,K}$ to this inverse system whose projection onto $S$ is given by $\pi_S$. Then this map is a homeomorphism of topological spaces.

Proof. The map is injective because every pair of points can be found in some rooted skeleton. The map is surjective because $\mathbb{D}_{\beta,K}$ is compact and surjects onto each rooted skeleton. The map is a homeomorphism because any continuous bijection from a quasicompact space to a Hausdorff space is a homeomorphism. (See also [BR10, Proposition 1.13] for an alternate treatment in the case where $K$ is algebraically closed and $\beta = 1$.)

Remark 4.2.11. Proposition 4.2.10 is a special case of the general phenomenon that Berkovich analytic spaces can be described as inverse limits of tropical spaces (see, for example, [Pay09]). For Berkovich curves, this inverse limit presentation is also closely related to semistable models; we will return to this point in §5.

Definition 4.2.12. For $x \in \mathbb{D}_{\beta,K}$, a branch of $\mathbb{D}_{\beta,K}$ at $x$ is a path-connected component of $\mathbb{D}_{\beta,K}\setminus\{x\}$. If $x$ is not the Gauss point, then there is a branch containing the Gauss point, called the upper branch of $\mathbb{D}_{\beta,K}$ at $x$. By Proposition 4.2.10, additional branches (called lower branches) exist according to the type of $x$ as follows:

(i) no lower branches;
(ii) infinitely many lower branches;
(iii) exactly one lower branch;
(iv) no lower branches.

For $S$ a rooted skeleton of $\mathbb{D}_{\beta,K}$ and $x \in S$, a branch of $S$ at $x$ is a branch of $X$ at $x$ meeting $S$. There are only finitely many such branches at any $x$.

Definition 4.2.13. Let $S$ be a rooted skeleton of $\mathbb{D}_{\beta,K}$. By a subdivision of $S$, we will mean a graph (in the combinatorial sense) with underlying topological space $S$.

We equip $S$ with the piecewise linear structure characterized as follows: a function $f : S \to \mathbb{R}$ is piecewise affine (with integral slopes) if and only if for each $x \in S$, the function $r \mapsto f(H(x, e^{-r}))$ is piecewise affine (with integral slopes) and constant for $r$ sufficiently large. Then for any piecewise affine function $f : S \to \mathbb{R}$, there exists a subdivision of $S$ such the restriction of $f$ to any edge of the subdivision is affine. We call such a subdivision a controlling graph of $f$.

It is meaningful to refer to the slope of a piecewise affine function $f : S \to \mathbb{R}$ along a branch of $S$ at a point $x$. Explicitly, the slope along the upper branch is the left slope of $r \mapsto f(H(x, e^{-r}))$ at $r_0 = -\log \rho(x)$ (or 0 in case $\rho(x) = 0$), while the slope along the lower branch containing $y \in S$ is the right slope of $r \mapsto f(H(y, e^{-r}))$ at $r_0$.

Definition 4.2.14. By the Berkovich open unit disc of radius $\beta$ over $K$, denoted $\mathbb{D}_{\beta,K}^\circ$, we will mean the branch of $\mathbb{D}_{\beta,K}$ at the Gauss point containing $\zeta_{0,0}$.

4.3 Radii of convergence

We now define the radii of optimal convergence for differential modules on discs, following Baldassarri [Bal10].

Hypothesis 4.3.1. Throughout this subsection, fix $\beta > 0$ and, except for within Definition 4.3.11 and Lemma 4.3.12, let $M$ be a differential module of rank $n \geq 0$ over $R_{[0,\beta]}$.
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Definition 4.3.2. For \( x \in \mathbb{D}_{\beta,K} \), put
\[
M_{x,0} = M \otimes_{R_{[0,\beta]}} \mathcal{H}(x)[t - t_x],
M_{x,\rho} = M \otimes_{R_{[0,\beta]}} \mathcal{H}(x)[(t - t_x)/\rho] \quad (\rho \in (0, \beta));
\]
these can be viewed as differential modules as well. By a standard argument (see, for instance, [Ked10a, Theorem 7.2.1]), the natural map
\[
M_{x,0}^{D=0} \otimes_{\mathcal{H}(x)} \mathcal{H}(x)[t - t_x] \to M_{x,0}
\]
is an isomorphism. Define the sequence \( s_i(M, x) \) of radii of optimal convergence of \( M \) at \( x \) as follows: for \( i = 1, \ldots, n \), put
\[
s_i(M, x) = \sup \{ \rho \in [0, \beta) : \dim_{\mathcal{H}(x)}(M_{x,0}^{D=0} \cap M_{x,\rho}) \geq n - i + 1 \}.
\]
In other words, \( s_i(M, x) \) is the radius of the maximal open disc around \( t_x \) on which there exist \( n - i + 1 \) linearly independent horizontal sections of \( M \). For \( M \neq 0 \), we refer to \( s_1(M, x) \) also as the radius of convergence of \( M \) at \( x \).

Lemma 4.3.3. Let \( K' \) be an analytic field containing \( K \), and suppose that \( y \in \mathbb{D}_{\beta,K'} \) restricts to \( x \in \mathbb{D}_{\beta,K} \). Then
\[
s_i(M, x) = s_i(M \otimes_{R_{[0,\beta],K}} R_{[0,\beta],K'}, y) \quad (i = 1, \ldots, n).
\]

Proof. By replacing \( K \) with \( \mathcal{H}(x) \), we may reduce to the case \( x = \zeta_{0,0} \). The claim then comes down to the fact that formation of the kernel of the bounded \( K \)-linear endomorphism of the Banach space \( M \otimes_{R_{[0,\beta]}} R_{[0,\beta]} \) commutes with formation of the completed tensor product over \( K \) with \( K' \). This in turn reduces formally to the case where \( K' \) is the completion of a countably generated field extension of \( K \), in which case the claim is clear because \( K' \) admits a Schauder basis over \( K \) (see [BGR84, Proposition 2.7.2/3] or [Ked10a, Lemma 1.3.8]).

Remark 4.3.4. The intuition behind Definition 4.3.2 is that the elements of \( M_{x,0}^{D=0} \) are the formal horizontal sections of \( M \) centered at \( x \). In the language of [Ked10a] and preceding literature on \( p \)-adic differential equations, one would think of \( x \) as the generic point of a certain subdisc of \( \mathbb{D}_{\beta,K} \).

Following this intuition, one observes that for \( y = H(x, \sigma) \) for some \( \sigma > \rho(x) \), the discs of radius \( \rho \) centered at \( x \) and \( y \) coincide for all \( \rho \in (\sigma, \beta) \). Formally, for any field \( L \) containing both \( \mathcal{H}(x) \) and \( \mathcal{H}(y) \), we obtain a natural isomorphism \( L((t - t_x)/\rho) \cong L((t - t_y)/\rho) \). One consequence is that for \( i \in \{1, \ldots, n\} \), if \( s_i(M, x) > \rho(x) \), then \( s_i(M, x) = s_i(M, H(x, \rho)) \) for all \( \rho < s_i(M, x) \).

The relationship between radii of optimal convergence and intrinsic subsidiary radii (due in its original form to Young) is the following.

Definition 4.3.5. For \( x \in \mathbb{D}_{\beta,K} \) not of type 1, let \( F_x \) be a copy of \( \mathcal{H}(x) \) viewed as a differential field for the derivation \( d/dt \).

Lemma 4.3.6. For any \( x \in \mathbb{D}_{\beta,K} \) not of type 1, any analytic field \( K' \) containing \( K \), and any \( y \in \mathbb{D}_{\beta,K'} \) lifting \( x \) with \( \rho(y) = \rho(x) \) (which exists by Lemma 4.2.6), the spectral norms of \( d/dt \) on \( F_x \) and \( F_y \) coincide.

Proof. Since \( F_x \subseteq F_y \), the spectral norm of \( d/dt \) on \( F_x \) is no greater than that on \( F_y \). To prove the reverse inequality, by Lemma 4.2.6 we are free to enlarge \( K' \). We may thus reduce to the
cases where $K = \mathbb{C}$ and where $K' = \mathbb{C}$. In the former case, we have $F_y = F_x \otimes_K K'$ with the tensor product norm (see the proof of Lemma 4.2.6), so the desired inequality is clear.

To treat the latter case, it is sufficient to instead consider the case where $K'$ is a finite extension of $K$. In this case, $F_y$ is a direct summand of $F_x \otimes_K K'$, so the desired inequality is again clear. \hfill \Box

**Proposition 4.3.7.** For $x \in \mathbb{D}_{\beta,K}$ not of type 1, the intrinsic subsidiary radii of $M \otimes_{R[0,\beta]} F_x$ are given by

$$\min\{1, s_i(M, x)/\rho(x)\} \quad (i = 1, \ldots, n).$$

*Proof.* By Lemmas 4.3.3 and 4.3.6, we are free to lift $x$ as long as we do not change its diameter. This lifting being possible by Lemma 4.2.6, we may reduce to the case $x = \zeta_{0,\rho}$ for some $\rho \in (0,\beta]$. In this case, the claim follows from [Ked10a, Theorem 11.9.2]. \hfill \Box

One can also interpret Dwork’s transfer theorem in this language.

**Proposition 4.3.8.** For $M$ nonzero, for all $x \in \mathbb{D}_{\beta,K}$ and $\rho \in [0,\beta]$,  

$$s_1(M, H(x, \rho)) \leq s_1(M, x).$$

*Proof.* Using Lemma 4.3.3, we may reduce to the case where $x = \zeta_{0,\rho}$, in which case the claim asserts that $s_1(M, \zeta_{0,\rho}) \leq s_1(M, \zeta_{0,0})$ for any $\rho \in [0,\beta]$. If $s_1(M, \zeta_{0,\rho}) > \rho$, then this follows from Remark 4.3.4. If $s_1(M, \zeta_{0,\rho}) \leq \rho$, then by Proposition 4.3.7, $\rho^{-1}s_1(M, \zeta_{0,\rho})$ equals the intrinsic radius of $M \otimes_{R[0,\beta]} F_{\rho}$, so we may apply [Ked10a, Theorem 9.6.1] to conclude. \hfill \Box

**Remark 4.3.9.** The radius of convergence of $M$ at any $x \in \mathbb{D}_{\beta,K}$ is always positive. This can be deduced either from Proposition 4.3.8 or from Clark’s $p$-adic Fuchs theorem [Ked10a, Theorem 13.2.3]; the latter also covers the case of a regular singularity with $p$-adic non-Liouville exponent differences.

**Remark 4.3.10.** For $M$ nonzero, the properties of the intrinsic radius described in Definition 2.2.2 carry over to the radius of convergence, as follows:

(a) we have $s_1(M', x) = s_1(M, x)$;

(b) for any short exact sequence $0 \to M_1 \to M \to M_2 \to 0$,  

$$s_1(M, x) = \min\{s_1(M_1, x), s_1(M_2, x)\};$$

(c) for any $M_1, M_2$,  

$$s_1(M_1 \otimes M_2, x) \geq \min\{s_1(M_1, x), s_1(M_2, x)\},$$

with equality if $s_1(M_1, x) \neq s_1(M_2, x)$.

However, unlike for intrinsic subsidiary radii, these properties do not propagate to radii of optimal convergence despite the validity of Proposition 4.3.7. The difficulty already appears in (a): the existence of a horizontal section of $M$ on a large open disc does not imply the same for $M'$. A similar difficulty arises for (b) unless we restrict consideration to split exact sequences. No such difficulty arises for (c).

So far we have considered only radii of convergence on closed discs, but one can make similar definitions for open discs.

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Definition 4.3.11. For $M$ a differential module over $R_{(0,\beta)}$, we may similarly define $s_i(M, x)$ for $x \in \mathbb{D}_{\beta, K}^\circ$. Then Proposition 4.3.8 implies that for $M$ nonzero, for all $x \in \mathbb{D}_{\beta, K}^\circ$,

$$\limsup_{\rho \to \beta^-} s_1(M, H(x, \rho)) \leq s_1(M, x).$$

For open discs, we have the following key lemma.

Lemma 4.3.12. Let $M$ be a differential module over $R_{(0,\beta)}$. Suppose that for some $\gamma \in (0, \beta]$, there exists $m \in \{0, \ldots, n\}$ satisfying the following conditions.

(a) For $i = 1, \ldots, m$, $s_i(M, \zeta_{0, \rho})$ is constant and less than $\gamma$ for $\rho$ in some punctured left neighborhood of $\gamma$.

(b) For $i = m + 1, \ldots, n$, $\limsup_{\rho \to \gamma^-} s_i(M, \zeta_{0, \rho}) \geq \gamma$.

Then the restrictions of the functions $s_i(M, \cdot)$ to $\mathbb{D}_{\gamma, K}^\circ$ are constant for $i = 1, \ldots, n$.

Proof. Using Proposition 4.3.7 to see that the appropriate hypotheses are satisfied, we may decompose $M \otimes_{R_{(0,\beta)}} R_{(0,\gamma)} = M_0 \oplus M_1 \oplus \cdots$ as per Corollary 3.6.8.

Consider any $k > 0$. By Corollary 3.6.8 and Proposition 4.3.7, for $i \in \{1, \ldots, \text{rank}(M_k)\}$, for all $y \in \mathbb{D}_{\gamma, K}^\circ$ we have $\min\{\rho(y), s_i(M_k, y)\} = \min\{\rho(y), e^{-ck}\}$. For those $y$ with $\rho(y) > e^{-ck}$, we have

$$e^{-ck} = \min\{\rho(y), e^{-ck}\} = \min\{\rho(y), s_i(M_k, y)\}$$

and the right-hand side cannot equal $\rho(y)$, so we must have $s_i(M_k, y) = e^{-ck}$. For those $y$ with $\rho(y) \leq e^{-ck}$, we cannot have $s_i(M_k, y) > e^{-ck}$; otherwise, we could choose $\delta \in (e^{-ck}, s_i(M_k, y))$ and apply Remark 4.3.4 to see that $s_i(M_k, H(y, \delta)) = s_i(M_k, y) > e^{-ck}$, contradicting the previously established equality $s_i(M_k, H(y, \delta)) = e^{-ck}$. We thus have $s_i(M_k, y) \leq e^{-ck}$ on the other hand, for any $\delta \in (e^{-ck}, \gamma)$ we may apply Proposition 4.3.8 to obtain $e^{-ck} = s_i(M_k, H(y, \delta)) \leq s_i(M_k, y) \leq s_i(M_k, y)$. We conclude that $s_i(M_k, y)$ is constant for $y \in \mathbb{D}_{\gamma, K}^\circ$.

For $i = 1, \ldots, n$ and $y \in \mathbb{D}_{\gamma, K}^\circ$, we have

$$s_i(M \otimes_{R_{(0,\beta)}} R_{(0,\gamma)}, y) = \min\{s_i(M, y), \gamma\}. \quad (4.3.12.1)$$

For $i = 1, \ldots, m$, we must have $s_i(M, y) < \gamma$ or else Remark 4.3.4 would lead to a violation of hypothesis (a); moreover, from (4.3.12.1) and the previous paragraph, $\min\{s_i(M, y), \gamma\}$ is constant on $\mathbb{D}_{\gamma, K}^\circ$. We are thus done in case $m = n$, so we may assume that $m < n$ hereafter.

By Corollary 3.6.8, Proposition 4.3.7, and Proposition 4.3.8 (applied as in Definition 4.3.11), we have $s_1(M_0, y) \geq \gamma > \rho(y)$ for all $y \in \mathbb{D}_{\gamma, K}^\circ$. From this inequality plus (4.3.12.1), it follows that for $i = m + 1, \ldots, n$, we have $s_i(M, y) \geq \gamma$ for all $y \in \mathbb{D}_{\gamma, K}^\circ$. If there exists $y \in \mathbb{D}_{\gamma, K}^\circ$ for which $s_i(M, y) > \gamma$, then by Remark 4.3.4, $s_i(M, y)$ is constant on $\mathbb{D}_{\gamma, K}^\circ$; otherwise, $s_i(M, y)$ is evidently equal to the constant value $\gamma$ on $\mathbb{D}_{\gamma, K}^\circ$. This completes the proof.

4.4 Solvable modules

If one views solvability of a differential module on an annulus as a question about what happens as one approaches the generic point of the inner boundary, one is then led to an analogous concept in which one approaches an arbitrary point of a Berkovich disc. For points of type 2, this amounts to a cosmetic revision of §3.8, but at points of other types one has more precise results. The case of type 4 points is especially critical in order to eliminate such points from the controlling graph of $M$ (see Theorem 4.5.15).
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**Definition 4.4.1.** Choose $x \in \mathbb{D}_{\beta,K}$ with $x \neq \zeta_{0,\beta}$, so that $\rho(x) < \beta$. Put $r_0 = -\log \rho(x)$.

For $\rho \in (\rho(x), \beta]$, the points $H(x, \rho)$ are all of types 2 and 3 (because they are not minimal). Moreover, as $\rho \to \rho(x)^+$ these points form a net converging to $x$.

For any $\gamma, \delta$ with $\rho(x) < \gamma \leq \delta \leq \beta$, the subset of $\mathbb{D}_{\beta,K}$ consisting of points dominated by $H(x, \rho)$ for some $\rho \in [\gamma, \delta]$ has the form $\mathcal{M}(R_{x,[\gamma,\delta]})$ for some Banach algebra $R_{x,[\gamma,\delta]}$ over $K$.

More precisely, this subset is an *affinoid subdomain* of $\mathbb{D}_{\beta,K}$ in the sense of Definition 5.1.1. Even more precisely, if $K = \mathbb{C}$ and $\gamma, \delta \in |\mathbb{C}^\times|$, the set in question is an annulus.

For $\delta \in (\rho(x), \beta]$, define

$$R_{x,(\rho(x),\delta]} = \bigcap_{\gamma \in (\rho(x), \delta]} R_{x,[\gamma,\delta]};$$

it is equivalent to run the intersection over $\gamma \in (\rho(x), \delta] \cap |\mathbb{C}^\times|$. Define the *Robba ring* at $x$ as the ring

$$\mathcal{R}_x = \bigcup_{\delta \in (\rho(x), \beta]} R_{x,(\rho(x),\delta]};$$

it is equivalent to run the union over $\delta \in (\rho(x), \beta] \cap |\mathbb{C}^\times|$. All of these rings may be viewed as differential rings for the derivation $d/dt$.

**Definition 4.4.2.** For $N$ a differential module of rank $n > 0$ over $\mathcal{R}_x$, the germ of the function $-\log s_i(N, H(x, e^{-r}))$ in a left neighborhood of $r_0$ is well defined for $i = 1, \ldots, n$. We may thus say that $N$ is *solvable at* $x$ if

$$\limsup_{r \to r_0^-} -\log s_i(N, H(x, e^{-r})) - r \leq 0.$$  

In this case, as in Definition 3.8.3, there exist nonnegative rational numbers $b_1(N, x) \geq \cdots \geq b_n(N, x)$ such that for $i = 1, \ldots, n$, at the level of germs we have

$$\max\{r, -\log s_i(N, H(x, e^{-r}))\} = r + b_i(N, x)(r_0 - r).$$

**Remark 4.4.3.** For $x$ of type 2, after making a finite extension of $K$ to force $K$ to be integrally closed in $H(x)$, we may obtain an isomorphism $\mathcal{R}_x \cong \mathcal{R}_\alpha$ for $\alpha = \rho(x)$ by translating $x$ to $\zeta_{0,\alpha}$. We may thus transfer statements about $\mathcal{R}_\alpha$, such as Theorem 3.8.21, directly to the setting of solvable modules over $\mathcal{R}_x$.

For $x$ of other types, the behavior of a solvable module over $\mathcal{R}_x$ is much more restricted, especially in the case of a module obtained by base extension from $R_{[0,\beta]}$. It is most convenient to postpone discussion of this point until after we have Theorem 4.5.15 in hand; see §4.6. However, one key case is needed for the proof of Theorem 4.5.15, so we include it here; see Lemma 4.4.5.

**Lemma 4.4.4.** Assume that $p > 0$. Let $x \in \mathbb{D}_{\beta,K}$ be a point of type 4 for which $\rho(x) \in |\mathbb{C}^\times|$. Let $N$ be a differential module over $\mathcal{R}_x$ of rank $n$ which is solvable at $x$. Then $b_i(N, x) \in [0, 1]$ for $i = 1, \ldots, n$.

**Proof.** Using Lemma 4.3.3 and the fact that Berkovich’s classification is preserved by passage from $K$ to $\mathbb{C}$ (see Proposition 4.2.7), we may assume without loss of generality that $K = \mathbb{C}$ and $\rho(x) = 1$. We may also assume that $n > 0$.

Let $L_1, L_2$ be two copies of $H(x)$, and let $L_3$ be a complete extension of both (obtained by choosing an element of $\mathcal{M}(L_1 \otimes_K L_2)$). Let $t_1, t_2$ be the copies of $t_x$ in $L_1, L_2$. For $i = 1, 2$, let $\mathcal{R}_{(i)}$ be a copy of $\mathcal{R}_1$ (that is, the ring $\mathcal{R}_\alpha$ with $\alpha = 1$) over $L_i$ in the variable $t - t_i$. Let $\mathcal{R}_{(3)}$ be a copy of $\mathcal{R}_1$ over $L_3$ in the variable $t - t_1$, and equip $\mathcal{R}_{(3)}$ with the map from $\mathcal{R}_{(1)}$ sending $t - t_1$ to $t - t_2$. 

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to \( t - t_1 \) and the map from \( \mathcal{R}(2) \) sending \( t - t_2 \) to \( t - t_1 + (t_1 - t_2) \). For \( i = 1, 2 \), we may identify the residue field of \( \mathcal{R}(i) \) with \( \kappa_K((u_i)) \) for \( u_i = (t - t_i)^{-1} \), and then apply Theorem 3.8.21 and Corollary 3.8.22 to produce the minimal finite étale extension \( S_i \) of \( \mathcal{R}_{\text{int}}^{(i)} \) over which \( N \otimes_{\mathcal{R}_{\text{int}}^{(i)}} S_i \) satisfies the Robba condition. By the uniqueness in Corollary 3.8.22, we must then have an isomorphism

\[
S_1 \otimes_{\mathcal{R}_{\text{int}}^{(1)}} \mathcal{R}_{\text{int}}^{3} \cong S_2 \otimes_{\mathcal{R}_{\text{int}}^{(2)}} \mathcal{R}_{\text{int}}^{3}
\]

(4.4.4.1)

which commutes with the cocycle condition. This implies (e.g., by faithfully flat descent) that \( S_1 \) admits an action of the group of \( \kappa_K \)-linear substitutions on \( \kappa_K((u_1)) \) of the form \( t \mapsto t + c \) with \( c \in \kappa_K \). Let \( L \) be the residue field of \( S_1 \); applying Proposition 1.2.6, we may deduce that the highest upper numbering ramification break of \( L \) as a finite extension of \( \kappa_K((u_1)) \) is at most 1. By Corollary 3.8.22, this implies that \( b_1(N, x) \leq 1 \) and hence \( b_i(N, x) \leq 1 \) for \( i = 1, \ldots, n \).

**Lemma 4.4.5.** Assume that \( p > 0 \). Let \( M \) be a differential module over \( R_{[0, \beta]} \) of rank \( n \). Let \( x \) be a point of type 4 for which \( \rho(x) \in \lfloor \mathbb{C}^* \rfloor \). Put \( N = M \otimes_{R_{[0, \beta]}} R_x \). If \( N \) is solvable at \( x \), then \( b_i(N, x) \in \{0, 1\} \) for \( i = 1, \ldots, n \).

**Proof.** Let \( j \in \{0, \ldots, n\} \) be any index for which \( b_1(N, x), \ldots, b_j(N, x) > 0 \). For \( r \) in some left neighborhood of \( -\log \rho(x) \), the function

\[
\sum_{i=1}^{j} -\log s_i(M, H(x, e^{-r}))
\]

is affine with nonpositive slope by Proposition 3.6.3(a, d). However, this slope is equal to

\[
\sum_{i=1}^{j} (1 - b_i(N, x)),
\]

each summand of which is nonnegative by Lemma 4.4.4. This proves the claim. \( \square \)

**Remark 4.4.6.** It is tempting to argue directly that the isomorphism (4.4.4.1) from the proof of Lemma 4.4.5 implies by faithfully flat descent that \( S_1 \) descends to a finite étale algebra over \( \mathcal{R}_{\text{int}}^x \). One obstruction to this approach is that it is unclear whether the maps \( \mathcal{R}_{\text{int}}^x \rightarrow \mathcal{R}_{\text{int}}^{(i)} \) are flat.

### 4.5 Controlling graphs for radii of convergence

Using Proposition 4.3.7, we can give a partial translation of Proposition 3.6.3 into the language of radii of optimal convergence. This reproduces and improves a result of Pulita [Pul14, Theorem 4.7]; see Remark 4.5.16. Throughout this subsection, retain Hypothesis 4.3.1.

**Definition 4.5.1.** For \( x \in \mathbb{D}_{\beta, K} \), put

\[
f_i(M, x) = -\log s_i(M, x) \quad (i \in \{1, \ldots, n\})
\]

and \( F_i(M, x) = f_1(M, x) + \cdots + f_i(M, x) \). Note that unlike the functions \( f_i(M, r) \) considered in §3.6, the function \( f_i(M, x) \) may take values less than \( -\log \rho(x) \). We are thus led to define the truncated functions

\[
\overline{s}_i(M, x) = \min\{\rho(x), s_i(M, x)\}
\]
\[
\overline{f}_i(M, x) = -\log \overline{s}_i(M, x).
\]
Remark 4.5.2. By Proposition 4.3.7, the function $f_i(M, r)$ of §3.6 coincides with the function $\overline{f}_i(M, H(\zeta_{0,0}, e^{-r}))$. This will allow us to apply Proposition 3.6.3 to obtain information about the functions $f_i$.

**Proposition 4.5.3.** For any $x \in \mathbb{D}_{\beta,K}$, $s_1(M, x)$ belongs to the divisible closure of $|\mathcal{H}(x)^\times|$.

**Proof.** By making the canonical base extension as in Remark 4.2.5, we may reduce to the case where $x$ is of type 1. By Lemma 4.3.3, we may further reduce to the case where $K = \mathbb{C}$ and $x = \zeta_{0,0}$.

By Remark 4.5.2 and Proposition 3.6.6, the function $\overline{f}_i(M, H(x, e^{-r}))$ is piecewise of the form $ar + b$ with $a \in \mathbb{Q}$ and $b \in \log |K^\times|$. Put $r_0 = -\log s_1(M, x)$. By Proposition 4.3.8, $r_0$ is the smallest value for which $\overline{f}_i(M, H(x, e^{-r})) = r$ for all $r \geq r_0$. We thus have $r_0 = -b/a$ for some $a \in \mathbb{Q}$ and $b \in \log |K^\times|$, proving the claim.

**Lemma 4.5.4.** For $i = 1, \ldots, n$, if $s_i(M, x) > \rho(x)$ for some $x$, then $s_i(M, x)$ is constant on some neighborhood of $x$.

**Proof.** This is immediate from Remark 4.3.4.

**Lemma 4.5.5.** For $i = 1, \ldots, n$, the restriction of $f_i(M, \cdot)$ to any skeleton of $\mathbb{D}_{\beta,K}$ is piecewise affine.

**Proof.** It suffices to check that for any $x \in \mathbb{D}_{\beta,K}$ not of type 1, the function $g_i$ given by $g_i(r) = f_i(M, H(x, e^{-r}))$ is piecewise affine (the same then holds for points of type 1 by Lemma 4.5.4 and Remark 4.3.9). We first verify that $\max\{r, g_i(r)\} = \overline{f}_i(M, H(x, e^{-r}))$ is piecewise affine. Using Lemma 4.3.3, we may reduce to the case where $x = \zeta_{0,\alpha}$ for some $\alpha > 0$, in which case the claim follows from Proposition 3.6.3(a).

Given that $\max\{r, g_i(r)\}$ is piecewise affine, it follows that $g_i$ is piecewise affine at any $r_0$ for which $g_i(r_0) > r_0$. At a value $r_0$ where $g_i(r_0) < r_0$, by Lemma 4.5.4, $g_i$ is constant in a neighborhood of $r_0$. It thus suffices to check piecewise affinity at an arbitrary value $r_0$ at which $g_i(r_0) = r_0$.

We first consider a right neighborhood of $r_0$. If the values of $r$ in this neighborhood for which $g_i(r) < r$ fail to accumulate at $r_0$, then in some smaller neighborhood we have $g_i(r) = r$ identically. Otherwise, for each value $r_1$ at which $g_i(r_1) < r_1$, by the previous paragraph $g_i$ is constant for $r \geq r_1$. It follows that $g_i$ is constant for $r > r_0$ and the constant value must be at most $r_0$. If it were strictly less than $r_0$, we would have $g_i(r_0) < r_0$ by Remark 4.3.4, contrary to hypothesis; we thus have $g_i(r) = r_0$ for $r \geq r_0$. This proves affinity to the right of $r_0$.

We next consider a left neighborhood of $r_0$. If there exists any $r_1$ in this neighborhood for which $g_i(r_1) < r_1$, then as above, $g_i$ would be constant for $r \geq r_1$. But then we would have $r_0 = g_i(r_0) = g_i(r_1) < r_1 < r_0$, a contradiction. Hence $g_i(r) = r$ identically in this neighborhood. This proves affinity to the left of $r_0$.

**Remark 4.5.6.** By Lemma 4.5.5, it makes sense to refer to the slopes of $f_i(M, \cdot)$ or $\overline{f}_i(M, \cdot)$ along any branch of $\mathbb{D}_{\beta,K}$.

**Definition 4.5.7.** For $x \in \mathbb{D}_{\beta,K}$, define the spectral cutoff of $M$ at $x$ to be the largest value $m(x) \in \{0, \ldots, n\}$ such that $s_i(M, x) < \rho(x)$ for $i = 1, \ldots, m(x)$.

**Lemma 4.5.8.** Let $U$ be a lower branch of $\mathbb{D}_{\beta,K}$ at some point $x$. Suppose that for $i = 1, \ldots, m(x)$, the slope of $\overline{f}_i(M, \cdot)$ along $U$ (which exists by Lemma 4.5.5) is equal to zero. Then $s_i(M, x) = s_i(M, y)$ for $y \in U$; $i = 1, \ldots, n$. 

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Proof. By Lemma 4.5.5, for any $y \in U$ we have $s_i(M,H(y,\rho)) \to s_i(M,x)$ as $\rho \to \rho(x)^-$. This implies that the conditions of Lemma 4.3.12 are satisfied, so $s_i(M,y)$ is constant for $y \in U$, and on the other hand that this constant value is equal to $s_i(M,x)$. \hfill \square

Lemma 4.5.9. Let $S$ be a rooted skeleton of $\mathbb{D}_{\beta,K}$. Let $T$ be the interior of an edge in a subdivision of $S$. Suppose that for $i = 1, \ldots, n$, $\mathcal{F}_i(M,\cdot)$ is affine on $T$. Then

$$s_i(M,y) = s_i(M,\pi_S(y)) \quad (i = 1, \ldots, n; y \in \pi_S^{-1}(T)).$$

Proof. For $x \in T$ and $i = 1, \ldots, m(x)$, by Proposition 3.6.3(c, d) and Remark 4.5.2, the slope of $\mathcal{F}_i(M,\cdot)$ along any lower branch of $x$ other than the one meeting $T$ is equal to zero. The claim thus follows from Lemma 4.5.8. \hfill \square

Lemma 4.5.10. For any $x \in \mathbb{D}_{\beta,K}$, along all but finitely many lower branches of $\mathbb{D}_{\beta,K}$ at $x$, the slope of $\mathcal{F}_i(M,\cdot)$ is 0 for $i = 1, \ldots, m(x)$.

Proof. This is immediate from Proposition 3.6.3(c). \hfill \square

Lemma 4.5.11. For any $x \in \mathbb{D}_{\beta,K}$, there exist a skeleton $S$ of $\mathbb{D}_{\beta,K}$ and an open neighborhood $I$ of $\pi_S(x)$ such that the restrictions of $s_1(M,\cdot), \ldots, s_n(M,\cdot)$ to $\pi_S^{-1}(I)$ factor through $\pi_S$. Moreover, we may choose $S$ to have no generators of type 3.

Proof. By Lemma 4.5.10, along all but finitely many lower branches of $\mathbb{D}_{\beta,K}$ at $x$, the slope of $\mathcal{F}_i(M,\cdot)$ is 0 for $i = 1, \ldots, m(x)$. Choose $S$ to pass through $x$ and meet each of the remaining lower branches of $X$ at $x$; this can always be done without using generators of type 3 because any point of type 3 dominates some points of type 2 by Proposition 4.2.7. By Lemma 4.5.5, we can find a subdivision of $S$ such that for $i = 1, \ldots, n$, $\mathcal{F}_i(M,\cdot)$ is affine on each edge of the subdivision meeting $x$. Let $I$ be the union of the interiors of these edges, together with $x$. For $y \in \pi_S^{-1}(I)$, we have $s_i(M,y) = s_i(M,\pi_S(y))$ by Lemma 4.5.8 (if $\pi_S(y) = x$) or Lemma 4.5.9 (if $\pi_S(y) \neq x$). \hfill \square

Lemma 4.5.12. For $x \in \mathbb{D}_{\beta,K}$ of type 4, for $i = 1, \ldots, n$, in some left neighborhood of $\rho(x)$, the function

$$\rho \mapsto \min\{\omega \rho, s_i(M,H(x,\rho))\}$$

is either constant or identically equal to $\omega \rho$.

Proof. Apply Corollary 2.1.6 to construct $v \in M$ which is a cyclic vector in $M \otimes R_{[0,\beta]} \text{Frac}(R_{[0,\beta]})$, and write $D^n(v) = a_0v + \cdots + a_{n-1}D^{n-1}(v)$ for some $a_0, \ldots, a_{n-1} \in \text{Frac}(R_{[0,\beta]})$. Since $x$ is of type 4, for $i = 0, \ldots, n-1$, the function $y \mapsto y(a_i)$ is constant in some neighborhood of $x$. By Proposition 2.2.6, this yields the desired result. \hfill \square

Lemma 4.5.13. For $x \in \mathbb{D}_{\beta,K}$ of type 4, for $i = 1, \ldots, n$, in some left neighborhood of $\rho(x)$, the function $\rho \mapsto \pi_i(M,H(x,\rho))$ is either constant or identically equal to $\rho$.

Proof. This is immediate from Lemma 4.5.12 if $p = 0$, so we may assume that $p > 0$; we may also assume that $K = \mathbb{C}$. Let $h$ be the smallest nonnegative integer for which $s_i(M,x) \notin (\omega^p)^{h-1}(\rho(x),\rho(x))$ for $i = 1, \ldots, n$. We proceed by induction on $h$.

Put $r_0 = -\log \rho(x)$; since $x$ is of type 4, we have $r_0 > -\log \beta$. Let $j \in \{0, \ldots, n\}$ be the largest value for which $s_i(M,x) \leq \omega \rho(x)$ for $i = 1, \ldots, j$. Since the functions $r \mapsto \mathcal{F}_i(M,H(x,e^{-r}))$ are continuous by Lemma 4.5.5, we may apply Lemma 4.5.12 to produce $r_1 \in (-\log \beta, r_0)$ such that
for \( i = 1,\ldots,j \), the function \( r \mapsto f_i(M, H(x, e^{-r})) \) is constant for \( r \in [r_1, r_0] \). By moving \( r_1 \) toward \( r_0 \), we may also ensure that \( \rho(x) > \omega e^{-r_1} \) and \( s_i(M, H(x, e^{-r})) > \omega e^{-r_1} \) for \( i > j \). By rescaling \( t \), we may further ensure that \( r_1 < 0 < r_0 \).

By Proposition 4.2.7, we can find \( z \in \mathbb{C} \) such that \( H(x, 1) = \zeta_{z,1} \). There is no harm in applying a translation on the disc to reduce to the case \( z = 0 \). If we put \( \beta' = e^{-r_1} \), then by Proposition 3.6.7, the restriction of \( M \) to \( \mathbb{D}_{\beta',K} \) splits as a direct sum \( M_1 \oplus M_2 \) with rank(\( M_1 \)) = \( j \) and \( f_i(M, e^{-r}) = f_i(M_1, e^{-r}) \) for \( i = 1,\ldots,j \) and \( r \in (r_1, 0] \). By Corollary 3.6.5, the original claim holds with \( M \) replaced by the restriction of \( M_1 \) to \( \mathbb{D}_{1,K} \).

Let \( N \) be the restriction of \( M_2 \) to \( \mathbb{D}_{1,K} \); it now suffices to prove the original claim with \( M \) replaced by \( N \). We may assume that \( j < n \), as otherwise there is nothing to check. We first check the claim for \( N \) in case \( s_{i+1}(M, x) \geq \rho(x) \), which in particular will cover the base case \( h = 0 \) of the induction. If \( \rho(x) \notin |\mathbb{C}^x| \), then Proposition 4.5.3 implies that \( s_i(N, x) > \rho(x) \) for all \( i \), so the desired result follows by Lemma 4.5.4. If instead \( \rho(x) \in |\mathbb{C}^x| \), then the desired result follows by Lemma 4.4.5.

We next check the claim for \( N \) when \( s_{i+1}(M, x) < \rho(x) \); note that by construction we also have \( \omega \rho(x) < s_{i+1}(M, x) \). Let \( \psi : \mathbb{D}_{1,K}^0 \to \mathbb{D}_{1,K}^0 \) be the map for which \( \psi^*(t) = (t+1)^p - 1 \). Put \( y = \psi(x) \); it is a point of type 4 with \( \rho(y) = \rho(x)^p \). Let \( N' \) be the off-center Frobenius descendant of \( N \) in the sense of Proposition 3.5.5 with \( \lambda = 1 \). By that proposition, \( \pi_{(p-1)(n-j)+i}(N', z) = \pi_i(N, z)^p \) for \( i = 1,\ldots,n-j \) and \( z \in \mathbb{D}_{1,K} \) with \( \rho(z) > \omega \). Since we assumed that \( \rho(x) > \omega \beta' > \omega \), we have \( \pi_{(p-1)(n-j)+i}(N', H(y, \rho^p)) = \pi_i(N, H(x, \rho))^p \) for \( i = 1,\ldots,n-j \) and \( \rho \in [\rho(x), 1] \). We may thus deduce the claim for \( N \) from the corresponding claim for \( N' \), to which we may apply the induction hypothesis because we have decreased the value of \( h \).

**Lemma 4.5.14.** For \( x \in \mathbb{D}_{\beta,K} \) of type 1 or 4, for \( i = 1,\ldots,n \), the function \( s_i(M, \cdot) \) is constant on some neighborhood of \( x \).

**Proof.** For \( x \) of type 1, the claim follows from Remarks 4.3.4 and 4.3.9. For \( x \) of type 4, Lemma 4.5.13 implies that the hypothesis of Lemma 4.5.8 holds for some open disc containing \( x \), yielding the claim in this case.

**Theorem 4.5.15.** (a) There exists a strict skeleton \( S \) of \( \mathbb{D}_{\beta,K} \) such that \( s_1(M, \cdot),\ldots,s_n(M, \cdot) \) factor through \( \pi_S \).

(b) For \( i = 1,\ldots,n \), \( f_i(M, \cdot) \) is piecewise affine with slopes in \( \frac{1}{n} \mathbb{Z} \cup \cdots \cup (1/n) \mathbb{Z} \). Moreover, \( F_n(M, \cdot) \) has integral slopes.

(c) There is a unique minimal graph \( G \) in \( \mathbb{D}_{\beta,K} \) which is a controlling graph for all of the functions \( f_i(M, \cdot) \). Moreover, the vertices of \( G \) are all of type 2 or 3. (We call \( G \) the controlling graph of \( M \).)

**Proof.** For each \( x \in \mathbb{D}_{\beta,K} \), apply Lemma 4.5.11 to construct a skeleton \( S_x \) of \( \mathbb{D}_{\beta,K} \) and an open neighborhood \( I_x \) of \( \pi_S(x) \) such that the restrictions of \( s_1(M, \cdot),\ldots,s_n(M, \cdot) \) to \( \pi_{S_x}^{-1}(I_x) \) factor through \( \pi_{S_x} \). Since \( \pi_{S_x}^{-1}(I_x) \) is open in the compact space \( \mathbb{D}_{\beta,K} \), we can choose finitely many points \( x_i \in \mathbb{D}_{\beta,K} \) such that, if we relabel \( S_x, I_x \) as \( S_i, I_i \), then the open sets \( \pi_{S_i}^{-1}(I_i) \) cover \( \mathbb{D}_{\beta,K} \). Let \( S \) be the union of the \( S_i \); for \( y \in \pi_{S_i}^{-1}(I_i) \), we have \( \pi_S(y) = \pi_{S_i}(\pi_S(y)) \) and so \( s_i(M, y) = s_i(M, \pi_{S_i}(y)) = s_i(M, \pi_S(y)) \). This proves (a) except that \( S \) might include some generators of types 1 or 4 (generators of type 3 are excluded by Lemma 4.5.11). However, by Lemma 4.5.14, if \( x \) is a generator of type 1 or 4, then the functions \( s_i(M, \cdot) \) are constant in a neighborhood of \( x \), so we may replace \( x \) with a point of type 2 in this neighborhood which dominates \( x \). We thus deduce (a).
From (a), we deduce piecewise affinity using Lemma 4.5.5. To deduce integrality of slopes, we apply Proposition 3.6.3(b) at points $x$ where $s_i(M,x) < \rho(x)$ and Lemma 4.5.4 at points $x$ where $s_i(M,x) = \rho(x)$ identically, but on any such segment $f_i(M,x)$ has slope one. We thus deduce (b).

Using (a) and (b), we deduce the existence of the minimal controlling graph $G$ and the fact that none of its vertices is of type 1 or 4. This yields (c).

**Remark 4.5.16.** The weaker form of Theorem 4.5.15 in which strictness of the skeleton is not asserted is the essential content of [Pul14, Theorem 4.7] applied to a disc: more precisely, parts (i) (finiteness) and (ii) (integrality) of that result are included in Theorem 4.5.15. The proof in [Pul14] is a bit different, making use of a combinatorial criterion for piecewise affinity. A proof in terms of $p$-adic potential theory is given in [PP12b]. Another proof, essentially a streamlined version of the above arguments, is given in [BK]. None of the analyses in [BK, PP12b, Pul14] includes any special study of type 4 points, as these are treated by base extension to convert them into other types. Consequently, the techniques of those papers cannot by themselves exclude vertices of type 4 from the controlling graph, which here is made possible by the analysis in §4.4.

Note that [Pul14, Theorem 4.7] gives a finer description of the controlling graph than appears here. It also includes weak analogues of the convexity, subharmonicity, and monotonicity assertions from Proposition 3.6.3 (although with a change of sign convention, so convexity becomes concavity and subharmonicity becomes superharmonicity). In [Pul14, Theorem 4.7] these statements are used in an essential way to prove finiteness; however, given Theorem 4.5.15, they can be deduced directly from Proposition 3.6.3.

Note also that [BK, PP12a, PP12b, Pul14] consider not just discs but more general curves. We will return to this more general case in §5.

### 4.6 More on solvable modules

With Theorem 4.5.15 in hand, we can now fill out the discussion of solvable modules over $R_x$ initiated in §4.4. We also point out a link with our previous work on semistable reduction for overconvergent $F$-isocrystals [Ked11b].

**Hypothesis 4.6.1.** Throughout this subsection, let $M$ be a differential module over $R_{[0,\beta]}$ of rank $n$, choose $x \in \mathbb{D}_{\beta,K}$, put $M_x = M \otimes_{R_{[0,\beta]}} R_x$, and let $N$ be a subquotient of $M_x$ which is solvable at $x$.

**Remark 4.6.2.** For $x$ of type 3, Theorem 4.5.15 forces $N$ to satisfy the Robba condition; if $N = M_x$, then $N$ is forced to be trivial by Proposition 4.3.8. For $x$ of type 1, we can say even more: Proposition 4.3.8 and Theorem 4.5.15 together imply that $M_x$ itself is a trivial differential module, as then is $N$.

For $x$ of type 4, we have the following refinements of Lemma 4.4.5.

**Proposition 4.6.3.** Suppose that $x$ is of type 4:

(a) if $\rho(x) \in \mathbb{C}^\times$, then $b_i(N,x) \in \{0,1\}$ for all $i$;
(b) if $\rho(x) \notin \mathbb{C}^\times$, then $N$ is trivial, so $b_i(N,x) = 0$ for all $i$.

**Proof.** By Theorem 4.5.15 (or Lemma 4.5.14), the functions $s_i(M,\cdot)$ are constant in a neighborhood of $x$. This immediately implies (a). To deduce (b), note that we must have $s_1(M,x) \neq \rho(x)$ by Proposition 4.5.3. Since the $s_i(M,\cdot)$ are constant, we may apply Proposition 3.6.7 to decompose $M$ in a neighborhood of $x$ as a direct sum $M' \oplus M''$ with

\[ M_x = (M_1' \oplus M_2') \otimes_{R_{[0,\beta]}} R_x, \]

where $s_1(M_1',x) = \rho(x)$ and $s_i(M_2',x) = 0$ for $i > 1$. This implies $N_x = (N_1' \oplus N_2') \otimes_{R_{[0,\beta]}} R_x$, and since $N_1'$ is trivial, we have $N_2'$ trivial by Proposition 4.3.8, as desired.
s_i(M', x) < \rho(x) \text{ for all } i \text{ and } s_i(M'', x) \geq \rho(x) \text{ for all } i; \text{ by applying Proposition 4.5.3 again,}
we see that in fact } s_i(M', x) > \rho(x) \text{ for all } i. \text{ In particular, } M'' \text{ is trivial on some neighborhood of } x; \text{ moreover, the projection of } N \text{ onto } M' \otimes \mathcal{R}_x \text{ must be zero. It follows that } N \text{ is trivial, yielding (b). \hfill \Box}

**Theorem 4.6.4.** Assume that } K = \mathbb{C}, \text{ } x \text{ is of type 4, and } \rho(x) = 1. \text{ For each } c \in \kappa_K, \text{ choose a lift } \tilde{c} \text{ of } c \text{ to } \sigma_K, \text{ and let } Q_c \text{ be the differential module over } \mathcal{R}_x \text{ free on one generator } \nu \text{ such that } D(\nu) = \tilde{c} \nu. \text{ }

(a) For each irreducible subquotient } P \text{ of } N, \text{ there exists } c \in \kappa_K \text{ such that } P \otimes Q_c \text{ satisfies the Robba condition.

(b) There exists a finite étale extension } S \text{ of } \mathcal{R}_x \text{ of the form}

\[ S = \mathcal{R}_x[z_1, \ldots, z_m]/(z_1^p - z_1 - a_1 t, \ldots, z_m^p - z_m - a_m t) \]

for some nonnegative integer } m \text{ and some } a_1, \ldots, a_m \in \sigma_K^x \text{ such that } N \otimes \mathcal{R}_x S \text{ is trivial.}

**Proof.** By Proposition 4.6.3 we have } b_1(P) \in \{0,1\}. \text{ If } b_1(P) = 0 \text{ we take } c = 0; \text{ otherwise, by} \ [Ked10a, \text{ Theorem 12.7.2}], \text{ we can choose } c \text{ so that } b_1(P \otimes Q_c) < 1, \text{ and then by Proposition 4.6.3 again we have } b_1(P \otimes Q_c) = 0. \text{ This proves (a).}

Given (a), to prove (b), Theorem 4.5.15 and Proposition 3.6.7 allow us to reduce to the case where } M_x \text{ itself is solvable at } x; \text{ we may then further reduce to the case where } N = M_x. \text{ In this case, the proof of Theorem 3.8.21 provides } S \text{ such that } N \otimes \mathcal{R}_x S \text{ satisfies the Robba condition. However, by induction on } m, \text{ we see that there is an isomorphism}

\[ R_{[0, \beta]^{p-m}} \cong R_{[0, \beta]^{p}}[z_1, \ldots, z_m]/(z_1^p - z_1 - a_1 t, \ldots, z_m^p - z_m - a_m t) \]  \hfill (4.6.4.1)

sending } t \text{ to } z_m. \text{ This isomorphism gives rise to a map } \psi : \mathbb{D}_{\beta}^{p-m}_K \to \mathbb{D}_{\beta}^K \text{ by mapping } R_{[0, \beta]} \text{ into the right-hand side of (4.6.4.1) and then crossing to the left-hand side. The inverse image of } x \text{ under this map is a single point } y. \text{ By construction, } N \otimes \mathcal{R}_x S \cong \psi^*M \otimes_{R_{[0, \beta]^{p-m}}} \mathcal{R}_y \text{ satisfies the Robba condition. By Proposition 4.3.8, } \psi^*M \text{ is trivial in a neighborhood of } y, \text{ so } N \otimes \mathcal{R}_x S \text{ is also trivial.} \hfill \Box

**Corollary 4.6.5.** Assume that } x \text{ is of type 4. Then any subquotient of } N \text{ satisfying the Robba condition is trivial, and hence admits the zero tuple as an exponent.}

**Proof.** If } \rho(x) \notin |\mathbb{C}^x|, \text{ then } N \text{ is trivial by Proposition 4.6.3(b), so any subquotient of } N \text{ satisfying the Robba condition is also trivial and hence admits the zero tuple as an exponent. If } \rho(x) \in |\mathbb{C}^x|, \text{ we may assume that } K = \mathbb{C} \text{ and } \rho(x) = 1. \text{ Set notation as in the proof of Theorem 4.6.4(b), again reducing to the case where } N = M_x. \text{ In this case, the Tannakian category of differential modules over } \mathcal{R}_x \text{ generated by } N \text{ admits a fibre functor computing horizontal sections over } S, \text{ for which the automorphism group is an elementary abelian } p\text{-group. In particular, } N \text{ splits as a direct sum of irreducible submodules whose } p\text{th tensor powers are trivial. Consequently, to check that a subquotient of } N \text{ satisfying the Robba condition is trivial, it suffices to check the case of a irreducible submodule } P \text{ for which } P^{\otimes p} \text{ is trivial; this case follows from Corollary 3.4.25.} \hfill \Box

**Remark 4.6.6.** Note that the isomorphism in (4.6.4.1) depends critically on having linear powers of } t \text{ on the right-hand side; otherwise, we would end up with something other than a disc, so Dwork’s transfer theorem (Proposition 4.3.8) would not apply. This is why it is necessary to invest the hard work to first prove } b_1(N, x) \in \{0,1\} \text{ in order to deduce Corollary 4.6.5.
Remark 4.6.7. The above arguments, including the proof of Lemma 4.4.5, are loosely inspired by the arguments made in [Ked11b, §5]. However, the correspondence turns out to be somewhat less close than we had originally expected, primarily because the process of transposing the arguments exposed an error in [Ked11b]. We now describe this error and how it may be remedied using results from this paper.

The error appears in the second sentence of the proof of [Ked11b, Lemma 5.6.2]: it is not the case that the property of being terminally presented is stable under tame alterations. That is because the tame alteration $x \mapsto x^m$ is ramified along the segment joining zero to the Gauss point; consequently, after pulling back a terminally presented module along a tame alteration, one encounters a change of slope at the point where one branches off from the ramification locus. In the continuation of the proof, the tame alteration is erroneously used to force the group $\tau(I_1)$, which initially is the semidirect product of the $p$-group $\tau(W'_1)$ with a cyclic group of order prime to $p$, to become equal to $\tau(W'_1)$.

To correct the proof, it suffices to establish that the equality $\tau(I_1) = \tau(W'_1)$ holds initially, so that no tame alteration is needed and the rest of the argument may proceed unchanged. To verify this, choose $\rho$ as in [Ked10a, Lemma 4.7.4]; by that lemma, $|\cdot|_{\rho^n,s_0}$ defines a point of $M(\ell(x))$ of type 4. We may thus apply Corollary 4.6.5 to deduce that any subquotient of the cross-section $M_{\rho}$ which satisfies the Robba condition admits the zero tuple as an exponent. This implies that $\tau(W'_1)$ has no nontrivial quotient of prime-to-$p$ order, and so $\tau(I_1) = \tau(W'_1)$ as desired.

One might prefer to incorporate some of the intermediate arguments from this paper into the proof method of [Ked11b], but this seems difficult. The plan of attack in [Ked11b] is to pick out an Artin–Schreier extension that reduces the image of the monodromy representation, which requires tame ramification to be ruled out first. By contrast, the method here is to use Artin–Schreier extensions only to lower the ramification numbers; only when this stops being possible is the presence of tame ramification ruled out.

A more satisfying resolution would be to use additional results of this paper, especially Theorem 4.6.4, to shortcut many of the complicated proofs in [Ked11b, §5]. We leave this as an exercise for the interested reader.

5. Berkovich curves

To conclude, we globalize our setup to include more general Berkovich curves. We now adopt the full language of Berkovich analytic spaces, as in [Ber90, Ber93].

5.1 Analytic spaces

Definition 5.1.1. A strictly affinoid algebra (respectively an affinoid algebra) over $K$ is a commutative Banach algebra over $K$ isomorphic to a quotient of the completion of some polynomial ring $K[T_1,\ldots,T_n]$ for the Gauss norm (respectively the $(r_1,\ldots,r_n)$-Gauss norm for some $r_1,\ldots,r_n > 0$).

Let $A$ be a (strictly) affinoid algebra over $K$. A (strictly) affinoid subdomain of $M(A)$ is a closed subset $U$ for which the category of bounded $K$-linear homomorphisms $A \to B$ of (strictly) affinoid algebras whose restriction maps carry $M(B)$ into $U$ has an initial element. Any such initial homomorphism $A \to B$ is then flat and induces a homeomorphism $M(B) \cong U$ [Ber90, Proposition 2.2.4]; in particular, a strictly affinoid subdomain is also an affinoid subdomain.

Note that $M(A)$ admits a neighborhood basis of affinoid subdomains, because any rational subdomain is an affinoid subdomain. For $x \in M(A)$, define the local $A$-algebra $A_x$ as the direct
limit of the representing homomorphisms $A \to B$ over all affinoid subdomains of $\mathcal{M}(A)$ which are neighborhoods of $x$. We define the structure sheaf $\mathcal{O}$ on $\mathcal{M}(A)$ so that for $U$ an open subset of $\mathcal{M}(A)$, $\mathcal{O}(U)$ consists of the functions $f : U \mapsto \prod_{x \in \mathcal{M}(A)} A_x$ such that for each $x \in U$, there exist a homomorphism $A \to B$ and an element $g \in B$ such that:

- the map $A \to B$ represents an affinoid subdomain of $\mathcal{M}(A)$ contained in $U$ and containing a neighborhood of $x$;
- for each $y \in U$, $f(y)$ is the image of $g$ in $A_y$.

By Tate’s theorem, the natural map $A \to \Gamma(\mathcal{M}(A), \mathcal{O})$ is a bijection. By Kiehl’s theorem, coherent sheaves over $\mathcal{O}$ correspond to finite $A$-modules via the functor of global sections.

**Definition 5.1.2.** A good (strictly) $K$-analytic space is a locally ringed space which is locally isomorphic to an open subspace of the Gel’fand spectrum of a (strictly) affinoid algebra over $K$. These are the analytic spaces considered in [Ber90]; they have the property that any point has a neighborhood basis consisting of affinoid spaces.

**Remark 5.1.3.** In [Ber93], the more general notion of a (strictly) $K$-analytic space is considered, in which it is only required that each point have a neighborhood basis consisting of a finite union of affinoid spaces (glued in a suitable way). In this paper, we can get away with considering only good spaces because any curve over $K$ is good [deJ95, Corollary 3.4].

### 5.2 Curves and triangulations

We next introduce some of the combinatorial structure of a Berkovich analytic curve over $K$.

One way to explain this is using semistable models, as in [Bal10, BK]. Here, we take an alternate approach using triangulations introduced by Ducros [Duc14], so as to avoid leaving the realm of analytic spaces; this follows the example of [Pul14, PP12a, PP12b]. There is also a link to tropicalization; see Remark 5.2.7.

**Definition 5.2.1.** For $K'$ an analytic field containing $K$ and $X$ a good $K$-analytic space, let $X_{K'}$ denote the base extension of $K$ to $K'$. For $X = \mathcal{M}(A)$, we have $X_{K'} = \mathcal{M}(A \widehat{\otimes}_K K')$.

Let $\Omega_X$ denote the sheaf of continuous Kähler differentials on $X$. We say that $X$ is rig-smooth of pure dimension $n$ if for every analytic field $K'$ containing $K$, $\Omega_{X_{K'}}$ is locally free of rank $n$.

By a curve over $K$, we will mean a good $K$-analytic space $X$ which is separated (i.e., the diagonal morphism is a closed immersion) and rig-smooth of pure dimension 1. In particular, $X$ is paracompact.

**Definition 5.2.2.** Let $X$ be a curve over $K$. For $x \in X$, we declare $x$ to be of type 1, 2, 3, 4 if the signature of $x$ is respectively $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(0,0,0)$. These cases are exhaustive by Proposition 4.2.7 plus Noether normalization for strictly affinoid algebras [BGR84, Corollary 6.1.2/2].

**Definition 5.2.3.** An open disc over $K$ is a $K$-analytic space isomorphic to $\bigcup_{\gamma \in (0, \beta]} \mathcal{M}(R_{[0,\gamma]})$ for some $\beta > 0$. An open annulus over $K$ is a $K$-analytic space isomorphic to $\bigcup_{\alpha < \gamma < \delta < \beta} \mathcal{M}(R_{[\gamma, \delta]})$ for some $0 < \alpha < \beta$.

A virtual open disc (respectively virtual open annulus) is a connected $K$-analytic space whose base extension to $\mathbb{C}$ is a disjoint union of open discs (respectively open annuli). By the skeleton of a virtual open annulus over $K$, we mean the set of points not contained in a virtual open disc. For the standard open annulus $\bigcup_{\alpha < \gamma < \delta < \beta} \mathcal{M}(R_{[\gamma, \delta]})$ within $\mathbb{D}_{\beta,K}$, the skeleton is the set $\{\zeta_{\rho, \beta} : \rho \in (\alpha, \beta)\}$; in general, the skeleton of a virtual open annulus is an open segment.
General introduction to the subject

**Definition 5.2.4.** Let \( X \) be a curve over \( K \). A **weak strict triangulation** (respectively **weak triangulation**) of \( X \) is a locally finite subset \( S \) of \( X \) consisting of points of type 4 (respectively of types 2 or 3) such that any connected component of \( X \setminus S \) is a virtual open disc or a virtual open annulus. The union of the skeleta of the connected components of \( X \setminus S \) which are virtual open annuli forms a locally finite graph \( \Gamma_S \), called the **skeleton** of the weak triangulation. The points of \( \Gamma_S \) are all of types 2 or 3.

**Remark 5.2.5.** The definition of weak triangulation used here is the same as in [Pul14] but is somewhat more permissive than the one used in [Duc14], in which it is required that \( X \setminus S \) be relatively compact. Omitting this condition makes it possible for \( \Gamma_S \) to fail to meet some connected components of \( X \), e.g., if there is a component which is itself a virtual open disc. If \( \Gamma_S \) does meet every connected component of \( X \), then there is a natural continuous retraction \( \pi_S : X \to \Gamma_S \) taking any \( x \in \Gamma_S \) to itself and taking any \( x \in X \setminus \Gamma_S \) to the unique point of \( \Gamma_S \) in the closure of the connected component of \( X \setminus \Gamma_S \) containing \( x \).

**Theorem 5.2.6.** Any (strictly) analytic curve over \( K \) admits a weak (strict) triangulation.

**Proof.** See [Duc14, Théorème 5.1.14].

**Remark 5.2.7.** There is also an approach to the structure theory of analytic curves via tropicalization, i.e., consideration of the projections defined by evaluation at finitely many functions on the curve. For discussion of the case \( K = \mathbb{C} \), including a proof of Theorem 5.2.6 in that context, see [BPR12, §5].

**Definition 5.2.8.** Let \( X \) be a curve and let \( x \in X \) be a point of type 2. Then the residue field \( \kappa_{\mathcal{H}(x)} \) is the function field of an algebraic curve over \( \kappa_K \); we denote the genus of this function field by \( g(x) \) and call it the **genus** of \( x \). For any weak triangulation \( S \) of \( X \), the type 2 points of \( X \setminus S \) are all of genus zero; by Theorem 5.2.6, it follows that the type 2 points of \( X \) of positive genus form a locally finite set.

**Definition 5.2.9.** Let \( X \) be a curve. By a **branch** of \( X \) at a point \( x \in X \), we mean a local path-connected component of \( X \setminus \{x\} \) at \( x \). Depending on the type of \( x \), branches exist as follows:

- (a) exactly one branch;
- (b) infinitely many branches, corresponding to all but finitely many places of the function field \( \kappa_{\mathcal{H}(x)} \);
- (c) either zero, one, or two branches;
- (d) exactly one branch.

Given a weak triangulation \( S \) of \( X \), we say that a branch \( U \) of \( X \) at \( x \) is **skeletal** (or \( S \)-**skeletal** in case of ambiguity) if the closure of \( U \cap \Gamma_S \) contains \( x \); such branches can only exist if \( x \in \Gamma_S \).

We say that \( x \in X \) is **external** if it is of type 2 and its branches do not correspond to all of the places of \( \kappa_{\mathcal{H}(x)} \) or if it is type 3 and it has fewer than two branches; otherwise, we say that \( x \) is **internal**. For any weak triangulation \( S \) of \( X \), every point of \( X \setminus \Gamma_S \) is internal, as is every point of \( \Gamma_S \) lying in the interior of an edge; by Theorem 5.2.6, it follows that the external points of \( X \) form a locally finite set.

**Example 5.2.10.** For \( X \) an affinoid space, the external points of \( X \) are precisely the points of the **Shilov boundary**, the minimal subset of \( X \) for which the maximal modulus principle holds.

**Remark 5.2.11.** If \( X \) is a strictly affinoid space, then the points of the Shilov boundary are all of type 2. Consequently, for any strictly analytic curve \( X \), the external points of \( X \) are all of type 2.
so every point of type 3 has exactly two branches. However, for more general analytic spaces, a point of type 3 may have one or even zero branches. For instance, take $X$ to be the annulus $\mathcal{M}(R_{[\alpha,\beta]})$ for some $\alpha, \beta \in (0, +\infty) \setminus \mathbb{C}^*$. If $\alpha < \beta$, then $\zeta_{0,\alpha}$ and $\zeta_{0,\beta}$ are points of type 3 each with only one branch. If $\alpha = \beta$, then $\mathcal{M}(R_{[\alpha,\alpha]})$ consists only of a single point $\zeta_{0,\alpha}$ of type 3, which in particular has zero branches.

### 5.3 Convergence of local horizontal sections

We next study the convergence of local horizontal sections on analytic curves. As in the case of discs, we end up with a global statement about the behavior of radii of convergence of differential modules on analytic curves; this statement recovers the main results of [Pul14, PP12a, PP12b].

**Hypothesis 5.3.1.** For the remainder of the paper, let $X$ be a curve over $K$ equipped with a weak triangulation $S$ and let $M$ be a vector bundle over $X$ of constant rank $n > 0$ equipped with a connection. (Since $X$ is of dimension 1, the connection is automatically integrable.)

**Remark 5.3.2.** One interesting case excluded by our hypotheses is that where $X$ is an affine line and $S$ is empty. In this case, the radii of convergence should be allowed to be infinite, but we do not want to worry about this. For a more comprehensive treatment, see, for instance, [BK].

In order to define analogues of the radii of optimal convergence, one must make reference to the chosen triangulation. This has the same effect as the choice of a semistable model in [Bal10].

**Definition 5.3.3.** For $x \in \Gamma_S$, define $s_1(M, S, x), \ldots, s_n(M, S, x)$ as the intrinsic subsidiary radii of $M$ in order, and put $\rho_S(x) = 1$.

For $x \in X \setminus \Gamma_S$, lift $x$ to a point $y \in X_C$, identify the connected component of $(X \setminus \Gamma_S)_C$ containing $x$ with an open disc of some radius $R$, then define $s_1(M, S, x), \ldots, s_n(M, S, x)$ as the functions $s_1(M, y)/R, \ldots, s_n(M, y)/R$ as in Definition 4.3.11. We use the same identification (again dividing by $R$) to define the diameter $\rho_S(x)$. These definitions do not depend on the choice of $y$ or $R$, and are stable under enlarging $K$.

For $x \in X$, define the **spectral cutoff** of $M$ as the largest $m(x) \in \{0, \ldots, n\}$ such that $s_i(M, S, x) < \rho_S(x)$ for $i = 1, \ldots, m(x)$.

In order to analyze these functions, it will be useful to consider them first along individual branches.

**Definition 5.3.4.** Choose $x \in \Gamma_S$ of type 2, let $U$ be a branch of $X$ at $x$, and let $v$ be the corresponding place of $\kappa_{H(x)}$. Choose $t \in O_{X,x}$ with $x(t) = 1$ whose image $\tilde{t}$ in $\kappa_{H(x)}$ is a uniformizer of $v$ (i.e., its $v$-valuation is the positive generator of the value group). Then for $\beta \in (0, 1)$ sufficiently close to one, $t$ defines an isomorphism between the space of $y \in U$ with $y(t) \in (\beta, 1)$ and the open annulus $\beta < |t| < 1$ in the $t$-line. We can use this isomorphism to define the class of functions $f : X \to \mathbb{R}$ which are affine along $U$ in a neighborhood of $x$, and to associate to each such function a slope (in the direction away from $x$); neither of these definitions depends on the choice of $t$.

**Lemma 5.3.5.** Set notation as in Definition 5.3.4. Then for $i = 1, \ldots, m(x)$, the function $\log s_i(M, S, \cdot)$ is affine along $U$ and its limit at $x$ (approached from within $U$) equals $\log s_i(M, S, x)$.

**Proof.** For $x \notin S$ this is immediate from Proposition 3.6.3(a). For $x \in S$ with $g(x) = 0$, we may also apply Proposition 3.6.3(a) over the ring $R^{an}_{(\alpha, 1)}$. For $x \in S$ with $g(x) \neq 0$, we obtain a differential module over a ring $S$ which can be written as a finite étale algebra over $R^{an}_{(\alpha, 1)}$ of
some degree \(d > 0\) such that \(S \otimes_{\mathbb{P}^{n}_{\mathbb{A}_{1}}} F_1 \cong \mathcal{H}(x)\) is a finite unramified extension of \(F_1\). If we restrict scalars from \(S\) to \(\mathbb{P}^{n}_{\mathbb{A}_{1}}\), the multiset of intrinsic subsidiary radii does not change except that each multiplicity gets multiplied by \(d\). We may thus apply Proposition 3.6.3(a) in this case also.

We have the following analogue of Proposition 3.6.3(c). A more detailed exposition of the geometry used in this argument will be given in [BK].

**Theorem 5.3.6.** Choose \(x \in X\) of type 2. Let \(c(x)\) be the number of skeletal branches of \(X\) at \(x\). (Note that if \(x \notin \Gamma_S\), then \(g(x) = c(x) = 0\).)

(a) For \(i = 1, \ldots, m(x)\), the function \(\log s_i(M, S, -)\) is affine of slope 0 along all but finitely many branches of \(X\) at \(x\). In particular, we may form the sum \(\mu_i\) of the slopes of the function \(\sum_{j=1}^{i} \log s_j(M, S, -)\) along all of the branches of \(X\) at \(x\) (in the directions away from \(x\)).

(b) If \(x \notin \Gamma_S\), then \(\mu_i \leqslant 0\) for \(i = 1, \ldots, m(x)\).

(c) If \(x \in \Gamma_S\) is internal, then \(\mu_i \leq (2g(x) - 2 + c(x))i\) for \(i = 1, \ldots, m(x)\).

(d) In (b) and (c), equality holds if \(i = m(x)\). Equality also holds if \(i < n\) and \(s_i(M, S, x) < s_{i+1}(M, S, x)\).

**Proof.** We may assume that \(K = \mathbb{C}\), so that \(\kappa_K\) is algebraically closed. If \(x \notin \Gamma_S\), by rescaling we may reduce the claims to an instance of Proposition 3.6.3(c), so we may assume hereafter that \(x \in \Gamma_S\).

Suppose first that \(X\) is contained in the affine line; in this case, we may follow the proof of [Ked10a, Theorem 11.3.2(c)]. Namely, using Frobenius pushforwards as in Definition 3.5.2 (and using Propositions 2.3.5 and 3.5.5), we may reduce to the case where \(s_i(M, S, x) < \omega \rho_S(x)\). In this case, the claims follow by first using Corollary 2.1.6 to choose an element of \(M_x\) which is a cyclic vector for \(M_x \otimes_{O_{X,x}} \text{Frac}(O_{X,x})\) for the derivation \(d/dt\), then applying Proposition 2.2.6.

We now treat the case of general \(X\). Let \(C\) be a smooth projective connected curve over \(\kappa_K\) with function field \(\kappa_{\mathcal{H}(x)}\). Choose a nonconstant \(\tilde{f} \in \kappa_{\mathcal{H}(x)}\) of degree \(d > 0\), then choose \(f \in O_{X,x}\) with \(x(f) = 1\) lifting \(\tilde{f}\) which is unramified at each point corresponding to a branch named in (a). Note that removing part of \(X\) contained in a branch adds \(i\) to both sides of the desired inequality and is thus harmless; we can thus ensure that \(f\) defines a finite étale map \(X \to X'\) for \(X'\) a subspace of the affine line. Put \(x' = f(x')\) and let \(S'\) be the image of \(S\). For each branch \(U'\) of \(X'\) at \(x\), the slope of \(\sum_{j=1}^{d_i} \log s_j(f_*M, S', U')\) can be computed as follows. Let \(P'\) be the point of \(C\) corresponding to \(U'\). For each point \(P \in \{P'\}^{-1}(P')\) with multiplicity \(m\) and ramification number \(e\) (so that \(e = m\) if the ramification at \(P\) is tame), let \(U\) be the corresponding branch of \(X\) at \(x\); we then get a contribution of \(1 - e\) plus the slope of \(\sum_{j=1}^{d_i} \log s_j(M, S, U)\) (as may be verified using Frobenius descendants). We thus deduce the claim from the previous case plus the Riemann–Hurwitz formula. \(\square\)

To show that the functions \(s_i(M, S, -)\) can be computed using some triangulation, we use the following criterion.

**Lemma 5.3.7.** Let \(T\) be a triangulation containing \(S\) with the following properties.

(a) The set \(\Gamma_T\) meets every connected component of \(X \setminus \Gamma_S\). In particular, the retraction \(\pi_T\) exists (see Remark 5.2.5).
(b) Along each edge of $\Gamma_T$, the functions $\log s_i(M,S,)\cdot$ are affine for $i = 1, \ldots, n$.

(c) For each $x \in T$, for $i = 1, \ldots, m(x)$, the slope of $\log s_i(M,S,\cdot)$ along any nonskeletal branch of $X$ at $x$ is 0. Then for $i = 1, \ldots, n$, $\log s_i(M,S,\cdot)$ factors as the retraction $\pi_T$ followed by a piecewise affine function on $\Gamma_T$.

Proof. Note that (b) implies that (c) holds also for $x \in \Gamma_T$ by Proposition 3.6.3(c, d). We may thus deduce the claim using Lemma 4.3.12 (applied after enlarging $K$ to turn a virtual open disc into a true open disc) and Lemma 5.3.5.

We then obtain the following generalization of Theorem 4.5.15, which recovers the main results of [PP12a, PP12b, Pul14].

Theorem 5.3.8. There exists a triangulation $T$ containing $S$ such that $\Gamma_T$ meets every connected component of $X \setminus \Gamma_S$ (so the retraction $\pi_T$ exists by Remark 5.2.5) and each function $\log s_i(M,S,\cdot)$ is zero along each $T$-nonskeletal branch of $X$ at $x$. By Proposition 3.6.3(a), we may draw an open star in $\Gamma_T$ around $x$ such that on each edge, the functions $\log s_i(M,S,)\cdot$ are affine for $i = 1, \ldots, n$. On this star, the conditions of Lemma 5.3.7 are satisfied, so the desired result follows.

One can also change the functions to match the new triangulation without disturbing the conclusion.

Definition 5.3.9. We say that a triangulation $T$ is controlling for $M$ if the functions $s_i(M,T,\cdot)$ also factor as the retraction $\pi_T$ followed by some piecewise affine functions on $\Gamma_T$. That is, we must be able to take $T = S$ in the conclusion of Theorem 5.3.8.

Corollary 5.3.10. In the notation of Theorem 5.3.8, the triangulation $T$ is controlling.

Proof. This follows from Theorem 5.3.8 and the fact that conditions (a, b) of Lemma 5.3.7 can be stated in terms of intrinsic subsidiary radii, and so remain valid if we replace $S$ by $T$.

Corollary 5.3.11. Let $T$ be a triangulation containing $S$ with the following properties.

(a) The set $\Gamma_T$ meets every connected component of $X \setminus \Gamma_S$.

(b) Along each edge of $\Gamma_T$, the functions $\sum_{i=1}^n \log s_i(M,S,\cdot)$ and $\sum_{i=1}^n \log s_i(\operatorname{End}(M),S,\cdot)$ are affine for $i = 1, \ldots, n$.

(c) For each $x \in T$, the slope of $\sum_{i=1}^{m(x)} \log s_i(M,S,\cdot)$ along any nonskeletal branch of $X$ at $x$ is zero. Then for $i = 1, \ldots, n$, $\log s_i(M,S,\cdot)$ factors as the retraction $\pi_T$ followed by a piecewise affine function on $\Gamma_T$. In particular, by Corollary 5.3.10, $T$ is controlling.

Proof. It suffices to verify the conditions of Lemma 5.3.7. Condition (a) is true by hypothesis. Condition (b) holds by Lemma 3.7.3. To check condition (c), note that for $i = 1, \ldots, m(x)$, the slope of $\log s_i(M,S,\cdot)$ at $x$ is nonnegative by Proposition 4.3.8, but the sum of these slopes is zero so each slope individually must equal zero.
Corollary 5.3.12. There exists a strict triangulation $T$ which is controlling for $M$.

Proof. We construct an increasing sequence of triangulations $T_0, \ldots, T_n$ such that for $i = 0, \ldots, n$, the retraction $\pi_{T_i}$ exists and for $j = 1, \ldots, i$, the functions $\log s_j(M, T_i, \cdot)$ factor as $\pi_{T_i}$ followed by a piecewise affine function on $\Gamma_{T_j}$. To begin, let $T_0$ be any strict triangulation of $M$ for which the retraction $\pi_{T_0}$ exists. Given $T_i$ for some $i \in \{0, \ldots, n - 1\}$, by Theorem 5.3.8 there exists a triangulation $T_{i+1}$ containing $T_i$ such that $\log s_{i+1}(S, T_i, \cdot)$ factors as $\pi_T$ followed by a piecewise affine function on $\Gamma_{T_i}$. If $i = 0$, then Proposition 4.5.3 ensures that $T_{i+1}$ can be chosen to be strict. If $i > 0$, we may make the same argument after applying Proposition 3.6.7 to separate the first $i - 1$ radii in each disc. \hfill \Box

Remark 5.3.13. The methods of [BK, PP12a, PP12b, Pul14], when considered without reference to this paper, can only prove a weaker version of Theorem 5.3.8: they only provide a controlling triangulation over a sufficiently large analytic field $K'$ containing $K$. As in Remark 4.5.16, the problem is that this triangulation may involve vertices which project to type 4 points of the original curve, which our methods are able to rule out. In the language of [Bal10], we are able to exhibit a controlling strictly semistable model already over $\mathbb{C}$, whereas the methods of [PP12a, PP12b, Pul14] provide such a model only over a possibly larger algebraically closed analytic field containing $\mathbb{C}$.

Remark 5.3.14. One can also give a variant of Theorem 5.3.8 for meromorphic (possibly irregular) connections; in this case, one must allow triangulations to have vertices at type 1 points (namely the poles of the connection). This result is described in [BK].

5.4 Clean decompositions

One has an analogue of the spectral decomposition for the stalk of $M$ at a point $x \in X$. Using Theorem 5.3.8, we can extend this decomposition to specific subspaces of $X$.

Lemma 5.4.1. Choose $x \in X$ of type 2 or 3.

(a) There exists a unique direct sum decomposition $M_x = \bigoplus_i N_i$ whose base extension to $\mathcal{H}(x)$ is the spectral decomposition.

(b) There exists a finite étale extension $S$ of $\mathcal{O}_{X,x}$ such that $M_x \otimes \mathcal{O}_{X,x} S$ admits a direct sum decomposition whose base extension to $\mathcal{H}(x) \otimes \mathcal{O}_{X,x} S$ is a refined decomposition.

Proof. Part (a) follows by using the pushforward argument from the proof of Theorem 5.3.6 to reduce to the case where $X$ is contained in the affine line over $K$; this case is the Dwork–Robba decomposition theorem [DR77, § 4, Theorem, p. 20], or can alternatively be derived by following the proof of [Ked10a, Theorem 12.3.2]. Part (b) follows similarly upon noting that the local ring $\mathcal{O}_{X,x}$ is henselian. \hfill \Box

Theorem 5.4.2. Let $T$ be a controlling triangulation for $M$.

(a) For $x \notin \Gamma_T$, let $U$ be the branch of $\pi_T(x)$ containing $x$. Then the restriction of $M$ to $U$ splits as a direct sum in which for each summand $N$, there exists a constant $c > 0$ such that $s_i(N, T, y) = c$ for $i = 1, \ldots, \text{rank}(N)$ and $y \in U$.

(b) For $x \in \Gamma_T$, let $E$ be the open star around $x$ (i.e., the union of $x$ with the interiors of the edges of $\Gamma_T$ incident upon $x$) and put $U = \pi_T^{-1}(E)$. Then there exists a unique direct sum decomposition of $M$ whose base extension to $\mathcal{H}(x)$ is the spectral decomposition.

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Proof. Part (a) is immediate from Proposition 3.6.7. To obtain (b), first apply Lemma 5.4.1 to obtain a decomposition over an uncontrolled open neighborhood $V$ of $x$. Note that $V$ already contains all but finitely many branches of $X$ at $x$. For each remaining branch $W$, it remains to construct a second decomposition which agrees with the first one on $V \cap W$. If $W$ is $T$-nonskeletal, this is immediate from (a). If $W$ is $T$-skeletal, we may apply Lemma 3.7.3 to obtain a decomposition in which each summand has a unique spectral radius along $E \cap W$. Each summand has a unique limiting spectral radius at $x$. If we group summands by limiting spectral radius, the resulting decomposition agrees with the original one on $V \cap W$, as desired. \qed

Remark 5.4.3. The conclusion of Theorem 5.4.2(b) is best possible in certain senses. For one, one cannot ensure that the base extension of the decomposition to $H(y)$ is the spectral decomposition at any $y \in E \setminus \{x\}$, because of the coarsening step in the proof of Theorem 5.4.2. Similarly, one cannot extend the decomposition to another vertex of $\Gamma_T$.

Remark 5.4.4. The decompositions appearing in Theorem 5.4.2 are analogues of the good formal structures for formal meromorphic connections described in [Ked10b, Ked11a]. Additional analogues in the $p$-adic setting also appear in [KX10]. The decompositions given here can be used to obtain a global index formula for connections on analytic curves, in the style of the work of Robba [Rob75, Rob76, Rob84, Rob85] and Christol and Mebkhout [CM93, CM97, CM00, CM01]. Such a formula will appear in a forthcoming paper of Baldassarri and the author.

Remark 5.4.5. Using these results, it is tempting to look for a more global version of Theorem 3.8.21. When $p > 0$, one might even guess that every connection étale-locally satisfies the Robba condition. However, this guess is incorrect as shown by Remark 2.3.18, and it is not immediately obvious to us how to salvage the statement.

One motivation for doing so would be to show that the behavior of radii of convergence for connections arising from discrete representations of the geometric fundamental group, which can be explained in terms of Faber’s Berkovich-theoretic ramification locus [Fab13a, Fab13b], is in fact completely representative of the general case.

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References


And09 Y. André, Slope filtrations, Confluentes Math. 1 (2009), 1–85.


Structure of connections on nonarchimedean curves

BK F. Baldassarri and K. S. Kedlaya, Harmonic functions attached to meromorphic connections on non-archimedean curves, in preparation.


Fab13a X. Faber, Topology and geometry of the Berkovich ramification locus for rational functions, Manuscripta Math. 142 (2013), 439–474.


Structure of connections on nonarchimedean curves


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