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A bound of the number of weighted blow-ups to compute the minimal log discrepancy for smooth 3-folds

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Abstract

We study a pair consisting of a smooth 3-fold defined over an algebraically closed field and a "general" $\mathbb{R}$-ideal. We show that the minimal log discrepancy ("mld" for short) of every such a pair is computed by a prime divisor obtained by at most two weighted blow-ups. This bound is regarded as a weighted blow-up version of Mustaţă–Nakamura's conjecture. We also show that if the mld of such a pair is not less than 1, then it is computed by at most one weighted blow-up. As a consequence, ACC of mld holds for such pairs.

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1. *Introduction*

Throughout this paper, the base field k of varieties is an algebraically closed field of arbitrary characteristic. We study pairs $(A, \mathfrak{a})$ consisting of a smooth variety A of dimension $N > 1$ and an "$\mathbb{R}$-ideal" $\mathfrak{a}$ which means $\mathfrak{a} = \mathfrak{a}_1^{e_1} \cdots \mathfrak{a}_r^{e_r}$, where $\mathfrak{a}_i$'s are non-zero coherent ideal sheaves on A and $e = (e_1, \ldots, e_r) \in \mathbb{R}^r_{>0}$. We fix a closed point $0 \in A$.

The minimal log discrepancy ("mld" for short) $\mathrm{mld}(0; A, \mathfrak{a})$ is an important invariant to measure the singularity of the pair $(A, \mathfrak{a})$ at 0 and plays important roles in birational geometry. We consider every prime divisor over A with the center at 0 and construct a "good model" of the divisor to approximate the mld. The prototype is as follows:

THEOREM 1·1 ([9, 6]). *Assume $N = 2$. For every prime divisor E over A with the center at* 0*, there exists a prime divisor F obtained by one weighted blow-up with the center at* 0 *satisfying*

$$a(E; A, \mathfrak{a}) \geq a(F; A, \mathfrak{a}),$$

for every $\mathbb{R}$-ideal $\mathfrak{a}$ such that $a(E; A, \mathfrak{a}) \geq 0$.

The inequality in the theorem implies that F is a better divisor to approximate the mld. Therefore the theorem states that every prime divisor over A with the center at 0 has a better

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divisor which is obtained in a simple procedure. Here, we note that F is constructed from E and does not depend on the choice of an $\mathbb{R}$-ideal $\mathfrak{a}$.

Actually, in the paper [9] and [6], the main theorem is not stated in this form, but its proof shows Theorem 1·1. The paper [9] is for char $k = 0$, and the paper [6] is for char $k = p > 0$ and the main statements of both papers are in the following form:

COROLLARY 1·2 ([9, 6]). *Assume $N = 2$. Then, for every pair $(A, \mathfrak{a})$, the minimal log discrepancy* $\mathrm{mld}(0; A, \mathfrak{a})$ *is computed by a prime divisor obtained by one weighted blow-up.*

The corollary follows from the theorem immediately. See, for example, the proof of Corollary 1·9 in Section 5.

When we consider the case $N = 3$, we can see that one weighted blow-up is not sufficient to obtain a prime divisor computing the mld (see Example 3·3). On the other hand, in the example we can also show that the mld is computed by a prime divisor obtained by two weighted blow-ups. So it is natural to expect the following conjecture:

CONJECTURE 1·3. *Assume $N \geq 3$. For every prime divisor E over A with the center at 0, there exists a prime divisor F centered at 0 obtained by at most $N - 1$ weighted blow-ups satisfying*

$$a(E; A, \mathfrak{a}) \geq a(F; A, \mathfrak{a}),$$

for every $\mathbb{R}$-ideal $\mathfrak{a}$ such that $a(E; A, \mathfrak{a}) \geq 0$.

As an immediate consequence of the conjecture, we obtain the following:

CONJECTURE 1·4 (Corollary of Conjecture 1·3). *Assume $N \geq 3$. Then, for every pair $(A, \mathfrak{a})$, the minimal log discrepancy* $\mathrm{mld}(0; A, \mathfrak{a})$ *is computed by a prime divisor obtained by at most $N - 1$ weighted blow-ups.*

One of the motivations of the conjectures is that it is considered as a "weighted blow-up version" of Mustaţă–Nakamura Conjecture (MN-Conjecture for short):

CONJECTURE 1·5 (MN-Conjecture [13].) *Fix N and the exponent e of $\mathbb{R}$-ideals. Then, there exists a number $\ell_{N,e} \in \mathbb{N}$ depending only on N and e such that for any $\mathbb{R}$-ideal $\mathfrak{a}$ with the exponent e the minimal log discrepancy* $\mathrm{mld}(0; A, \mathfrak{a})$ *is computed by a prime divisor obtained by at most $\ell_{N,e}$ times blow-ups. Here, the blow-up means the "usual blow-up", i.e., blow-up with the center at an irreducible reduced closed subset.*

If this conjecture holds, then ACC Conjecture for these pairs holds ([13]), so it seems to be a significant conjecture. On the other hand, MN-Conjecture is equivalent to a reasonable conjecture on arc spaces ([5]), so it makes sense to study it.

Note that MN-Conjecture requires to fix an exponent e, while the weighted blow-up versions (Conjecture 1·3, 1·4) do not require it. Assume Conjecture 1·3 holds, it is also an interesting question whether the weights of the blow-ups can be bound uniformly in terms of exponents. This will strengthen the MN-Conjecture.

Another motivation of Conjecture 1·3 is for the project to bridge between positive characteristic and characteristic 0 ([5]). In [5], we have:

LEMMA 1·6. *Let $\mathfrak{a}$ be an $\mathbb{R}$-ideal on a smooth variety A_k over k ($\mathrm{char}\, k = p > 0$) and E a prime divisor over $(A_k, 0_k)$ computing* $\mathrm{mld}(0_k; A_k, \mathfrak{a})$.

If there exists an $\mathbb{R}$-ideal $\widetilde{\mathfrak{a}}$ on a smooth variety $A_{\mathbb{C}}$ over $\mathbb{C}$ and a prime divisor $\widetilde{E}$ over $(A_{\mathbb{C}}, 0_{\mathbb{C}})$, where $0_{\mathbb{C}} \in A_{\mathbb{C}}$ such that

1. $\widetilde{\mathfrak{a}}(\mathrm{mod}\, p) = \mathfrak{a}$ *(see [5] for the definition of* $(\mathrm{mod}\, p)$*)*
2. $a(\widetilde{E}; A_{\mathbb{C}}, \widetilde{\mathfrak{a}}) \le a(E; A_k, \mathfrak{a})$,

then, $\mathrm{mld}(0_{\mathbb{C}}; A_{\mathbb{C}}, \widetilde{\mathfrak{a}}) = \mathrm{mld}(0_k; A_k, \mathfrak{a})$.

Remark 1·7. In particular, if such $\widetilde{\mathfrak{a}}$ and $\widetilde{E}$ exist for every $\mathfrak{a}$ and E and assume that $\mathrm{mld}(0_k; A_k, \mathfrak{a})$ is computed by a divisor, then the set of $\mathrm{mld}(0_k; A_k, \mathfrak{a})$'s is contained in the set of $\mathrm{mld}(0_{\mathbb{C}}; A_{\mathbb{C}}, \mathfrak{b})$'s. Therefore, if we fix the exponent e and the dimension N of A_k, then the number of the values $\Lambda_e := \{\mathrm{mld}(0_k, A_k, \mathfrak{a}) \mid \mathfrak{a} \text{ is a } \mathbb{R}\text{- ideal with the exponent } e\}$ is finite for char $k > 0$, because it is proved to be finite in characteristic 0 by [8]. Similarly, if ACC holds in characteristic 0, then it also holds in positive characteristic.

Now, the problem is to construct appropriate $\widetilde{E}$ and $\widetilde{\mathfrak{a}}$ for given E and $\mathfrak{a}$. If Conjecture 1·3 holds, we can reduce this problem to a divisor F of special type (i.e., obtained by at most $N-1$ weighted blow-ups), which seems easier to handle.

The main results of this paper are the following:

THEOREM 1·8. *Assume $N = 3$. For every prime divisor E over A with the center at 0, there exists a prime divisor F centered at 0 obtained by at most two weighted blow-ups satisfying*

$$a(E; A, \mathfrak{a}) \ge a(F; A, \mathfrak{a}),$$

for every "general" $\mathbb{R}$-ideal $\mathfrak{a}$ for E such that $a(E; A, \mathfrak{a}) \ge 0$.

The terminology "general" will be defined in Definition 4·9. The weighted blow-ups will be constructed by "squeezed" blow-ups (see, Definition 4·4) depending only on E and it works for every general ideal. Here, "general" is necessary, because there exists an example of non-general ideal such that two squeezed blow-ups do not give the required divisor in the theorem (cf. Example 5·5). But it does not give a counter example for Conjecture 1·3, indeed for the example there exists another sequence of weighted blow-ups to obtain the required divisor (see, also Example 5·5).

As a corollary we obtain:

COROLLARY 1·9. *Assume $N = 3$. Then, for every pair $(A, \mathfrak{a})$ with a "general" $\mathbb{R}$-ideal $\mathfrak{a}$, the minimal log discrepancy* $\mathrm{mld}(0; A, \mathfrak{a})$ *is computed by a prime divisor obtained by at most two weighted blow-ups.*

It is known as the Zariski's sequence that every prime divisor E over A with the center at 0 is obtained by successive usual blow-ups from A, such that the centers of blow-ups are the center of E on each step ([11, VI, 1·3]). The following corollary shows that in some cases, we obtain the two weighted blow-ups to compute the mld by just looking at the center of the second blow-up in the Zariski's sequence.

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COROLLARY 1·10 (Corollary 5·9). *Assume* $N = 3$. *Let* E *be a prime divisor over* A *computing* $\mathrm{mld}(0; A, \mathfrak{a})$ *for a pair* $(A, \mathfrak{a})$. *Let* $A_1 \longrightarrow A$ *be the first usual blow-up with the center at* 0 *in the Zariski's sequence. Assume that the center* $C \subset A_1$ *of* E *is a curve of degree* ≥ 2 *in the exceptional divisor* $E_1 \simeq \mathbb{P}^2$. *Then a weighted blow-up which is called "squeezed blow-up" at* C *gives a divisor computing* $\mathrm{mld}(0; A, \mathfrak{a})$.

Note that in this case the first blow-up is also a squeezed blow-up. Example 3·3 is just in this case. In Section 5, we show a more general corollary. On the other hand, if we restrict to the case $\mathrm{mld} \geq 1$, then we have the following:

THEOREM 1·11. *Assume* $N = 3$. *Then, for every general pair* $(A, \mathfrak{a})$ *with* $\mathrm{mld}(0; A, \mathfrak{a}) \geq 1$, *the minimal log discrepancy is computed by a prime divisor obtained by one weighted blow-up.*

COROLLARY 1·12. *Assume* $N = 3$. *In*

$$\Lambda = \{(A, \mathfrak{a}) \mid \ \mathrm{mld}(0; A, \mathfrak{a}) \geq 1 \ \textit{with general} \ \ \mathfrak{a}\}$$

the Mustaţă–Nakamura Conjecture holds and also the ACC Conjecture holds for char $k \geq 0$. *Here, ACC Conjecture means that the set of* $\mathrm{mld}(0; A, \mathfrak{a})$ *for the pairs in the subset* $\Lambda_J \subset \Lambda$ *consisting of* $\mathbb{R}$*-pairs with the exponents in* $J \subset \mathbb{R}_{>0}$ *satisfies the Ascending Chain Condition. Here,* J *is a DCC set.*

The corollary follows from Theorem 1·11 in the same way as in the proof of [**6**, corollary 1·6], since the mld is computed by one weighted blow-up.

This paper is organised as follows: in Section 2 we prepare basic terminologies which will be used in this paper. In Section 3 we discuss about weighted blow-up at a (not necessarily closed) smooth point and basic formula on weighted projective space, that is the exceptional divisor appearing in a weighted blow-up. In Section 4 we construct an appropriate regular system of parameter (RSP for short) with the weight, in order to make a weighted blow-up. In Section 5 we give the proofs of the main results.

2. *Preliminaries*

Let A be an N-dimensional smooth variety defined over an algebraically closed field k. We fix a closed point $0 \in A$.

Definition 2·1. We call E a *prime divisor over* A, if there is a proper birational morphism $\varphi: A' \longrightarrow A$ from a normal variety A' on which E is an irreducible divisor. The generic point $P \in A$ of the image $\varphi(E)$ is called the *center of* E on A. In this case, we sometimes call E a prime divisor over (A, P).

Definition 2·2. For a prime divisor E over a non-singular variety A, let $\varphi: A' \longrightarrow A$ be a proper birational morphism with normal A' such that E appears on A'. Let k_E (or sometimes written as $k_{E/A}$) be the coefficient of the relative canonical divisor $K_{A'/A}$ at E and v_E the valuation defined by the prime divisor E. Here, note that k_E ($k_{E/A}$) does not depend on the choice of A'.

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Let $\mathfrak{a}$ be an $\mathbb{R}$-ideal on A as in the beginning of the first section and e_i's are the exponents. The *log discrepancy* of the pair $(A, \mathfrak{a})$ at E is defined as

$$a(E; A, \mathfrak{a}) := k_E - \sum_i e_i v_E(\mathfrak{a}_i) + 1$$

and the *minimal log discrepancy* of the pair at a closed point 0 is defined as

$$\mathrm{mld}(0; A, \mathfrak{a}) := \inf\{a(E; A, \mathfrak{a}) \mid E \text{ prime divisor over } A \text{ with the center at } 0\}$$

It is known that for $N \geq 2$, either $\mathrm{mld}(0; A, \mathfrak{a}) \geq 0$ or $\mathrm{mld}(0; A, \mathfrak{a}) = -\infty$ holds. For $N = 1$, we define $\mathrm{mld}(0; A, \mathfrak{a}) = -\infty$ if the left-hand side is negative, by abuse of notation, because it is convenient to describe the Inversion of adjunction.

Definition 2·3. We say that a prime divisor E over A with the center at 0 *computes* $\mathrm{mld}(0; A, \mathfrak{a})$
if either $a(E; A, \mathfrak{a}) = \mathrm{mld}(0; A, \mathfrak{a})$ (when the right-hand side is ≥ 0)
or $a(E; A, \mathfrak{a}) < 0$ (when the mld is $-\infty$).

Remark 2·4. Assume there exists a log resolution of the pair $(A, \mathfrak{a}\mathfrak{m}_0)$, where $\mathfrak{m}_0$ is the maximal ideal defining $0 \in A$. If $\mathrm{mld}(0; A, \mathfrak{a}) \geq 0$, then, on every such resolution there is a prime divisor computing $\mathrm{mld}(0; A, \mathfrak{a})$. If $\mathrm{mld}(0; A, \mathfrak{a}) = -\infty$ and $Z(\mathfrak{a}) \subset A$ contains an irreducible component of codimension one, there may not exist a prime divisor computing the mld among the exceptional divisors appearing in a given log resolution (cf. [3, proposition 7·2]). But in this case, if we construct an appropriate log resolution of $(A, \mathfrak{a}\mathfrak{m}_0)$ by taking more blowing-ups from the given one, a prime divisor computing $\mathrm{mld}(0; A, \mathfrak{a})$ appears on that. Therefore, for char $k = 0$ or $N \leq 3$, every pair $(A, \mathfrak{a})$ has a prime divisor computing $\mathrm{mld}(0; A, \mathfrak{a})$, since there is a log resolution for every pair.

3. *Weighted blow-ups and weighted projective spaces*

In this section A is always a smooth variety of dimension $N \geq 2$ defined over an algebraically closed field k and $P \in A$ is a (not necessarily closed) point.

Definition 3·1. Let $x_1, \ldots, x_c$ be an RSP of a regular local ring R with the algebraically closed residue field and $w_1, \ldots, w_c$ be positive integers with $\gcd(w_1, \ldots, w_c) = 1$. For $n \in \mathbb{N}$, denote by $\mathcal{I}_n$ the ideal in R generated by all monomials $x_1^{s_1} \cdots x_c^{s_c}$ such that $\sum_{i=1}^{c} s_i w_i \geq n$. The *weighted blow-up* of Spec R with $wt_w(x_1, \ldots, x_c) = (w_1, \ldots, w_c)$ is the canonical projection:

$$\mathrm{Proj}_A(\oplus_{n\in\mathbb{N}} \mathcal{I}_n) \longrightarrow A := \mathrm{Spec}\, R.$$

The exceptional divisor E for the weighted blow-up is called *a prime divisor obtained by a weighted blow-up* of A at P.

More generally, let $P \in A$ be a smooth point with the not-necessarily-algebraically closed residue field K. Let $\overline{K}$ be the algebraic closure of the residue field of $\mathcal{O}_{A,P}$. A *weighted blow-up of A at the point P* is the canonical morphism induced from a weighted blow-up $\overline{A} \longrightarrow \mathrm{Spec}\, \overline{K}\widehat{\mathcal{O}}_{A,P}$ for some RSP $x_1, \ldots, x_c$ of $\overline{K}\widehat{\mathcal{O}}_{A,P}$ with $wt_w(x_1, \ldots, x_c) = (w_1, \ldots, w_c)$ for some $(w_1, \ldots, w_c) \in \mathbb{Z}^c_{>0}$, where $\overline{K}\widehat{\mathcal{O}}_{A,P}$ is the extension of the formal power series ring

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$\widehat{\mathcal{O}}_{A,P}$ over K to the one over $\overline{K}$. Let $\overline{E}$ be the prime divisor obtained by the weighted blow-up $\overline{A} \longrightarrow$ Spec $\overline{K}\widehat{\mathcal{O}}_{A,P}$. The prime divisor E over A with the center at P corresponding to $\overline{E}$ is called a *prime divisor obtained by a weighted blow-up* of A at P. Note that if $\overline{E}$ gives a valuation $\overline{v}$ and the valuation ring $\mathcal{O}_{\overline{v}}$, the prime divisor E corresponds to the valuation v whose valuation ring is $K(A) \cap \mathcal{O}_{\overline{v}}$.

Note that weighted blow-ups are only defined at smooth points.

Here, we show a 3-dimensional example that the minimal log discrepancy is not computed by a divisor obtained by only one weighted blow-up, but computed by a divisor obtained by two weighted blow-ups.

The following are well known, for example see [10, remark 2·6, lemma 2·7].

Remark 3·2. Let $P \in A$ be a point of a smooth variety with the residue field K.

(1) The set of prime divisors over A with the center at P corresponds bijectively to the set of prime divisors over $\widehat{A} :=$ Spec $\widehat{\mathcal{O}}_{A,P}$ with the center at the closed point. Moreover, if prime divisors E and $\widehat{E}$ correspond under the above bijection, then for every $\mathbb{R}$-ideal $\mathfrak{a}$ on A we have $v_E(\mathfrak{a}) = v_{\widehat{E}}(\mathfrak{a})$ and also $a(E; A, \mathfrak{a}) = a(\widehat{E}, \widehat{A}, \mathfrak{a}\mathcal{O}_{\widehat{A}})$.

(2) Let $K' \supset K$ be a field extension and $A' :=$ Spec $K'\widehat{\mathcal{O}}_{A,P}$. Then, there is a surjective map from the set of prime divisors over A' with the center at the closed point to the set of prime divisors over A with the center at P. If prime divisors E' and E correspond by the above surjective map, then it follows $a(E'; A', \mathfrak{a}\mathcal{O}_{A'}) = a(E; A, \mathfrak{a})$ for every $\mathbb{R}$-ideal $\mathfrak{a}$ on A.

Example 3·3. Assume char $k \neq 2, 5$. Let $A := \mathbb{A}^3_k$ and $\mathfrak{a} = (f)^{7/10}$, where

$$f = (x^2 + y^2 + z^2)^2 + x^5 + y^5 + z^5.$$

Then, a divisor computing $\mathrm{mld}(0; A, \mathfrak{a}) = 0$ is not obtained by one weighted blow-up ([12, exercise 6·45]).

On the other hand, there is a sequence of weighted blow-ups

$$A_2 \xrightarrow{\varphi_2} A_1 \xrightarrow{\varphi_1} A,$$

where φ_1 is the usual blow-up at 0 and φ_2 is a weighted blow-up with weight (1, 2) at the generic point of the curve $x^2 + y^2 + z^2 = 0$ on $E_1 = \mathbb{P}^2_k$. Here, E_1 is the exceptional divisor for φ_1. The exceptional divisor E_2 for φ_2 computes $\mathrm{mld}(0; A, \mathfrak{a}) = 0$

The following lemma for a weighted projective space with a special weight is used for our main results. The statement is easily generalised to higher dimensional case, but for simplicity of notation we state here only for 2-dimensional case.

LEMMA 3·4. *Let $r \leq s$ be positive integers such that* $\gcd(r, s) = 1$. *Let $g \in k[x_1, x_2, x_3]$ be a weighted homogeneous polynomial with respect to the weight $w = (w(x_1), w(x_2), w(x_3)) = (r, r, s)$ and $Q \in \mathbb{P}_k(r, r, s)$ a closed point not contained in the coordinate planes, i.e., $Q \notin (x_1 \cdot x_2 \cdot x_3 = 0)$. Let $\ell \in k[x_1, x_2, x_3]$ be a weighted homogeneous polynomial of* $\deg_w(\ell) = r$ *such that $\ell(Q) = 0$. If $\ell \not| g$, then it follows*

$$r \cdot s \cdot \mathrm{ord}_Q(g) \leq r \cdot s \cdot \mathrm{ord}_Q(g|_L) \leq \deg_w g,$$

where $L \subset \mathbb{P}_k(r, r, s)$ is the divisor defined by $\ell = 0$ in $\mathbb{P}_k(r, r, s)$.

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Proof. As $\mathrm{ord}_Q g \leq \mathrm{ord}_Q(g\,|_\ell\,)$, the first inequality is trivial. We will show the second inequality. Let $G \subset \mathbb{P}_k(r, r, s)$ be the subscheme defined by $g = 0$ on $\mathbb{P}_k(r, r, s)$. Let

$$\pi : \mathbb{P}^2_k \twoheadrightarrow \mathbb{P}(r, r, s), (X_1, X_2, X_3) \mapsto (X_1^r, X_2^r, X_3^s) = (x_1, x_2, x_3)$$

be the canonical covering. Then, as π^*L and π^*G has no common irreducible components, Bezout's theorem on $\mathbb{P}^2$ implies

$$\pi^*L \cdot \pi^*G = \deg \pi^*\ell \cdot \deg \pi^*g = \deg_w \ell \cdot \deg_w g = r \cdot \deg_w g, \tag{1}$$

In case char $k = 0$ or char $k = p > 0$ and $p \not| r \cdot s$, the morphism π is étale around Q. Therefore, $\pi^{-1}(Q)$ consists of $r^2 \cdot s$ closed points $\{Q_i \mid i = 1, \ldots, r^2 \cdot s\}$ whose analytic neighbourhoods of π^*G and π^*L are isomorphic to those of G and L at Q, respectively. Then, by (1) we obtain

$$r^2 \cdot s \cdot \mathrm{ord}_Q(g\,|_L\,) = \sum_{i=1}^{r^2 s} \mathrm{ord}_{Q_i}(\pi^*g\,|_{\pi^*L}\,) \leq \pi^*L \cdot \pi^*G = r \cdot \deg_w g,$$

which yields the required inequality.

In case $p|r$, denote $r = p^e \cdot q$ (gcd $(p, q) = 1$). Then, the fiber $\pi^{-1}(Q)$ consists of $q^2 \cdot s$ closed points, as a topological space. For a closed point Q_i $(i = 1, \ldots, q^2 \cdot s)$ in the fiber $\pi^{-1}(Q)$ we obtain

$$\mathfrak{m}_Q \mathcal{O}_{\mathbb{P}^2} \subset \mathfrak{m}_{Q_i}^{p^e},$$

where $\mathfrak{m}_Q$ and $\mathfrak{m}_{Q_i}$ are the maximal ideals of $Q \in \mathbb{P}(r, r, s)$ and of $Q_i \in \mathbb{P}^2$, respectively. Let $C \subset \mathbb{P}^2$ be the subscheme with the reduced structure of π^*L. Then, we have

$$\mathfrak{m}_{L,Q} \mathcal{O}_C \subset \mathfrak{m}_{C,Q_i}^{p^e},$$

where $\mathfrak{m}_{L,Q}$ and $\mathfrak{m}_{C,Q_i}$ are the maximal ideals of $Q \in L$ and of $Q_i \in C$, respectively. Therefore, for every $i = 1, \ldots, q^2 \cdot s$ it follows

$$p^e \cdot \mathrm{ord}_Q(g \mid L) \leq \mathrm{ord}_{Q_i}(\pi^*g)\,|_C\,.$$

Now, there are $q \cdot s$ points Q_i lying on C. Then, by Bezout's theorem on $\mathbb{P}^2$ for C and π^*G, we obtain

$$q \cdot s \cdot p^e \mathrm{ord}_Q(g\,|_L\,) \leq q \cdot s \cdot \mathrm{ord}_{Q_i}(\pi^*g)\,|_C \leq C \cdot \pi^*G = \deg_w g.$$

Here noting that $q \cdot s \cdot p^e = r \cdot s$, this is the required inequality.

In case $p|s$, the proof is similar.

4. *Squeezed systems and squeezed blow-ups*

Let A be a variety of dimension $N \geq 2$ over an algebraically closed field k.

Definition 4·1. Let $P \in A$ be a smooth point (not necessarily closed), K the residue field, and E a prime divisor over A with the center at P. Denote the algebraic closure of K by $\overline{K}$. An RSP $\{x_1, \ldots, x_c\}$ of $\overline{K}\widehat{\mathcal{O}}_{A,P}$ at the closed point is called a *squeezed system* for E at P, if $v_i := v_E(x_i)$ $(i = 1, \ldots, c)$ satisfy:

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(1) $v_1 = \cdots = v_{c-1} \leq v_c$;

(2) $v_1 := \min\{v_E(x) \mid x \in \mathfrak{m} \setminus \mathfrak{m}^2\}$;

(3) $v_c := \max\{v_E(x) \mid x \in \mathfrak{m} \setminus \mathfrak{m}^2\}$;

where $\overline{K}\widehat{\mathcal{O}}_{A,P}$ is the extension of the coefficient field K of the formal power series ring $\mathcal{O}_{A,P}$ to $\overline{K}$, and $\mathfrak{m} \subset \overline{K}\widehat{\mathcal{O}}_{A,P}$ is the maximal ideal.

In this case,

$$v' := (v'_1, \ldots, v'_c) = \frac{(v_1, \ldots, v_c)}{\gcd(v_1, \ldots, v_c)}$$

is called a *squeezed weight* for E at P.

Let E and $v' = (v'_1, \ldots, v'_c)$ be as above. In this case, we call E a prime divisor of squeezed type v'.

Note that the squeezed weight for E is determined by a prime divisor but squeezed system is not uniquely determined by the prime divisor E.

Remark 4·2. For every A, P and E as in Definition 4·1, there exists a squeezed system of $\overline{K}\widehat{\mathcal{O}}_{A,P}$. Indeed, it is obvious that there is $x_1 \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $v(x_1)$ is the minimal value among $\{v_E(x) \mid x \in \mathfrak{m} \setminus \mathfrak{m}^2\}$. Existence of the maximal $v(x_c)$ among the set is proved by Zariski's subspace theorem (cf. [1, (10·6)]). Now, we extend $\{x_1, x_c\}$ to an RSP $\{x_1, x_2, \ldots, x_c\}$ of $\mathcal{O}_{A,P}$. Here, if $v_E(x_i) > v_E(x_1)$ for $2 \leq i \leq r-1$, replace x_i by $x_1 + x_i$. Then, we obtain a squeezed system $\{x_1, x_2, \ldots, x_c\}$.

Actually in [9] and [6], the proofs of Theorem 1·1 show the following:

Example 4·3 (Theorem 1·1). For every prime divisor E over a smooth surface A with the center at 0 such that $a(E; A, \mathfrak{a}) \geq 0$ for an $\mathbb{R}$-ideal $\mathfrak{a}$ on A. Then, the exceptional divisor E_1 obtained by a squeezed blow-up for E satisfies

$$a(E; A, \mathfrak{a}) \geq a(E_1; A, \mathfrak{a}).$$

Definition 4·4. Let A, P and E as above and let $\{x_1, \ldots, x_c\}$ be a squeezed system for E and $v' = (v'_1, \ldots, v'_r)$ be the squeezed weight. We call the weighted blow-up of weight v' with respect to the coordinate system $\{x_1, \ldots, x_c\}$ a *squeezed blow-up* for E.

Remark 4·5. As in the definitions, a squeezed system is a RSP in the local ring with extended coefficient field. A squeezed system is not in general a RSP of the original local ring $\mathcal{O}_{A,P}$.

Example 4·6. Let $A_K :=$ Spec $K[[y, z]]$ and $A_{\overline{K}} :=$ Spec $\overline{K}[[y, z]]$, where $\overline{K}$ is the algebraic closure of K. Take an element $a \in \overline{K} \setminus K$ and let $\phi \in K[T]$ be the minimal polynomial of a. Let $\varphi_1 : A_1 \longrightarrow A_K$ be the usual blow-up at the closed point of A_K. Then the exceptional divisor E_1 is the projective line $\mathbb{P}^1_K$ with the homogeneous coordinates $\{y, z\}$. Denote the homogenised polynomial of ϕ by $\Phi(y, z) := z^{\deg \phi}\phi(y/z)$. Take the blow-up $\varphi_2 : A_2 \longrightarrow A_1$ with the center at the closed subscheme C defined by the ideal $(\Phi(y, z))$ on E_1. As the proper transforms of any curves defined by linear forms $\ell = cy + dz = 0$ $(c, d \in K)$ on A_1 do not intersect to C, it follows $v_{E_2}(\ell) = 1$. Therefore, every RSP $\{f_1, f_2\}$ of $K[[y,z]]$ satisfies $v_E(f_1) = v_E(f_2) = 1$.

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On the other hand, take the base change $\psi: A_{\overline{K}} \longrightarrow A_K$ by the field extension $\overline{K} \supset K$. Let $z' := y - az \in \overline{K}[[y, z]]$. Then, the proper transform of the curve defined by $z' = 0$ contains the point $(a{:}1) \in \mathbb{P}^1_{\overline{K}} = \overline{E}_1$ where $\overline{E}_1$ is the exceptional divisor of the blow-up at the closed point of $A_{\overline{K}}$. As $(a{:}1) \in \overline{E}_1$ satisfies $\Phi(y, z) = 0$, the proper transform of $z' = 0$ intersects the center of the second blow-up induced from φ_2. One can see that $v_E(z') > 1$, and therefore a squeezed system cannot be taken from $K[[y,z]]$.

Now we are going to define "general" ideal.

Definition 4·7. Let E be a prime divisor over A of squeezed type (v'_1, v'_2, v'_3) (note that $v'_1 = v'_2$) and let E_1 be the exceptional divisor obtained by the squeezed blow-up with respect to a squeezed system $\{x_1, x_2, x_3\}$.

An irreducible curve $B \subset E_1 = \mathbb{P}(v'_1, v'_2, v'_3)$ with the following properties is called a *bad curve* for E on E_1.

(1) B is a curve of degree v'_1 with respect to (v'_1, v'_2, v'_3). (In the discussions on a weighted projective space, "degree" always means degree with respect to (v'_1, v'_2, v'_3), and it is sometimes denoted by $\deg_{v'}$.)

(2) B contains the center of E.

LEMMA 4·8. *Under the setting of Definition 4·7, the following hold*:

(i) *A bad curve does not always exist. More precisely a bad curve does not exist if and only if one of the following holds*;

  (*a*) *the squeezed weight is (1, 1, 1)*; *or*

  (*b*) *the squeezed weight* (v'_1, v'_2, v'_3) *satisfies* $v'_1 < v'_3$ *and the center of* E *on* A_1 *is a curve of* $\deg_{v'} > v'_1$ *on* $E_1 \simeq \mathbb{P}(v'_1, v'_2, v'_3)$; *or*

  (*c*) $E = E_1$.

(ii) *If a bad curve exists, then it is unique in* E_1.

Proof. It is clear that if $E = E_1$, then the center of E on E_1 is the generic point, so there is no bad curve on E_1. We exclude this trivial case in the following discussions. In case the squeezed blow-up is the usual blow-up, then the exceptional divisor does not have a bad curve. Because if B is a bad curve, it is defined by linear form $\ell = \sum_i a_i X_i = 0$ with $a_3 \neq 0$, where $\{X_1, X_2, X_3\}$ is the projective coordinate system on $E_1 = \mathbb{P}^2$ corresponding to the squeezed system $\{x_1, x_2, x_3\}$ on $\mathcal{O}_{A,0}$. This is a contradiction to the fact that (1, 1, 1) is the squeezed system, as we obtain another RSP $\{x_1, x_2, \ell(x_1)\}$ such that

$$v_E(x_1) < v_E(\ell(x_i)). \tag{2}$$

Here, we give the proof of this inequality, as this kind of discussion is used frequently in this paper.

Let $\varphi_1 : A_1 \longrightarrow A$ be the squeezed blow-up and $\psi: \widetilde{A} \longrightarrow A_1$ a birational morphism on which E appears. Denote the composite $\varphi_1 \circ \psi$ by φ. Let D be the proper transform of $Z(\ell(x_i)) \subset A$ in A_1, then $D \cap E_1$ contains the center of E on A_1 by the assumption. Note that we can express

$$(\varphi_1^* \ell(x_i)) = rE_1 + D, \quad (r = v_{E_1}(\ell(x_i))).$$

Here, we remind the reader that $v_E(\ell(x_i))$ is the coefficient of the divisor $(\varphi^*\ell(x_i)) = \psi^*(rE_1 + D)$ at the component E. The center of E on A_1 is contained in D, therefore the contribution from $\psi^*(D)$ to $v_E(\ell(x_i))$ is positive. Therefore, $v_E(\ell(x_i)) > rv_E(E_1) = v_{E_1}(\ell(x_i))v_E(E_1) = v_E(x_1)$. This shows the inequality (2).

For the case where E_1 is an exceptional divisor of a squeezed blow-up with respect to (v'_1, v'_2, v'_3) with $v'_1 < v'_3$, if the center C of E on E_1 is a curve of degree $> v'_1$, then there is no bad curve. Because, a curve of degree v'_1 cannot contain a curve of degree $> v'_1$. This gives the proof of "if" part of (i).

Assume a bad curve exists on E_1. When the center of E on E_1 is a curve, then it should coincide with the bad curve by the definition, therefore the center should be of degree v'_1. When the center of E on E_1 is a closed point P, then a bad curve should contain P. Express the point P by the homogeneous coordinates (a,b,c) with $a, b, c \in k$. Then a curve of degree v'_1 containing P is defined by $bX_1 - aX_2 = 0$. Now we obtain the uniqueness of the bad curve on E_1. This completes the proof of "only if" part of (i) and the proof of (ii).

Definition 4·9. Let E be a prime divisor over a smooth variety A with the center at a closed point 0. An $\mathbb{R}$-ideal $\mathfrak{a}$ is called *general for E* if there exists a squeezed blow-up $A_1 \longrightarrow A$ for E with the exceptional divisor E_1 satisfying the following:

(1) $\operatorname{ord}_B \mathfrak{a}_{A_1}\mathcal{O}_{E_1} \leq 1$, where B is the bad curve on E_1 and $\mathfrak{a}_{A_1}$ is the weak transform of $\mathfrak{a}$ at A_1. If there is no bad curve on E_1, then we account it as the inequality automatically holds;

(2) in addition, if $a(E; A, \mathfrak{a}) < a(E_1; A, \mathfrak{a})$ and the center P of E on A_1 is a smooth closed point, then there exists a squeezed blow-up $A_2 \longrightarrow A_1$ for E at P. Let E_2 be the exceptional divisor. Then, $\operatorname{ord}_{B'} I_L \mathfrak{a}_{A_2}\mathcal{O}_{E_2} \leq 1$, where B' is the bad curve on E_2, $\mathfrak{a}_{A_2}$ is the weak transform of $\mathfrak{a}$ at A_2 and I_L is the defining ideal of the intersection $L := E_2 \cap E'_1$ in E_2. Here, E'_1 is the proper transform of E_1 on A_2. If there is no bad curve on E_2, then we account it as the inequality automatically holds.

We say that a pair $(A, \mathfrak{a})$ is *general* if the $\mathbb{R}$-ideal $\mathfrak{a}$ is general for a prime divisor computing $\operatorname{mld}(0; A, \mathfrak{a})$. Here, the weak transform $\mathfrak{a}_{iA_2}$ of an ideal $\mathfrak{a}_i \subset \mathcal{O}_A$ on A_2 is defined as

$$\mathfrak{a}_i\mathcal{O}_{A_2} = \mathfrak{a}_{iA_2}\mathcal{O}_{A_2}(-v_{E_1}(\mathfrak{a}_i)E_1 - v_{E_2}(\mathfrak{a}_i)E_2).$$

The weak transform $\mathfrak{a}_{A_2}$ of an $\mathbb{R}$-ideal $\mathfrak{a}$ on A is defined as the canonical extension of the one for an ideal of $\mathcal{O}_A$ (see, for example [9]).

Remark 4·10. In (2), we assume smoothness of the center P of E on A_1. But it turns out that it always holds by Lemma 5·1.

Remark 4·11. The definition of generality of an $\mathbb{R}$-ideal is rather complicated. However, one can see that under a fixed exponent, the inequalities of orders at specific curves of E_1 and E_2 are open conditions in the space of regular functions of A, which is the reason why we call the ideal $\mathfrak{a}$ "general". The following gives a sufficient condition for generality of the ideal.

Under the same symbols as in Definition 4·9, the $\mathbb{R}$-ideal $\mathfrak{a}$ is general for E if one of the following hold:

(1) there is no bad curve on E_1 or E_2;

(2) assume the bad curves $B \subset E_1$ and $B' \subset E_2$ exist. $\mathrm{ord}_B \mathfrak{a}_{A_1} \mathcal{O}_{E_1} = 0$, and $\mathrm{ord}_{B'} \mathfrak{a}_{A_2} \mathcal{O}_{E_2} = 0$.

5. *Proofs of the main results*

For the proofs of the main theorems we need the following lemma which guarantees that the second weighted blow-up is possible.

LEMMA 5·1. *Let E be a prime divisor over a smooth N-fold A ($N \geq 2$) with the center at the closed point 0. Let $\{x_1, \ldots, x_N\}$ be a RSP at 0. Let $v_i := v_E(x_i)$, $v := (v_1, \ldots, v_N)$ and define*

$$v' := (v'_1, \ldots, v'_N) = \frac{(v_1, \ldots, v_N)}{\gcd v}.$$

Let $\varphi_1 \colon A_1 \longrightarrow A$ be the weighted blow-up with respect to $\{x_1, \ldots, x_N\}$ with weight v'. Denote the exceptional divisor of φ_1 by E_1. Assume $E \neq E_1$ and let C be the center of E on A_1 and $P \in C$ the generic point of C.

Then,

$$P \in E_1 \setminus \left\{ \bigcup (X_i = 0) \right\} \subset E_1 = \mathbb{P}(v'_1, \ldots, v'_N),$$

where X_i is a homogeneous coordinate function corresponding to x_i. In particular, P is smooth on A_1 and also on E_1.

Proof. Assume that the statement does not hold, then we may assume that P is in the hyperplane defined by $X_1 = 0$ in $E_1 = \mathbb{P}(v')$. There exists at least one homogeneous coordinate function X_i such that P does not lay in the hyperplane defined by $X_i = 0$. Then we obtain:

$$v_E(x_i) = v_{E_1}(x_i) \cdot v_E(E_1) = v'_i \cdot v_E(E_1);$$
$$v_E(x_1) = v_{E_1}(x_1) \cdot v_E(E_1) + \mathrm{ord}_P X_1 \geq v'_1 \cdot v_E(E_1) + 1.$$

This is a contradiction to the fact that

$$v_E(x_1) : v_E(x_i) = v'_1 : v'_i.$$

The following lemma is a basic idea appeared in [9].

LEMMA 5·2. *Let $\mathfrak{a}$ be an $\mathbb{R}$-ideal on A with $a(E; A, \mathfrak{a}) \geq 0$. Let $A' \longrightarrow A$ be a proper birational morphism with normal A', and D an irreducible divisor on A' with the same center on A as that of E. Assume $a(D; A, \mathfrak{a}) > a(E; A, \mathfrak{a})$ and the generic point P of the center of E on A' is smooth and not contained in the other exceptional divisors for $A' \longrightarrow A$.*

Then, we have

$$\mathrm{mld}(P; D, \mathfrak{a}_{A'} \mathcal{O}_D) < 0, \quad \text{in particular}$$

$$\mathrm{ord}_P \mathfrak{a}_{A'} \mathcal{O}_D > 1,$$

where $\mathfrak{a}_{A'}$ is a weak transform of $\mathfrak{a}$ on A'.

Proof. First we express the log discrepancy at E as follows:

$$\begin{aligned} a(E;A,\mathfrak{a}) &= k_{E/A} + 1 - v_E(\mathfrak{a}) \\ &= k_{E/A'} + k_{D/A}\cdot v_E(D) + 1 - v_D(\mathfrak{a})\cdot v_E(D) - v_E(\mathfrak{a}_{A'}) \\ &= a(E;A', I_D\cdot \mathfrak{a}_{A'}) + v_E(D)\cdot a(D;A,\mathfrak{a}), \end{aligned} \tag{3}$$

where $k_{E/A'}$ is the coefficient of the relative canonical divisor $K_{\widetilde{A}/A'}$ at E and I_D is the defining ideal of D in A'. Then, by the assumption, it follows $a(E;A', I_D\cdot \mathfrak{a}_{A'}) < 0$ and therefore we obtain

$$\mathrm{mld}(P;A', I_D\cdot \mathfrak{a}_{A'}) = -\infty.$$

By Inversion of adjunction ([3, 7]) we obtain $\mathrm{mld}(P;D,\mathfrak{a}_{A'}\cdot \mathcal{O}_D) = -\infty$. Hence, it follows $\mathrm{ord}_P(\mathfrak{a}_{A'}\cdot \mathcal{O}_D) > 1$ as claimed.

Setting for the proof of Theorem 1·8.

Let E be a prime divisor over a smooth 3-fold A with the center at a closed point 0. Let $\mathfrak{a}$ be a general $\mathbb{R}$-ideal on A such that $a(E;A,\mathfrak{a}) \geq 0$. Let

$$\varphi_1 : A_1 \longrightarrow A$$

be a squeezed blow-up for E satisfying the condition (1) in Definition 4·9. Let the squeezed system $\{x_1, x_2, x_3\}$ and the weight $v' = ({v_1}', v_2', v_3')$ correspond to the squeezed blow-up φ (note that $v_1' = v_2'$). Denote the exceptional divisor for φ by E_1. If $a(E_1;A,\mathfrak{a}) \leq a(E;A,\mathfrak{a})$, then E_1 is the required prime divisor F in the theorem. Therefore, from now on, we assume that the inequalities $a(E_1;A,\mathfrak{a}) > a(E;A,\mathfrak{a}) \geq 0$ hold.

LEMMA 5·3. *Let A, E and E_1 be as above. If $\mathfrak{a}$ is general for E and the inequalities $a(E_1;A,\mathfrak{a}) > a(E;A,\mathfrak{a}) \geq 0$ hold, then we obtain the following*:

(i) $0 < a(E_1;A,\mathfrak{a}) < 1$;

(ii) $v' = (1,1,n)$ *with* $n \geq 1$ *or* $v' = (2,2,3)$.

(*a*) *In case (1, 1, n) the center of E on A_1 is a curve in $E_1 = \mathbb{P}(1,1,n)$ of degree $n+1$.*

(*b*) *In case (2, 2, 3) the center of E on A_1 is either a curve of degree* 6 *or a closed point in $E_1 = \mathbb{P}(2,2,3)$.*

Proof. Let $f^e = f_1^{e_1}\cdots f_r^{e_r} \in \mathfrak{a}$ be a general element, *i.e.*, $v_{E_1}(\mathfrak{a}) = \sum_i e_i\cdot \deg_{v'}(\mathrm{in}_{v'} f_i)$, where $\mathrm{in}_{v'} f$ is the initial part of f with respect to the weight v'.

We divide the proof into two cases according to the dimension of the center of E on A_1. Let $P \in A_1$ be the generic point of the center of E on A_1.

Case 1. $\dim \overline{\{P\}} = 1$.

Let $C := \overline{\{P\}}$ defined by $\ell = 0$ on $E_1 = \mathbb{P}(v')$, where ℓ is homogeneous of degree $\geq v_1'$ with respect to the weight v'.

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The $\mathbb{R}$-divisor on E_1 induced from a general element $f^e = f_1^{e_1} \cdots f_r^{e_r}$ is expressed as follows:

$$\left(\prod \mathrm{in}_{v'} f_i^{e_i}\right) = \alpha C + \sum_j \gamma_j C_j, \quad \text{with} \quad \alpha > 1, \gamma_i \in \mathbb{R}_{>0}$$

Here, note that $\alpha > 1$ follows from Lemma 5·2. As $\mathfrak{a}$ is general, C is not a bad curve, therefore its degree is greater than v'_1. Then, $\deg_{v'} \ell \geq v'_1 v'_3$, because ℓ is an irreducible weighted homogeneous polynomial in x_1, x_2, x_3 of weight v'_1, v'_1, v'_3 not contained in the coordinate hyperplanes in $E_1 \simeq \mathbb{P}(v')$. (Note that such a polynomial with smallest degree is in the form $ax_1^{v'_3} + bx_2^{v'_3} + cx_3^{v'_1}$.) Then, we have:

$$v_{E_1}(\mathfrak{a}) = \sum_i e_i \cdot \deg_{v'}(\mathrm{in}_{v'} f_i) = \deg_{v'}(\alpha C + \sum_j \gamma_j C_j) > \deg_{v'} C = \deg_{v'} \ell \geq v'_1 v'_3.$$

By the assumption $a(E_1; A, \mathfrak{a}) > a(E; A, \mathfrak{a}) \geq 0$, it follows

$$0 \leq a(E_1; A, \mathfrak{a}) = 2v'_1 + v'_3 - v_{E_1}(\mathfrak{a}) < 2v'_1 + v'_3 - v'_1 \cdot v'_3. \tag{4}$$

The possibilities of (v'_1, v'_1, v'_3) are only $(1, 1, n)$ with $n \in \mathbb{N}$ and $(2, 2, 3)$. In case $(2, 2, 3)$, by (4) we have $a(E_1; A, \mathfrak{a})) < 2 \cdot 2 + 3 - 2 \cdot 3 = 1$. Then, in this case we have (i) and (b) of (ii).

In case $(1, 1, n)$ for $n \in \mathbb{N}$, we have $\deg_{v'} \ell \geq n + 1$. Indeed, if not, we have $\deg_{v'} \ell = n$ and $\ell = X_3 + h(X_1, X_2)$ for a nonzero homogeneous polynomial h of degree n. As E has the center at the curve $\ell = 0$, in the same way as the proof of (2) we have

$$v_E(x_3 + h(x_1, x_2)) > v_E(x_3),$$

and also $x_3 + h(x_1, x_2) \in \mathfrak{m}_0 \setminus \mathfrak{m}_0^2$ which is a contradiction to the maximality of $v_E(x_3)$. Therefore, in this case also we have $a(E_1; A, \mathfrak{a})) < 2 + n - (n + 1) = 1$, which shows (i) and (a) of (ii).

Case 2. $\dim \overline{\{P\}} = 0$

We can take $P = (1 \colon a \colon b) \in E_1 = \mathbb{P}(v')$ $(a, b \neq 0)$ as the homogeneous coordinate of the point P by Lemma 5·1.

First we will show that $v'_1 \neq 1$. To see this, assume that $v'_1 = 1$. Then a curve $bX_1^{v'_3} - X_3 = 0$ contains P, therefore

$$v_E(bx_1^{v'_3} - x_3) > v_E(x_3) = v_3,$$

and also $bx_1^{v'_3} - x_3 \in \mathfrak{m}_0 \setminus \mathfrak{m}_0^2$ which is a contradiction to the maximality of $v_E(x_3)$.

Now we may assume that $v'_1 \geq 2$. Then, of course $v'_1 < v'_3$ and the curve B defined by $aX_1 - X_2 = 0$ contains P. Note that B is the bad curve.

Take a general element $f^e = f_1^{e_1} \cdots f_r^{e_r} \in \mathfrak{a}$ such that $v_{E_1}(\mathfrak{a}) = v_{E_1}(f^e) = \deg_{v'}(\mathrm{in}_{v'} f^e)$. The $\mathbb{R}$-divisor on $E_1 = \mathbb{P}(v')$ induced from a general element $f^e = f_1^{e_1} \cdots f_r^{e_r}$ is expressed as follows:

$$\left(\prod \mathrm{in}_{v'} f_i^{e_i}\right) = \alpha B + \sum_j \gamma_j C_j, \quad \text{with} \quad \alpha, \gamma_i \in \mathbb{R}_{>0}. \tag{5}$$

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By generality of $\mathfrak{a}$, we have $\alpha \le 1$. By Lemma 5·2, we have $\mathrm{mld}(P; E_1, \mathfrak{a}_{A_1}\mathcal{O}_{E_1}) = -\infty$. By the description (5) of the divisor defined by a general element f^e, we have

$$-\infty = \mathrm{mld}(P; E_1, \mathfrak{a}_{A_1}\mathcal{O}_{E_1}) = \mathrm{mld}(P;E_1, I_B^{\alpha} \cdot \prod_i I_{C_i}^{\gamma_i}) \ge \mathrm{mld}(P;E_1, I_B \cdot \prod_i I_{C_i}^{\gamma_i})$$
$$= \mathrm{mld}(P; B, (\prod_i I_{C_i}^{\gamma_i})\mathcal{O}_B).$$

Hence, it follows $\mathrm{ord}_P(\prod_i I_{C_i}^{\gamma_i})\mathcal{O}_B > 1$. Applying Lemma 3·4 to the curve B of degree v'_1, we obtain

$$1 < \mathrm{ord}_P(\prod_i I_{C_i}^{\gamma_i})\mathcal{O}_B \le \frac{\sum \gamma_i \deg_{v'} C_i}{v'_1 v'_3} \le \frac{v_{E_1}(f^e)}{v'_1 v'_3} \le \frac{2v'_1 + v'_3}{v'_1 v'_3},$$

Here, for the third inequality, we use

$$\sum \gamma_i \deg_{v'} C_i \le v_{E_1}(f^e) - \alpha {v_1}'.$$

Then, the only possibility of v' satisfying these inequalities is (2, 2, 3) and we also have $v_{E_1}(\mathfrak{a}) = v_{E_1}(f^e) > 2 \cdot 3$ which completes the proof of (i) and (ii) in case $\dim \overline{\{P\}} = 0$.

COROLLARY 5·4 (Theorem 1·11). *Let A be a smooth variety of dimension 3 over an algebraically closed field k. For any general pair $(A, \mathfrak{a})$ with $\mathrm{mld}(0; A, \mathfrak{a}) \ge 1$ the minimal log discrepancy is computed by a prime divisor obtained by one weighted blow-up.*

Proof. As $a(E_1; A, \mathfrak{a}) \ge \mathrm{mld}(0; A, \mathfrak{a}) \ge 1$, the inequality $a(E_1; A, \mathfrak{a}) > a(E; A, \mathfrak{a})$ does not hold by (i) in Lemma 5·3.

Proof of Theorem 1·8. Let A_1, E_1 be as in the setting above. Assuming $0 \le a(E; A, \mathfrak{a}) < a(E_1; A, \mathfrak{a})$, we will prove that $a(E; A, \mathfrak{a}) \ge a(E_2; A, \mathfrak{a})$ for a divisor E_2 obtained by the second "blow-up" constructed below in Case 1 and Case 2.

Let $P \in E_1 \subset A_1$ be the center of E. First, for every prime divisor D over A_1 with the center at P and with the inequality $a(D; A, \mathfrak{a}) > a(E; A, \mathfrak{a}) \ge 0$, we observe that

$$a(D; A_1, \mathfrak{a}_{A_1}) \ge 0. \tag{6}$$

Indeed, we have an expression of $a(D; A, \mathfrak{a})$ as follows:

$$a(D; A, \mathfrak{a}) = a(D; A_1, \mathfrak{a}_{A_1}) + v_D(E_1)(a(E_1; A, \mathfrak{a}) - 1).$$

As $a(D; A, \mathfrak{a}) \ge 0$ and $a(E_1; A, \mathfrak{a}) - 1 < 0$ (Lemma 5·3), we have $a(D; A_1, \mathfrak{a}_{A_1}) \ge 0$.

Case 1. $\dim \overline{\{P\}} = 1$

Let $\{y_1, y_2\}$ be a squeezed system for E on A_1 at P and E_2 the prime divisor obtained by the squeezed blow-up of A_1 at P with respect to $\{y_1, y_2\}$. Let $K := \mathcal{O}_{A_1,P}/\mathfrak{m}_{A_1,P}$ and $\overline{K}$ the algebraic closure of K. Let $A_{1K} :=$ Spec $\widehat{\mathcal{O}}_{A,P}$, $A_{1\overline{K}} :=$ Spec $\overline{K}\widehat{\mathcal{O}}_{A,P} =$ Spec $\overline{K}[[y_1, y_2]]$. Denote the both closed points of A_{1K} and of $A_{1\overline{K}}$ by 0. Here, we note that $\{y_1, y_2\}$ is not necessarily a squeezed system on $A_{1\overline{K}}$ for $\overline{E}$ as is shown in Example 4·6, but it does not matter. Because we are interested only in ideals which came from A_1 and in this case a squeezed system on A_1 for E works in the same way as in [9] and [6], which one can see below:

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Let $\tilde{A} \longrightarrow A_1$ be a log resolution of $(A_1, \mathfrak{a}\mathcal{O}_{A_1})$ on which E appears. Then, the base change $\tilde{\tilde{A}} \longrightarrow A_{1\overline{K}}$ by $A_{1\overline{K}} \longrightarrow A_1$ is also a log resolution of $(A_{1\overline{K}}, \mathfrak{a}\mathcal{O}_{A_{1\overline{K}}})$ on which the prime divisor $\overline{E}$ corresponding to E appears. Let $A_2 \longrightarrow A_1$ be the squeezed blow-up with respect to the squeezed system $\{y_1, y_2\}$ and E_2 the exceptional divisor. By definition, it means that $A_{2\overline{K}} \longrightarrow A_{1\overline{K}}$ is squeezed weighted blow-up with respect to the squeezed system $\{y_1, y_2\}$ and $\overline{E}_2$ be the exceptional divisor corresponding to E_2.

If $\overline{E} = \overline{E}_2$, then we have $E = E_2$ and we are done. So, we may assume that the center of $\overline{E}$ on $A_{2\overline{K}}$ is a point. Then the center $Q \in A_{2\overline{K}}$ is not on the proper transform of $\overline{E}_1$ on $A_{2\overline{K}}$. This is proved as follows:

Let $w = (r, s)$ be the weight of the squeezed system $\{y_1, y_2\}$ on A_1.

First, we show that $r = s$ does not happen. Assume $r = s$, *i.e.*, $w = (1, 1)$, then we can take an expression $Q = (a, b)$ of $Q \in \overline{E}_2 = \mathbb{P}^1_{\overline{K}}$ by homogeneous coordinates with $a, b \neq 0$. Let $z := by_1 - ay_2 \in \mathcal{O}_{A_{1\overline{K}}}$. As Q is the center of $\overline{E}$ on $\overline{E}_2 \subset A_{2\overline{K}}$ and satisfying $bY_1 - aY_2 = 0$ (Y_1, Y_2 are the homogeneous coordinates on $E_2 = \mathbb{P}^1_K$ corresponding to y_1, y_2.), it follows

$$z \in \mathfrak{m}_Q \setminus \mathfrak{m}_Q^2, \quad \text{and} \quad v_E(z) > v_E(y_1), v_E(y_2),$$

which is a contradiction to the fact that $\{y_1, y_2\}$ is a squeezed system. Now, we may assume that $r < s$. Let $h = 0$ be the defining equation of E_1 in A_1 around P, then $\overline{E}_1$ is also defined by $h = 0$ and it is smooth at the closed point $0 \in A_{1\overline{K}}$. Therefore, we have $\text{ord}_{y_1,y_2} h = 1$. Then the initial part of h with respect to w is one of the following:

(1) $\text{in}_w(h) = y_1$, (2) $\text{in}_w(h) = y_2$, (3) $\text{in}_w(h) = y_2 + a{y_1}^d$ ($a \in \overline{K}$, $w_1 d = w_2$). In the first two cases, $\overline{E}'_1 \mid_{\overline{E}_2}$ is in the zero locus of the coordinate functions, where $\overline{E}'_1$ is the proper transform of $\overline{E}_1$ on $A_{2\overline{K}}$. Therefore it does not contain the center Q of $\overline{E}$ by Lemma 5·1. In case (3), it follows $w = (1, d)$. If Q is in $\overline{E}'_1 \mid_{\overline{E}_2}$, then we have $y'_2 := y_2 + a{y_1}^d \in \mathfrak{m}_{A_{1\overline{K}},0} \setminus \mathfrak{m}^2_{A_{1\overline{K}},0}$ and $v_{\overline{E}}(y'_2) > v_{\overline{E}}(y_2)$ which is a contradiction to the assumption that $\{y_1, y_2\}$ is a squeezed system. Now, in any case we obtain that $Q \notin \overline{E}'_1$.

On the other hand, $a(E; A, \mathfrak{a})$ has another expression as follows:

$$a(E; A, \mathfrak{a}) = k_{E/A_1} + k_{E_1/A} \cdot v_E(E_1) + 1 - v_E(\mathfrak{a}).$$

It is sufficient to show that

$$a(\overline{E}; A, \mathfrak{a}) \geq a(\overline{E}_2; A, \mathfrak{a}).$$

Assume contrary, then

$$0 > \overline{a}(\overline{E}; A, \mathfrak{a}) - \overline{a}(\overline{E}_2; A, \mathfrak{a}) = a(\overline{E}; A_{2\overline{K}}, I_{\overline{E}_2} \cdot \mathfrak{a}_{A_{2\overline{K}}}) + (v_{\overline{E}}(\overline{E}_2) - 1) \cdot \overline{a}(\overline{E}_2; A, \mathfrak{a}), \qquad (7)$$

where $\mathfrak{a}_{A_{2\overline{K}}}$ is the weak transform of $\mathfrak{a}_{A_1}\mathcal{O}_{A_{1\overline{K}}}$. For the calculation of (7), we used

(i) $v_{\overline{E}}(\overline{E}_1) = v_{\overline{E}}(\overline{E}_2) v_{\overline{E}_2}(\overline{E}_1) + v_{\overline{E}}(\overline{E}'_1) = v_{\overline{E}}(\overline{E}_2) v_{\overline{E}_2}(\overline{E}_1)$.

Then the inequality (7) shows that $a(\overline{E}; A_{2\overline{K}}, I_{\overline{E}_2} \cdot \mathfrak{a}_{A_{2\overline{K}}}) < 0$ which implies

$$\text{mld}(Q; A_{2\overline{K}}, I_{\overline{E}_2} \cdot \mathfrak{a}_{A_{2\overline{K}}}) = -\infty.$$

Then, by Inversion of adjunction ([3, 7]), it follows

$$\text{mld}(Q; \overline{E}_2, \mathfrak{a}_{A_{2\overline{K}}} \cdot \mathcal{O}_{\overline{E}_2}) < 0$$

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which yields $\mathrm{ord}_Q((\mathfrak{a}_{A_1}\mathcal{O}_{A_{1\overline{K}}})_{A_{2\overline{K}}}\cdot\mathcal{O}_{\overline{E}_2})=\mathrm{ord}_Q(\mathfrak{a}_{A_{2\overline{K}}}\cdot\mathcal{O}_{\overline{E}_2})>1$.

Let (r,s) be the squeezed weight for $\overline{E}$ at the closed point $0\in A_{1\overline{K}}$, then

$$a(\overline{E},A_{1\overline{K}},\mathfrak{a}_{A_{1\overline{K}}})=a(E;A_1,\mathfrak{a}_{A_1})\geq 0,$$

where we the last inequality follows from (6). Now we reach the situation in Theorem 1·1 and apply the argument in ([**9**]) for the surface pair $(A_{1\overline{K}},\mathfrak{a}_{A_{1\overline{K}}})$, we obtain

$$1<\mathrm{ord}_Q((\mathfrak{a}_{A_1}\mathcal{O}_{A_{1\overline{K}}})_{A_{2\overline{K}}}\cdot\mathcal{O}_{\overline{E}_2})\leq\frac{v_{\overline{E}_2}(\mathfrak{a}_{A_1}\mathcal{O}_{A_{1\overline{K}}})}{r\cdot s}\leq\frac{r+s}{r\cdot s},\tag{8}$$

where we note that $\mathfrak{a}_{A_{2\overline{K}}}=(\mathfrak{a}_{A_1}\mathcal{O}_{A_{1\overline{K}}})_{A_{2\overline{K}}}$ and the third inequality follows from

$$r+s-v_{\overline{E}_2}(\mathfrak{a}_{A_1}\mathcal{O}_{A_{1\overline{K}}})=a(\overline{E}_2;A_{1\overline{K}},\mathfrak{a}_{A_1})=a(E_2;A_1,\mathfrak{a}_{A_1})\geq 0$$

by (6). The possible positive intergers $\{r,s\}$ satisfying (8) with gcd $(r,s)=1$ are only $\{1,s\}$. In this case let $z':=y_1^s-cy_2$, where $Q=(c,1)\in\overline{E}_2=\mathbb{P}(1,s)$, then $v_{\overline{E}}(z')>v_{\overline{E}}(y_2)$, which is a contradiction to that $\{y_1,y_2\}$ is a squeezed system for $\overline{E}$. Hence we obtain

$$\overline{a}(\overline{E};A,\mathfrak{a})\geq\overline{a}(\overline{E}_2;A,\mathfrak{a}),$$

which completes the proof of the theorem for Case 1.

Case 2. $\dim\overline{\{P\}}=0$

Since we are assuming $0\leq a(E;A,\mathfrak{a})<a(E_1;A,\mathfrak{a})$, by Lemma 5·3 only possibility of v' is $(2,2,3)$ and we have $0\leq a(E_1;A,\mathfrak{a})<1$.

Now take a squeezed blow-up $A_2\longrightarrow A_1$ of weight $w=(w_1,w_2,w_3)$ at P and let E_2 be the exceptional divisor. We may assume that the condition (2) in Definition 4·9 holds. Let $Q\in E_2$ be the center of E on A_2.

Let E_1' be the proper transform of E_1 on A_2. Denote the defining ideals of E_1' and E_2 in A_2 by $I_{E_1'}$ and I_{E_2}, respectively.

Then, we have the similar expansion of $a(E;A,\mathfrak{a})$ as in (3) as follows:

$$a(E;A,\mathfrak{a})=a(E;A_2,I_{E_1'}\cdot I_{E_2}\cdot\mathfrak{a}_{A_2})+v_E(E_2)a(E_2;A,\mathfrak{a})+v_E(E_1')a(E_1;A,\mathfrak{a}),\tag{9}$$

where $\mathfrak{a}_{A_2}$ is the weak transform of $\mathfrak{a}$ on A_2 and is also the weak transform of $\mathfrak{a}_{A_1}$ on A_2.

Case 2·1. $\dim\overline{\{Q\}}=0$:

We will prove $a(E_2;A,\mathfrak{a})\leq a(E;A,\mathfrak{a})$. Assume on the contrary that $a(E_2;A,\mathfrak{a})>a(E;A,\mathfrak{a})$. Then, by (9), we obtain

$$a(E;A_2,I_{E_1'}\cdot I_{E_2}\cdot\mathfrak{a}_{A_2})<0.\tag{10}$$

It implies that $\mathrm{mld}(Q;A_2,I_{E_1'}\cdot I_{E_2}\cdot\mathfrak{a}_{A_2})=-\infty$. Let $L:=E_1'\cap E_2$, by Inversion of adjunction, we obtain

$$\mathrm{mld}(Q;E_2,I_L\mathfrak{a}_{A_2}\mathcal{O}_{E_2})<0.$$

Let B' be the bad curve on E_2 (note that a bad curve exists in our case by Lemma 4·8). Then, we obtain

$$\mathrm{ord}_{B'}\mathfrak{a}_{A_2}\mathcal{O}_{E_2}\leq 1.\tag{11}$$

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Indeed, when $L = B'$, then generality of $\mathfrak{a}$ implies that $\mathrm{ord}_{B'}\mathfrak{a}_{A_2}\mathcal{O}_{E_2} = 0$, as $\mathrm{ord}_{B'} I_L = 1$. On the other hand, when $L \neq B'$, then $Q \notin L$ and therefore generality implies $\mathrm{ord}_{B'}\mathfrak{a}_{A_2}\mathcal{O}_{E_2} \leq 1$. Now, in the same way as Case 2 in the proof of Lemma 5·3, we obtain that the weight of the second squeezed blow-up is (2, 2, 3).

We will show a contradiction under this situation. In this case, we have

$$v_{E_2}(\mathfrak{a}_{A_1}) > 6, \quad \text{as well as} \quad v_{E_1}(\mathfrak{a}) > 6, \tag{12}$$

by applying (i) of Lemma 5·3 for $(A_1, \mathfrak{a}_{A_1})$, E_2 with the weight $w = (2, 2, 3)$ and also for $(A, \mathfrak{a})$, E_1 with the weight $v' = (2, 2, 3)$. As the squeezed system $\{y_1, y_2, y_3\}$ at $P \in A_1$ has weight (2, 2, 3), it follows $v_{E_2}(f) \leq 3 \cdot \mathrm{ord}_P f$ for every $f \in \mathfrak{a}_{A_1}$. Therefore we obtain

$$v_{E_2}(\mathfrak{a}_{A_1}) \leq 3 \cdot \mathrm{ord}_P \mathfrak{a}_{A_1} \leq 3 \cdot \mathrm{ord}_P \mathfrak{a}_{A_1}\mathcal{O}_{E_1}. \tag{13}$$

On the other hand, applying Lemma 3·4 to $E_1 = \mathbb{P}(2, 2, 3)$ and a general element of $\mathfrak{a}_{A_1} \cdot \mathcal{O}_{E_1}$, we obtain $1 < \mathrm{ord}_P \mathfrak{a}_{A_1}\mathcal{O}_{E_1} \leq v_{E_1}(\mathfrak{a})/2 \cdot 3$. Note that the first inequality follows from Lemma 5·2.

Then, it follows

$$7 = 2 + 2 + 3 = k_{E_1} + 1 \geq v_{E_1}(\mathfrak{a}) \geq 6 \cdot \mathrm{ord}_P \mathfrak{a}_{A_1}\mathcal{O}_{E_1}. \tag{14}$$

Using (12), (13) and (14) we obtain

$$\frac{7}{2} > 3 \cdot \mathrm{ord}_P \mathfrak{a}_{A_1}\mathcal{O}_{E_1} \geq v_{E_2}(\mathfrak{a}_{A_1}) > 6$$

which is a contradiction. Therefore $a(E_2; A, \mathfrak{a}) \leq a(E; A, \mathfrak{a})$ holds.

Case 2·2. $\dim \overline{\{Q\}} = 1$.

In the following, we will prove $a(E_2; A, \mathfrak{a}) \leq a(E; A, \mathfrak{a})$. Assume contrary, $a(E_2; A, \mathfrak{a}) > a(E; A, \mathfrak{a})$. The curve $\overline{\{Q\}}$ is not a bad curve, because if it is, then

$$-\infty = \mathrm{mld}(Q; A_2, I_{E_1'} \cdot I_{E_2} \cdot \mathfrak{a}_{A_2}) = \mathrm{mld}(Q; E_2, I_L \mathfrak{a}_{A_2}\mathcal{O}_{E_2})$$

implies $\mathrm{ord}_Q I_L \mathfrak{a}_{A_2}\mathcal{O}_{E_2} > 1$, while the generality of $\mathfrak{a}$ implies the converse inequality $\mathrm{ord}_Q I_L \mathfrak{a}_{A_2}\mathcal{O}_{E_2} = \mathrm{ord}_{B'} I_L \mathfrak{a}_{A_2}\mathcal{O}_{E_2} \leq 1$. We also have $\overline{\{Q\}} \neq L$. This is proved as follows.

Let $h' \in \mathcal{O}_{A_1}$ define E_1 around P. As P is smooth on E_1 and also on A_1, we have $\mathrm{ord} h' = 1$ with respect to RSP $\{y_1, y_2, y_3\}$ of $\mathcal{O}_{A_1}$ at P. Then, considering of the initial term of h' with respect to the weight w, we see that one of the following holds:

(1) L is a coordinate axis of $E_2 = \mathbb{P}(w)$;

(2) L is defined by $Y_1 + aY_2$ $(a \in k)$ in E_2;

(3) L is defined by $Y_3 + f(Y_1, Y_2)$ in E_2, where f is a homogeneous polynomial of degree d.

In the third case, the weight w must be $(1, 1, d)$. In this case, if $\overline{\{Q\}} = L$, it follows $y_3' := y_3 + f(y_1, y_2) \in \mathfrak{m}_{A_1,P} \setminus \mathfrak{m}_{A_1,P}^2$ and $v_E(y_3') > v_E(y_3)$, which is a contradiction to the maximality of $v_E(y_3)$. In case (1), $\overline{\{Q\}} \neq L$ because Q is not contained in the coordinate axes (Lemma 5·1). In case (2), L becomes the bad curve, therefore $\overline{\{Q\}} \neq L$, because $\overline{\{Q\}}$ is not the bad curve, as we saw above.

Now we obtain $Q \notin E'_1 \cap E_2$. By using this, we have

$$\mathrm{mld}(Q; A_2, I_{E_2} \cdot \mathfrak{a}_{A_2}) = \mathrm{mld}(Q; A_2, I_{E'_1} \cdot I_{E_2} \cdot \mathfrak{a}_{A_2}) = -\infty.$$

By Inversion of adjunction, we have

$$\mathrm{mld}(Q; E_2, \mathfrak{a}_{A_2}\mathcal{O}_{E_2}) = -\infty.$$

Then, we have $1 < \mathrm{ord}_Q \mathfrak{a}_{A_2} \cdot \mathcal{O}_{E_2}$

First we show that the squeezed weight $w = (r, r, s)$ for E at $P \in A_1$ is $(1, 1, n)$ for $n \in \mathbb{N}$. Let $C := \overline{\{Q\}}$ be defined by $\ell = 0$ in $E_2 = \mathbb{P}(r, r, s)$. If $w \neq (1, 1, n)$, then the other possible weight w is $(2, 2, 3)$. In this case the smallest possible value for the degree of ℓ on $\mathbb{P}(2, 2, 3)$ with respect to w is 6. Therefore, by $1 < \mathrm{ord}_Q \mathfrak{a}_{A_2} \cdot \mathcal{O}_{E_2}$,

$$v_{E_2}(\mathfrak{a}_{A_1}) \geq \deg_w \ell \cdot \mathrm{ord}_Q(\mathfrak{a}_{A_1})_{A_2} \geq 6 \cdot \mathrm{ord}_Q(\mathfrak{a}_{A_1})_{A_2} > 6.$$

Now we obtain the inequality (12). The inequalities (13) and (14) also hold in the present case. Therefore, we induce a contradiction and w must be $(1, 1, n)$. By Lemma 5·3, $deg_w \ell = 1 + n$.

Let $\{y_1, y_2, y_3\}$ be a squeezed system at $P \in A_1$ with the weight $(1, 1, n)$. Let $\{Y_1, Y_2, Y_3\}$ be the homogeneous coordinates of $E_2 = \mathbb{P}(1, 1, n)$ corresponding to $\{y_1, y_2, y_3\}$. As ℓ is irreducible of degree $1 + n$ with respect to the weight $(1, 1, n)$, we can express

$$\ell = Y_1 Y_3 - Y_2^{n+1}.$$

For simplicity, assume $\mathfrak{a} = \mathfrak{a}_1^{e_1}$ and take a general element $f \in \mathfrak{a}_1 \mathcal{O}_{A,0} \subset k[[x_1, x_2, x_3]]$, where $\{x_1, x_2, x_3\}$ is a squeezed system for E at $0 \in A$ of weight $(2, 2, 3)$. Then the weak transform f_{A_1} of f on A_1 is written as

$$f_{A_1} = (y_1 \cdot y_3 - y_2^{n+1})^r \cdot \ell' + g(y), \tag{15}$$

where ℓ' is weighted homogeneous and $g(y)$ is the term with the higher weight with respect to the weight $w = (1, 1, n)$.

Here, we may assume that $P = (1, 1, 1) \in E_1 = \mathbb{P}(2, 2, 3)$, then we can take a RSP at $P \in A_1$ by making use of the squeezed system $\{x_1, x_2, x_3\}$ of squeezed weight $(2, 2, 3)$ which gives the first weighted blow-up $\varphi_1 : A_1 \longrightarrow A$:

$$z_1 = \frac{x_1^3 - x_3^2}{x_3^2}, \quad z_2 = \frac{x_2^3 - x_3^2}{x_3^2}, \quad z_3 = x_3,$$

where x_3 defines E_1 in the neighborhood of P. Take the minimal $m \in \mathbb{N}$ such that

$$f = x_3^m \cdot f_{A_1} \in \mathcal{O}_{A,0} \subset k[[x_1, x_2, x_3]]. \tag{16}$$

We note that for $m \geq 2$,

$$\mathrm{ord}_0 x_3^m \cdot z_i = m \ \ (i = 1, 2), \qquad \mathrm{ord}_0 x_3^m \cdot z_3 = m + 1, \tag{17}$$

where ord_0 is the order with respect to the parameters x_1, x_2, x_3 in $\mathcal{O}_{A,0}$. Then, by (17),

$$\mathrm{ord}_0 f = \mathrm{ord}_0(x_3^m \cdot f_A) \geq m.$$

On the other hand if $x_3^s(y_1 y_3 - y_2^{n+1})^r \in \mathcal{O}_{A,0}$, it should be $s \geq 4r$. In fact, if a quadratic monomial $z_i z_j$ $(i, j \in \{1, 2\})$ appears in $y_1 y_3$ which is expressed as a function of z_1, z_2, z_3, then

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$s \geq 4r$. If such a monomial $z_i z_j$ $(i, j \in \{1, 2\})$ does not appear in $y_1 y_3$, then z_i $(i < 3)$ appears in y_2, because $\{z_1, z_2, z_3\}$ and $\{y_1, y_2, y_3\}$ are both RSP at $P \in A_1$. This yields $s \geq 2(n+1)r \geq 4r$.

Consider the initial part $(y_1 \cdot y_3 - y_2^{n+1})^r \cdot \ell'$ of f_{A_1} with respect to the weight $w = (1, 1, n)$. We know that $a(E_2; A_1, \mathfrak{a}_{A_1}) \geq 0$, therefore $v_{E_2}(f_{A_1}^{e_1}) = v_{E_2}(\mathfrak{a}_{A_1}^{e_1}) \leq k_{E_2/A_1} + 1 = n+2$. Then, it follows that

$$e_1(r(n+1) + \deg_w \ell') \leq n+2. \tag{18}$$

As $1 < \mathrm{ord}_Q \mathfrak{a}_{A_2} \mathcal{O}_{E_2}$, it follows $1 < \mathrm{ord}_Q (y_1 y_3 - y_2^{n+1})^{re_1}$ which yields $re_1 > 1$. By this and (18), we have $\deg_w \ell' < r$, therefore $\mathrm{ord}_P \ell' < r$ which yields that the factor of $z_3(= x_3)$ appears in ℓ' at most $r-1$ times. Hence, as (16) the inclusion $x_3^m (y_1 \cdot y_3 - y_2^{n+1})^r \cdot \ell' \in \mathcal{O}_{A,0}$ should hold, which implies $m \geq 4r - (r-1) = 3r+1$.

Then, $\mathrm{ord}_0 f = \mathrm{ord}_0(x_3^m \cdot f_{A_1}) \geq 3r+1$, and therefore, taking $e_1 r > 1$ into account, we have

$$\mathrm{ord}_0 \mathfrak{a}_1^{e_1} = \mathrm{ord}_0 f^{e_1} \geq e_1(3r+1) > 3.$$

Then, for every prime divisor D over A with the center at 0 has the discrepancy $a(D; A, \mathfrak{a}) < 0$, which is a contradiction to the condition that $a(E; A, \mathfrak{a}) \geq 0$.

The condition "general" is necessary as far as we use "squeezed" blow-ups to construct a required divisor in Theorem 1·8. Actually, we have a non-general ideal such that two squeezed blow-ups do not give the required divisor.

Example 5·5. Let $f = (x_1 - x_2)^2 + x_3^2 + x_1^4 \in k[x_1, x_2, x_3]$, $e = 6/5$ and $\mathfrak{a} = (f)^e$. Define E as follows:

g $\varphi_1 : A_1 \longrightarrow A$ be the weighted blow-up with weight $(1, 1, 2)$ with respect to the coordinates $\{x_1, x_2, x_3\}$. Let E_1 be the exceptional divisor of φ_1. Let $\varphi_2 : A_2 \longrightarrow A_1$ be the (usual) blow-up with the center at $E_1 \cap (f_{A_1} = 0)$, where (f_{A_1}) is the weak transform of (f) on A_1. Let E_2 be the exceptional divisor of φ_2. Let $\varphi_3 : \widetilde{A} \longrightarrow A_2$ be the (usual) blow-up with the center at $E_2 \cap (f_{A_2} = 0)$, where (f_{A_2}) is the weak transform of (f) on A_2. Let E be the exceptional divisor of φ_3. Then, φ_1 and φ_2 are squeezed blow-ups for E, $\mathfrak{a}$ is not general for E and the following hold:

$$0 = a(E; A, \mathfrak{a}) < a(E_2; A, \mathfrak{a}) = \frac{1}{5} < a(E_1; A, \mathfrak{a}) = \frac{3}{5}.$$

So, we can see that the squeezed blow-ups do not work for this ideal. But if we do not stick to squeezed blow-up, we can find two weighted blow-ups to obtain the required F in the theorem. Let $\{x_1', x_2', x_3'\}$ be another RSP defined by $x_i' = x_i$ $(i = 1, 3)$ and $x_2' = x_1 - x_2$. Then, $v_E(x_1') = 1, v_E(x_2') = 2, v_E(x_3') = 2$. (We can see that this RSP is not squeezed.) Now, let $\psi_1 : A_1' \longrightarrow A$ be the weighted blow-up with weight $(1, 2, 2)$ with respect to $\{x_1', x_2', x_3'\}$. Let E_1' be the exceptional divisor of ψ_1. Let $\psi_2 : A_2' \longrightarrow A_1'$ be the (usual) blow-up with the center at $E_1' \cap (f_{A_1'} = 0)$. Let E_2' be the exceptional divisor of ψ_2. Then, we can see that $E = E_2$ at the generic points. So, E itself is obtained by two weighted blow-ups.

The example suggests us that we may take an appropriate weighted blow-up to obtain the required F in the theorem, if $\mathfrak{a}$ is not general.

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COROLLARY 5·6 (Corollary 1·9). *Assume $N = 3$. Then, for every "general" pair $(A, \mathfrak{a})$, the minimal log discrepancy* $\mathrm{mld}(0; A, \mathfrak{a})$ *is computed by a prime divisor E obtained by at most two weighted blow-ups. More concretely, the blow-ups are squeezed blow-ups for E.*

Proof. When $\mathrm{mld}(0; A, \mathfrak{a}) \geq 0$, then apply the theorem for a divisor E computing the mld. When $\mathrm{mld}(0; A, \mathfrak{a}) = -\infty$, then in a similar way as in [**9**], take a prime divisor E computing the mld. Then by taking a positive real number $t < 1$ such that $a(E; A, \mathfrak{a}^t) = 0$ and apply Theorem 1·8.

COROLLARY 5·7. *Let E be a prime divisor over A with the center at 0 and $E_1 = \mathbb{P}(r, r, s)$ $(r, s \geq 1)$ the exceptional divisor of a squeezed blow-up for E. Assume that $a(E; A, \mathfrak{a}) \geq 0$ and the center of E on E_1 is a curve of degree $> r$, then there is a prime divisor F such that*

$$a(F; A, \mathfrak{a}) \leq a(E; A, \mathfrak{a})$$

holds for every $\mathbb{R}$-ideal $\mathfrak{a}$ and F is obtained by at most two weighted blow-ups.

Proof. We can see that there is no bad curve on E. Therefore, every $\mathbb{R}$-ideal $\mathfrak{a}$ is general for E.

The proof of the theorem shows also the following corollary.

COROLLARY 5·8. *Let E be a prime divisor over A with the center at 0 computing* $\mathrm{mld}(0; A, \mathfrak{a}) \geq 0$. *Let E' be the exceptional divisor of a weighted blow-up with weight $v := (r, s, t)$, where* $\gcd(r, s, t) = 1$. *Assume that the center C of E on E' is a curve of degree $d \geq r + s + t - 1$ If* $\mathrm{mld}(0; A, \mathfrak{a})$ *is not computed by E', then the mld is computed by the divisor obtained by one additional weighted blow-up at C.*

Proof. Let $A' \longrightarrow A$ be the weighted blow-up with weight (r,s,t). By the assumption, we have $a(E; A, \mathfrak{a}) < a(E'; A, \mathfrak{a})$. Then, by Lemma 5·2, we have $\alpha := \mathrm{ord}_P \mathfrak{a}_{A'} \mathcal{O}_{E'} > 1$, where P is the generic point of C. Therefore, we obtain $v_{E'}(\mathfrak{a}) = \alpha d > r + s + t - 1$, and therefore $a(E'; A, \mathfrak{a}) < 1$. Now, in the same way as Case 1 in the proof of Theorem 1·8, we obtain that the squeezed blow-up at P gives a divisor F satisfying $a(F; A, \mathfrak{a}) \leq a(E; A, \mathfrak{a}) = \mathrm{mld}(0; A, \mathfrak{a})$.

The following is a special case of the corollary above. Example 3·3 is in this case.

COROLLARY 5·9 (Corollary 1·10). *Let E be a prime divisor over A with the center at 0 computing* $\mathrm{mld}(0; A, \mathfrak{a}) \geq 0$. *Let E' be the exceptional divisor of the usual blow-up with the center at 0. Assume that the center C of E on E' is a curve of degree $d \geq 2$ Then,* $\mathrm{mld}(0; A, \mathfrak{a})$ *is computed by the divisor obtained by one additional weighted blow-up at C.*

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