A NOTE ON DIOPHANTINE APPROXIMATION

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1. Introduction

For convenience, we mention some notation and results, which we use very often:

(i) $[a_0, a_1, ..., a_n, ...]$ denotes an infinite simple continued fraction.

(ii) In a periodic continued fraction, for instance $[0, m, n, n^*, n, m, m^*]$, the convergent $[0, m, n, n, n, m, m, n, n, m, m]$ is written as $[0, m, n(n, n, m, m)_2]$ and likewise.

(iii) I denotes the set

$\{\xi/\xi = [a_0, a_1, ..., a_n, ...]$ such that $a_n \geqq 2$ from some point on$\}$.

(iv) If $\xi = [0, a_1, a_2, ..., a_n, ...]$, it is well known that

$$\left|\xi - \frac{p_n}{q_n}\right| = \frac{1}{\alpha_n q_n^2},$$

where α_n denotes $[0, a_n, ..., a_1] + [a_{n+1}, a_{n+2}, ...]$ and $p_n/q_n = [0, a_1, ..., a_n]$, the nth convergent to ξ; in particular $p_1/q_1 = 1/a_1$.

(v) Two irrational numbers ξ_1 and ξ_2 are equivalent (e.g. $\xi_1 \sim \xi_2$) if their simple continued fractions tally from some point on.

In (3), the chain of theorems for at least m approximations corresponding to the chain of theorems of Markov (1) is given. Theorems of the former chain becomes those of the latter if one allows $m \to \infty$.

In this paper, we discuss the third theorem of the former chain (namely Theorem 1 here) in a different manner. Following a technique of (3) we work out a lemma first and, on the basis of this lemma, we recover Theorem 1 and obtain another approximation theorem also. We now state all these results.

Lemma. *If ξ is equivalent to $[0, n^*, m, m, n^*]$, $(n > m)$, then the inequality*

$$\left|\xi - \frac{h}{k}\right| \leqq \frac{1}{A_p k^2}, \tag{A}$$

where $A_p = [0, n, m, m(n, n, m, m)_{p-1}] + [n^*, m, m, n^*]$, has at least p solutions in coprime integers h and k with $k > 0$. Further A_p cannot be improved as when $\xi = [0, n^*, m, m, n^*]$ or $[0, m^*, m, n, n^*]$ the inequality (A) has exactly p solutions.