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Note on the Kasparov Product of C^*-algebra Extensions

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Abstract. Using the Dadarlat isomorphism, we give a characterization for the Kasparov product of C^*-algebra extensions. A certain relation between $KK(A, \mathcal{Q}(B))$ and $KK(A, \mathcal{Q}(\mathcal{K}B))$ is also considered when B is not stable, and it is proved that $KK(A, \mathcal{Q}(B))$ and $KK(A, \mathcal{Q}(\mathcal{K}B))$ are not isomorphic in general.

1 Introduction

KK-theory was introduced in [8] by G. Kasparov, which generalizes both K-theory and extension theory of C^*-algebras. One can refer to [1, 7] for details. Recently, KK-theory has become more and more important in many branches of mathematics, such as functional analysis, algebra, and topology, etc. It is a powerful tool for classifying mathematical objects. It plays an especially crucial role in the classification of C^*-algebras and their extensions (see [5, 10–14, 16, 17, 20, 21]). It is also well known that KK-theory is famous for its extreme technicality and high complexity. Since it came into being, many mathematicians have tried to simplify it and to give some acceptable descriptions. As we know, there are several successful characterizations and pictures for KK-theory, for example, [3, 6, 7].

In the classification of C^*-algebras and their extensions, an effective approach is to turn Ext-groups into KK^1-groups, so that the Kasparov product and the UCT can be used to cope with questions on extensions. The Kasparov product is the soul of KK-theory. A special case of the Kasparov product is the pairing

$$KK^i(A,B) \otimes KK^j(B,C) \longrightarrow KK^{i+j}(A,C).$$

The Kasparov product has a wide range of applications, but it is still very hard to get a simple description for the general case. An advantage of the Cuntz picture is that the product of two elements is represented by composition of certain homomorphisms when $i = j = 0$. But there is no similar result for the case $i = j = 1$. It seems that one may use the suspension transformations to turn KK^1-groups into KK^0-groups and then obtain their product by the Cuntz picture. Unfortunately, the suspension transformations are induced by multiplying by a fixed extension (see [1, 19.2]). So there is no effective way to describe the product of two C^*-algebra extensions in KK-theory.

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One purpose of this paper is to give a characterization for the Kasparov product of two extensions. Our main tool is the Dadarlat isomorphism, which first appeared in [4]. It turns Ext-groups into KK^0-groups by the Busby invariants instead of the suspension transformations. With the help of the Dadarlat isomorphism, we represent the Kasparov product of two extensions as a composition of two homomorphisms (see Theorem 3.3 and 3.4).

On the other hand, the group $KK(A, \mathcal{Q}(B))$ is very useful in the classification of C^*-algebra extensions. Its quotient by pure extensions is denoted by $KL(A, \mathcal{Q}(B))$ when A satisfies the UCT. H. Lin has used $KL(A, \mathcal{Q}(B))$ to classify certain extensions in [12, 13]. When B is stable, by the Dadarlat isomorphism we have

$$KK\big(A, \mathcal{Q}(B)\big) = KK\big(A, \mathcal{Q}(\mathcal{K}B)\big) \cong \mathrm{Ext}(A, B).$$

But $KK(A, \mathcal{Q}(B))$ may be quite different from $KK(A, \mathcal{Q}(\mathcal{K}B))$ and $\mathrm{Ext}(A, B)$ when B is not stable. In this note, we also consider a relation between $KK(A, \mathcal{Q}(B))$ and $KK(A, \mathcal{Q}(\mathcal{K}B))$. One natural homomorphism is given, and we prove that it is not an isomorphism in general.

2 Preliminaries

In this section, we give some notations and results on KK-theory and C^*-algebra extensions. One can see [1–3, 6, 7, 9], *etc.*, for details.

Let A and B be C^*-algebras. Let

$$e\colon\ 0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0$$

be an extension of A by B with Busby invariant $\tau\colon A \to \mathcal{Q}(B)$, where $\mathcal{Q}(B) = M(B)/B$ is the corona algebra of B with the quotient map $\pi\colon M(B) \to \mathcal{Q}(B)$. The above extension e is called trivial if the exact sequence splits.

We call e essential if its Busby invariant τ is an injective homomorphism. Denote the set of all essential extensions by $\mathrm{E}(A, B)$.

Let $\mathcal{K}$ be the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space. For any C^*-algebra B, we call $\mathcal{K} \otimes B$ the stablization of B and often denote it by $\mathcal{K}B$. If $\mathcal{K}B \cong B$, then B is said to be stable.

When B is stable, the sum of two extensions τ_1 and τ_2 is defined to be the homomorphism $\tau_1 \oplus \tau_2$, where

$$\tau_1 \oplus \tau_2\colon A \longrightarrow \mathcal{Q}(B) \oplus \mathcal{Q}(B) \subseteq M_2\big(\mathcal{Q}(B)\big) \cong \mathcal{Q}(B).$$

Two extensions τ_1 and τ_2 are said to be stably unitarily equivalent if there are two trivial extensions σ_1 and σ_2 and a unitary $u \in \mathcal{Q}(B)$ such that $\tau_2 \oplus \sigma_2 = u(\tau_1 \oplus \sigma_1)u^*$.

Let $[e]$ or $[\tau]$ denote the stable unitary equivalence class of an extension e with Busby invariant τ. Denote by $\mathrm{Ext}(A, B)$ the set of equivalence classes of all essential extensions of A by $\mathcal{K} \otimes B$. It is easy to see that $\mathrm{Ext}(A, B)$ is a commutative semigroup with respect to the above addition.

If A is a separable nuclear C^*-algebra and B is a σ-unital C^*-algebra, then $\operatorname{Ext}(A, B)$ becomes an abelian group. It is known that $\operatorname{Ext}(A, B) \cong KK^1(A, B)$ by [8, Theorem 1, p. 562].

KK-theory was introduced by Kasparov in [8]. There are some nice pictures for KK-groups. One can see [3, 6, 7, 18] for details. The main properties of KK-groups are contained in the following theorem.

Theorem 2.1 (Kasparov [8]) (i) *For each C^*-algebra A, $KK^*(A, -)$ is a covariant functor from the category of C^*-algebra to the category of abelian groups.*
(ii) *For each C^*-algebra B, $KK^*(-, B)$ is a contravariant functor from the category of C^*-algebra to the category of abelian groups.*
(iii) *Bott Periodicity: for any A and B, there are several isomorphisms*

$$KK^1(A, B) \cong KK(A, SB) \cong KK(SA, B),$$

$$KK(A, B) \cong KK^1(A, SB) \cong KK^1(SA, B),$$

$$KK(S^2A, B) \cong KK(A, S^2B) \cong KK(SA, SB).$$

(iv) *Kasparov product: there is a natural product*

$$KK^i(A, B) \otimes KK^j(B, C) \longrightarrow KK^{i+j}(A, C).$$

(v) *Stability: the homomorphisms $A \to \mathcal{K} \otimes A$ and $B \to \mathcal{K} \otimes B$ induced by a minimal projection in $\mathcal{K}$ induce natural isomorphisms*

$$KK^*(\mathcal{K} \otimes A, B) \cong KK^*(A, B) \quad \text{and} \quad KK^*(A, B) \cong KK^*(A, \mathcal{K} \otimes B).$$

(vi) *Split exactness: $KK^*(A, -)$ and $KK^*(-, B)$ map split sequences of separable C^*-algebras to split sequences of abelian groups.*
(vii) *Homotopy invariance: any homotopic homomorphisms induce equal homomorphisms between KK-groups.*

Let A be a separable C^*-algebra. Recall that A satisfies the Universal Coefficient Theorem (UCT) if for any σ-unital C^*-algebras B, there is a short exact sequence

$$0 \to \operatorname{Ext}\big(K_*(A), K_*(B)\big) \to KK^*(A, B) \to \operatorname{Hom}\big(K_*(A), K_*(B)\big) \to 0.$$

Let $\mathcal{N}$ be the bootstrap class defined in [19]. Then A satisfies the UCT if $A \in \mathcal{N}$.

Let $e \in \mathrm{E}(A, B)$ and let C and D be C^*-algebras. Suppose that $\beta\colon B \to C$ is a surjective homomorphism and $\alpha \in \operatorname{Hom}(D, A)$. Then there are two induced extensions $\beta \circ e$ and $e \circ \alpha$ making the following diagrams commute respectively:

$$\begin{array}{rcccccccc} e \circ \alpha : 0 & \longrightarrow & B & \longrightarrow & E' & \longrightarrow & D & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow \alpha & & \\ e : 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0, \end{array}$$

$$\begin{array}{rcccccccc} e : 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow \beta & & \downarrow & & \| & & \\ \beta \circ e : 0 & \longrightarrow & C & \longrightarrow & E'' & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

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One can see [15] or [17] for more details.

3 Main Results

Let A, B, D be C^*-algebras. Recall from [1, 17.8.5] that there is a homomorphism

$$T_D\colon KK^*(A,B) \longrightarrow KK^*(A\otimes D, B\otimes D)$$

that is natural in each variable.

Let $S = C_0(0,1)$ and $C = C_0(0,1]$. Then there is an extension

$$\sigma_1\colon 0 \longrightarrow S \longrightarrow C \longrightarrow \mathbb{C} \longrightarrow 0.$$

Let u be the unilateral shift and $\mathcal{T}_0 = C^*(u^* - 1)$. Then we have an extension

$$\sigma_2\colon 0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T}_0 \longrightarrow S \longrightarrow 0.$$

Set

$$x_0 = [\sigma_1] \in KK^1(\mathbb{C}, S) \quad \text{and} \quad y_0 = [\sigma_2] \in KK^1(S, \mathbb{C}).$$

Then $T_B(x_0) \in KK^1(B, SB)$ and $T_B(y_0) \in KK^1(SB, B)$. By [1, 19.2] the suspension isomorphisms and Bott periodicity of KK-groups are induced by product with one of $T_B(x_0)$, $T_B(y_0)$, $T_A(x_0)$, and $T_A(y_0)$.

For any C^*-algebras A, B, and C, define

$$\beta_C\colon KK^1(B,C) \to KK(B, SC) \quad \text{by} \quad \beta_C(y) = y \otimes T_C(x_0)$$

and

$$\beta^A\colon KK^1(A,B) \to KK(SA, B) \quad \text{by} \quad \beta^A(x) = T_A(y_0) \otimes x.$$

Then the following diagram is commutative

$$\begin{array}{ccc} KK^1(A,B)\otimes KK^1(B,C) & \longrightarrow & KK(A,C) \\ \downarrow \beta^A \qquad \downarrow \beta_C & & \beta^A \downarrow \beta_C \\ KK(SA,B)\otimes KK(B,SC) & \longrightarrow & KK(SA,SC). \end{array}$$

Hence we obtain the product of two extensions formally, but when we notice that

$$\beta^A(x)\otimes\beta_C(y) = T_A(y_0)\otimes x\otimes y\otimes T_C(x_0),$$

we realize that it is a product of four extensions! So we should avoid the suspension isomorphisms when we look for an approach to describe products of extensions.

In the following lemma, we identify $\mathrm{Ext}(A,B)$ with $KK^1(A,B)$ in the sense of Kasparov when A is a separable nuclear C^*-algebra and B is a σ-unital C^*-algebra. This identification is very useful.

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Lemma 3.1 ([4, Proposition 4.2]) *Suppose that A is a separable nuclear C^*-algebra and B is a σ-unital C^*-algebra. Then there is a natural isomorphism $\Phi_B\colon \operatorname{Ext}(A,B) \to KK(A, \mathcal{Q}(\mathcal{K}B))$ such that $\Phi_B([e]) = KK(\tau_e)$ for any extension e with Busby invariant τ_e.*

Let $e_1\colon 0 \to \mathcal{K} \otimes B \to E_1 \to A \to 0$ and $e_2\colon 0 \to \mathcal{K} \otimes C \to E_2 \to B \to 0$ be two extensions with Busby invariants $\tau_1\colon A \to \mathcal{Q}(\mathcal{K}B)$ and $\tau_2\colon B \to \mathcal{Q}(\mathcal{K}C)$, respectively. For any C^*-algebra D, suppose that e_D is the extension $0 \to \mathcal{K}D \to M(\mathcal{K}D) \to \mathcal{Q}(\mathcal{K}D) \to 0$ with Busby invariant $\tau_D\colon \mathcal{Q}(\mathcal{K}D) \xrightarrow{\text{id}} \mathcal{Q}(\mathcal{K}D)$.

By six-term exact sequence in KK-theory, we obtain the Dadarlat isomorphisms

$$KK\big(A, \mathcal{Q}(\mathcal{K}D)\big) \overset{\otimes[e_D]}{\longrightarrow} KK^1(A,D) \quad \text{and} \quad KK^1\big(A, \mathcal{Q}(\mathcal{K}D)\big) \overset{\otimes[e_D]}{\longrightarrow} KK(A,D).$$

Hence, we have an isomorphism $\phi_C\colon KK^1(B,C) \to KK(B, \mathcal{Q}(\mathcal{K}C))$ such that

$$\phi_C([e_2]) = [e_2] \otimes [e_C]^{-1} = KK(\tau_2).$$

By the associativity of the Kasparov product, we have

$$[e_1] \otimes \phi_C([e_2]) = [e_1] \otimes \big([e_2] \otimes [e_C]^{-1}\big) = \big([e_1] \otimes [e_2]\big) \otimes [e_C]^{-1}.$$

Note that $[e_1] \otimes \phi_C([e_2]) = [e_1] \otimes KK(\tau_2) = [\tau_2 \circ e_1]$. Hence we have $\phi_C([e_1] \otimes [e_2]) = [e_1] \otimes \phi_C([e_2]) = [\tau_2 \circ e_1]$. So we obtain the following lemma.

Lemma 3.2 *Let A be a separable nuclear C^*-algebra. Then there is a commutative diagram*

$$\begin{array}{ccccc}
KK^1(A,B)\otimes & KK^1(B,C) & \longrightarrow & KK(A,C) \\
\big\downarrow \text{id} & \big\downarrow \phi_C & & \big\downarrow \phi_C \\
KK^1(A,B)\otimes & KK\big(B, \mathcal{Q}(\mathcal{K}C)\big) & \longrightarrow & KK^1\big(A, \mathcal{Q}(\mathcal{K}C)\big).
\end{array}$$

Theorem 3.3 *Let A be a separable nuclear C^*-algebra. Suppose that B, G are σ-unital C^*-algebras such that $KK(B,G) = \{KK(\varphi) \mid \varphi\colon B \longrightarrow G \text{ is injective}\}$. Then there is a multiplication (still denoted by)*

$$\otimes\colon KK\big(A, \mathcal{Q}(\mathcal{K}B)\big) \times KK(B,G) \longrightarrow KK\big(A, \mathcal{Q}(\mathcal{K}G)\big)$$

such that the following diagram commutes:

$$\begin{array}{ccccc}
KK^1(A,B) & \otimes KK(B,G) & \longrightarrow & KK^1(A,G) \\
\big\downarrow \Phi_B & \big\downarrow \text{id} & & \big\downarrow \Phi_G \\
KK\big(A, \mathcal{Q}(\mathcal{K}B)\big) & \otimes KK(B,G) & \longrightarrow & KK\big(A, \mathcal{Q}(\mathcal{K}G)\big).
\end{array}$$

Proof First, we will define a multiplication

$$\otimes\colon KK\big(A, \mathcal{Q}(\mathcal{K}B)\big) \times KK(B,G) \longrightarrow KK\big(A, \mathcal{Q}(\mathcal{K}G)\big).$$

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For any $y \in KK(B,G)$ there is a homomorphism $\phi' \colon \mathcal{K} \otimes B \to \mathcal{K} \otimes G$ with $KK(\phi') = y$. By [7, Theorem 1.3.16] there exists a quasi-unital homomorphism $\phi \colon \mathcal{K} \otimes B \to \mathcal{K} \otimes G$ such that ϕ is homotopic to ϕ'. Then there is a unique extension $\bar{\phi} \colon M(\mathcal{K}B) \to M(\mathcal{K}G)$. Denote the induced map by $\psi \colon \mathcal{Q}(\mathcal{K}B) \to \mathcal{Q}(\mathcal{K}G)$. Then we have a commutative diagram

$$\begin{array}{ccccccccc} e_B : 0 & \longrightarrow & \mathcal{K}B & \longrightarrow & M(\mathcal{K}B) & \longrightarrow & \mathcal{Q}(\mathcal{K}B) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \bar{\phi}} & & \downarrow{\scriptstyle \psi} & & \\ e_G : 0 & \longrightarrow & \mathcal{K}G & \longrightarrow & M(\mathcal{K}G) & \longrightarrow & \mathcal{Q}(\mathcal{K}G) & \longrightarrow & 0. \end{array}$$

For any $x \in KK(A, \mathcal{Q}(\mathcal{K}B))$, by the Dadarlat isomorphism there is an essential extension $e \colon 0 \to \mathcal{K}B \to E \to A \to 0$ with Busby invariant $\tau_e \colon A \to \mathcal{Q}(\mathcal{K}B)$ such that $x = KK(\tau_e) = \Phi_B([e])$. Then the diagram

$$\begin{array}{ccccccccc} e : 0 & \longrightarrow & \mathcal{K}B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \tau_e} & & \\ e_B : 0 & \longrightarrow & \mathcal{K}B & \longrightarrow & M(\mathcal{K}B) & \longrightarrow & \mathcal{Q}(\mathcal{K}B) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \bar{\phi}} & & \downarrow{\scriptstyle \psi} & & \\ e_G : 0 & \longrightarrow & \mathcal{K}B & \longrightarrow & M(\mathcal{K}B) & \longrightarrow & \mathcal{Q}(\mathcal{K}B) & \longrightarrow & 0 \end{array}$$

commutes. Hence $\phi \circ e = e_G \circ (\psi \circ \tau_e)$. Furthermore,

$$[e] \otimes KK(\phi) \otimes [e_G]^{-1} = KK(\tau_e) \otimes KK(\psi).$$

So there is an additive map $\otimes \colon KK(A, \mathcal{Q}(\mathcal{K}B)) \times KK(B,G) \to KK(A, \mathcal{Q}(\mathcal{K}G))$ with $x \otimes y = KK(\psi \circ \tau_e)$.

Next, we need to show that the above product is well defined.

Let $x \in KK(A, \mathcal{Q}(\mathcal{K}B))$ and suppose that $e_i \colon 0 \to \mathcal{K}B \to E_i \to A \to 0$ are essential extensions with Busby invariant τ_i such that $\Phi_B(e_1) = \Phi_B(e_2) = x$. Since Φ_B is an isomorphism, we have $[e_1] = [e_2]$ in $KK^1(A,B)$ and $KK(\tau_1) = KK(\tau_2)$ in $KK(A, \mathcal{Q}(\mathcal{K}B))$.

Let $y \in KK(B,G)$ and suppose that $\phi_i \colon \mathcal{K}B \to \mathcal{K}G$ are two quasi-unital homomorphisms such that $KK(\phi_1) = KK(\phi_2) = y$. Then there are homomorphisms $\bar{\phi}_i \colon M(\mathcal{K}B) \to M(\mathcal{K}B)$ and $\psi_i \colon \mathcal{Q}(\mathcal{K}B) \to \mathcal{Q}(\mathcal{K}G)$ such that

$$\begin{array}{ccccccccc} e_B : 0 & \longrightarrow & \mathcal{K}B & \longrightarrow & M(\mathcal{K}B) & \longrightarrow & \mathcal{Q}(\mathcal{K}B) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \bar{\phi}_i} & & \downarrow{\scriptstyle \psi_i} & & \\ e_G : 0 & \longrightarrow & \mathcal{K}G & \longrightarrow & M(\mathcal{K}G) & \longrightarrow & \mathcal{Q}(\mathcal{K}G) & \longrightarrow & 0 \end{array}$$

commutes. By the above proof, we have $\phi_i \circ e_i = e_G \circ (\psi_i \circ \tau_i)$ and then

$$[e_i] \otimes KK(\phi_i) \otimes [e_G]^{-1} = KK(\tau_i) \otimes KK(\psi_i).$$

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Since $KK(\phi_1) = KK(\phi_2)$ and $[e_1] = [e_2]$, we have

$$KK(\tau_1) \otimes KK(\psi_1) = KK(\tau_2) \otimes KK(\psi_2).$$

Hence $KK(\psi_1 \circ \tau_1) = KK(\psi_2 \circ \tau_2)$.

Finally, note that

$$\Phi_G\big([e] \otimes y\big) = \big([e] \otimes KK(\phi)\big) \otimes [e_G]^{-1}$$

and

$$\Phi_B([e]) \otimes y = [\tau_e] \otimes KK(\phi) = KK(\psi \circ \tau_e).$$

Since $[e] \otimes KK(\phi) = KK(\psi \circ \tau_e) \otimes [e_G]$, we have $\Phi_G([e] \otimes y) = \Phi_B([e]) \otimes y$. ■

Theorem 3.4 *Suppose that A is a separable nuclear C^*-algebra and B, C are σ-unital C^*-algebras. Then the multiplication defined in Theorem 3.3 is natural and makes the following diagram commute:*

$$\begin{array}{ccccc} KK^1(A,B) & \otimes & KK^1(B,C) & \longrightarrow & KK(A,C) \\ \downarrow \Phi_B & & \downarrow \Phi_C & & \downarrow \Phi_{\mathcal{Q}(\mathcal{K}C)} \circ \Phi_C \\ KK\big(A, \mathcal{Q}(\mathcal{K}B)\big) & \otimes & KK\big(B, \mathcal{Q}(\mathcal{K}C)\big) & \longrightarrow & KK\Big(A, \mathcal{Q}\big(\mathcal{K}\mathcal{Q}(\mathcal{K}C)\big)\Big). \end{array}$$

Proof By Lemma 3.2 and Theorem 3.3 (let $G = \mathcal{Q}(\mathcal{K}C)$), we have a commutative diagram

$$\begin{array}{ccccc} KK^1(A,B) & \otimes & KK^1(B,C) & \longrightarrow & KK(A,C) \\ \downarrow \mathrm{id} & & \downarrow \Phi_C & & \downarrow \Phi_C \\ KK^1(A,B) & \otimes & KK\big(B, \mathcal{Q}(\mathcal{K}C)\big) & \longrightarrow & KK^1\big(A, \mathcal{Q}(\mathcal{K}C)\big) \\ \downarrow \Phi_B & & \downarrow \mathrm{id} & & \downarrow \Phi_{\mathcal{Q}(\mathcal{K}C)} \\ KK\big(A, \mathcal{Q}(\mathcal{K}B)\big) & \otimes & KK\big(B, \mathcal{Q}(\mathcal{K}C)\big) & \longrightarrow & KK\Big(A, \mathcal{Q}\big(\mathcal{K}\mathcal{Q}(\mathcal{K}C)\big)\Big). \end{array}$$

Composing these homomorphisms, we obtain the commutative diagram required above.

It follows from Lemma 3.1 that the Dadarlat isomorphism is natural, and so is the multiplication defined in Theorem 3.3 by the above commutative diagram. ■

Remark 3.5 Since the vertical maps in the diagram in Theorem 3.4 are natural isomorphisms, one can view the product defined in Theorem 3.3 as a description of the Kasparov product of two C^*-algebra extensions. With this characterization, the Kasparov product of extensions is turned into a composition of two homomorphisms.

Next we consider the relation between $KK(A, \mathcal{Q}(B))$ and $KK(A, \mathcal{Q}(\mathcal{K}B))$ when B is σ-unital, but is nonunital and nonstable. The following results illustrate that there

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is a natural homomorphism between the two groups, but they are not isomorphic in general. In order to show these facts, we need to begin with some notations and definitions.

Define $\beta\colon B \to \mathcal{K}\otimes B$ by $b \mapsto e_{11}\otimes b$, where e_{11} is a minimal projection in $\mathcal{K}$. Similarly, we can define $\eta\colon M(B)\to\mathcal{K}\otimes M(B)$ and $\gamma\colon \mathcal{Q}(B)\to\mathcal{K}\otimes\mathcal{Q}(B)$. Let e_0 be the extension $0\to B\to M(B)\to\mathcal{Q}(B)\to 0$. By [6, 1.3] there are homomorphisms $\bar\beta\colon M(B)\to M(\mathcal{K}B)$ and $\widetilde\beta\colon\mathcal{Q}(B)\to\mathcal{Q}(\mathcal{K}B)$ such that the following diagram is commutative

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \longrightarrow & M(B) & \longrightarrow & \mathcal{Q}(B) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\bar\beta} & & \downarrow{\scriptstyle\widetilde\beta} & & \\
0 & \longrightarrow & \mathcal{K}B & \longrightarrow & M(\mathcal{K}B) & \longrightarrow & \mathcal{Q}(\mathcal{K}B) & \longrightarrow & 0.
\end{array}$$

On the other hand, tensoring e_0 by $\mathcal{K}$, we obtain an extension $\mathcal{K}\otimes e_0\colon 0\to\mathcal{K}\otimes B\to\mathcal{K}\otimes M(B)\to\mathcal{K}\otimes\mathcal{Q}(B)\to 0$. Let $\psi\colon\mathcal{K}\otimes\mathcal{Q}(B)\to\mathcal{Q}(\mathcal{K}B)$ be the Busby invariant of $\mathcal{K}\otimes e_0$. Then the following diagrams are commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}\otimes B & \longrightarrow & \mathcal{K}\otimes M(B) & \longrightarrow & \mathcal{K}\otimes\mathcal{Q}(B) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle 1_{\mathcal{K}}\otimes\beta} & & \downarrow{\scriptstyle 1_{\mathcal{K}}\otimes\bar\beta} & & \downarrow{\scriptstyle 1_{\mathcal{K}}\otimes\widetilde\beta} & & \\
0 & \longrightarrow & \mathcal{K}\otimes\mathcal{K}B & \longrightarrow & \mathcal{K}\otimes M(\mathcal{K}B) & \longrightarrow & \mathcal{K}\otimes\mathcal{Q}(\mathcal{K}B) & \longrightarrow & 0,
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \longrightarrow & M(B) & \longrightarrow & \mathcal{Q}(B) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & \mathcal{K}\otimes B & \longrightarrow & \mathcal{K}\otimes M(B) & \longrightarrow & \mathcal{K}\otimes\mathcal{Q}(B) & \longrightarrow & 0\\
 & & \Vert & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} & & \\
0 & \longrightarrow & \mathcal{K}B & \longrightarrow & M(\mathcal{K}B) & \longrightarrow & \mathcal{Q}(\mathcal{K}B) & \longrightarrow & 0.
\end{array}$$

By the suspension of KK-groups, we have

$$KK\big(A,\mathcal{Q}(B)\big)\cong KK^1\big(SA,\mathcal{Q}(B)\big),\quad KK\big(A,\mathcal{Q}(\mathcal{K}B)\big)\cong KK^1(SA,\mathcal{Q}(\mathcal{K}B)).$$

Define

$$\rho_i\colon KK^1\big(SA,\mathcal{Q}(B)\big)\longrightarrow KK^1\big(SA,\mathcal{Q}(\mathcal{K}B)\big)$$

by $\rho_1([e]) = [(1_{\mathcal{K}}\otimes\widetilde\beta)\circ e]$ and $\rho_2([e]) = [\psi\circ e]$ for any essential extension e, respectively.

Proposition 3.6 *Let $\rho_1,\rho_2\colon KK^1(SA,\mathcal{Q}(B))\to KK^1(SA,\mathcal{Q}(\mathcal{K}B))$ be the maps defined above. Then $\rho_1=\rho_2$.*

Proof For any $x\in KK^1(SA,\mathcal{Q}(B))$, there is an essential extension

$$e\colon 0\longrightarrow\mathcal{K}\mathcal{Q}(B)\longrightarrow E\longrightarrow SA\longrightarrow 0$$

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such that $[e] = x$ in $KK^1(SA, \mathcal{Q}(B))$. Define $\psi_1 \colon \mathcal{K}\mathcal{Q}(B) \to \mathcal{K}\mathcal{Q}(\mathcal{K}B)$ by $\psi_1(z) = e_{11} \otimes \psi(z)$. Set $e_1 = (1_{\mathcal{K}} \otimes \widetilde{\beta}) \circ e$ and $e_2 = \psi \circ e$. Hence $\rho_i(x) = [e_i]$ in $KK^1(SA, \mathcal{Q}(\mathcal{K}B))$ and there are two commutative diagrams

$$\begin{array}{ccccccccc}
e : 0 & \longrightarrow & \mathcal{K}\mathcal{Q}(B) & \longrightarrow & E & \longrightarrow & SA & \longrightarrow & 0 \\
 & & \big\downarrow {\scriptstyle 1_{\mathcal{K}} \otimes \widetilde{\beta}} & & \big\downarrow & & \big\| & & \\
e_1 : 0 & \longrightarrow & \mathcal{K}\mathcal{Q}(\mathcal{K}B) & \longrightarrow & E_1 & \longrightarrow & SA & \longrightarrow & 0,
\end{array}$$

$$\begin{array}{ccccccccc}
e : 0 & \longrightarrow & \mathcal{K}\mathcal{Q}(B) & \longrightarrow & E & \longrightarrow & SA & \longrightarrow & 0 \\
 & & \big\downarrow {\scriptstyle \psi_1} & & \big\downarrow & & \big\| & & \\
e_2 : 0 & \longrightarrow & \mathcal{K}\mathcal{Q}(\mathcal{K}B) & \longrightarrow & E_2 & \longrightarrow & SA & \longrightarrow & 0.
\end{array}$$

By the functoriality of the KK-groups, we have $\rho_1 = \widetilde{\beta}_*$ and $\rho_2 = \psi_*$, so they are group homomorphisms.

Since $\mathcal{K} \otimes B$ is an essential ideal in $\mathcal{K} \otimes M(B)$, ϕ is the inclusion map from $\mathcal{K} \otimes M(B)$ into $M(\mathcal{K}B)$. Set $p = e_{11} \otimes 1_{M(B)}$. By [6, 1.3] $\bar{\beta}$ is defined by

$$M(B) \xrightarrow{\beta_1} M\big(p(\mathcal{K} \otimes B)p\big) \cong pM(\mathcal{K} \otimes B)p \subset M(\mathcal{K} \otimes B),$$

where β_1 is the unique extension of β. Note that

$$M\big(p(\mathcal{K} \otimes B)p\big) \cong M(e_{11} \otimes B) \cong e_{11} \otimes M(B)$$

and $\beta(b) = e_{11} \otimes b$. Then $\bar{\beta}$ is the map $M(B) \cong e_{11} \otimes M(B) \subset M(\mathcal{K}B)$. Hence

$$1_{\mathcal{K}} \otimes \bar{\beta} \colon \mathcal{K}M(B) \longrightarrow e_{11} \otimes \mathcal{K}M(B) \subset \mathcal{K}M(\mathcal{K}B).$$

Define $\phi_1 \colon \mathcal{K}M(C) \to \mathcal{K}M(\mathcal{K}B)$ by $\phi_1(c) = e_{11} \otimes \phi(c)$. Then we have

$$\phi_1 : \mathcal{K}M(B) \subset M(\mathcal{K}B) \cong e_{11} \otimes M(\mathcal{K}B) \subset \mathcal{K}M(\mathcal{K}B).$$

Hence $\phi_1 = 1_{\mathcal{K}} \otimes \bar{\beta}$. Note that $1_{\mathcal{K}} \otimes \widetilde{\beta}$ and ψ_1 are the induced maps of $1_{\mathcal{K}} \otimes \bar{\beta}$ and ϕ_1, respectively. Therefore, we have $\rho_1 = \rho_2$. ∎

Suppose that B is a simple C^*-algebra with real rank zero, stable rank one, and weakly unperforated K_0-group. Let q be a nonzero projection in B. Set

$$T_q = \{\tau : \tau(q) = 1, \tau \text{ is a trace on } B\}.$$

Denote the set of affine functions on T_q by $\mathrm{Aff}(T_q)$. Then there is a homomorphism

$$\rho_B \colon K_0(B) \longrightarrow \mathrm{Aff}(T_q)$$

defined by $\rho_B([p]) = \tau(p)$ for every projection p in any matrix algebra over B.

Lemma 3.7 ([13, Theorem 1.4 and Corollary 1.5]) *Let B be as above. Then*

(i) $K_0(M(B)) \cong \mathrm{Aff}(T_q)$ *and* $K_1(M(B)) = \{0\}$;

(ii) $K_1(M(B)/B) = \mathrm{Ker}(\rho_B)$.

Theorem 3.8 *Let ρ_1 be as in Proposition 3.6. Then there is a C^*-algebra B such that ρ_1 is not an isomorphism.*

Proof By the UCT, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Ext}(K_*(SA), K_*(\mathcal{Q}(B))) & \to & KK^1(SA, \mathcal{Q}(B)) & \to & \mathrm{Hom}(K_*(SA), K_*(\mathcal{Q}(B))) & \to 0 \\ & \downarrow K_*(\widetilde{\beta}) & & \downarrow \widetilde{\beta}_* & & \downarrow K_*(\widetilde{\beta}) & \\ 0 \to & \mathrm{Ext}(K_*(SA), K_*(\mathcal{Q}(\mathcal{K}B))) & \to & KK^1(SA, \mathcal{Q}(\mathcal{K}B)) & \to & \mathrm{Hom}(K_*(SA), K_*(\mathcal{Q}(\mathcal{K}B))) & \to 0 \end{array}$$

If ρ_1 is an isomorphism, it follows from the above diagram that $K_*(\widetilde{\beta})\colon K_*(\mathcal{Q}(B)) \to K_*(\mathcal{Q}(\mathcal{K}B))$ is an isomorphism.

By the naturality of the boundary maps in six-term exact sequence in K-theory and the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & M(B) & \longrightarrow & \mathcal{Q}(B) & \longrightarrow & 0 \\ & & \downarrow \beta & & \downarrow \bar{\beta} & & \downarrow \widetilde{\beta} & & \\ 0 & \longrightarrow & \mathcal{K}B & \longrightarrow & M(\mathcal{K}B) & \longrightarrow & \mathcal{Q}(\mathcal{K}B) & \longrightarrow & 0, \end{array}$$

we obtain a commutative diagram

$$\begin{array}{ccc} K_*(\mathcal{Q}(B)) & \xrightarrow{\delta_B} & K_*(B) \\ \downarrow K_*(\widetilde{\beta}) & & \| \\ K_*(\mathcal{Q}(\mathcal{K}B)) & \xrightarrow{\delta_{\mathcal{K}B}} & K_*(\mathcal{K}B). \end{array}$$

Since the index map $\delta_{\mathcal{K}B}$ is an isomorphism, so is δ_B. But by Lemma 3.7 there exists B such that δ_B is not an isomorphism. Hence ρ_1 is not an isomorphism in general. ■

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