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SIMILARITY OF OPERATORS ON TENSOR PRODUCTS OF SPACES AND MATRIX DIFFERENTIAL OPERATORS

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Abstract

Let $\mathcal{H} = \mathbb{C}^n \otimes \mathcal{E}$ be the tensor product of a Euclidean space $\mathbb{C}^n$ and a separable Hilbert space $\mathcal{E}$. Our main object is the operator $G = I_n \otimes S + A \otimes I_{\mathcal{E}}$, where S is a normal operator in $\mathcal{E}$, A is an $n \times n$ matrix, and $I_n, I_{\mathcal{E}}$ are the unit operators in $\mathbb{C}^n$ and $\mathcal{E}$, respectively. Numerous differential operators with constant matrix coefficients are examples of operator G. In the present paper we show that G is similar to an operator $M = I_n \otimes S + \hat{D} \times I_{\mathcal{E}}$ where $\hat{D}$ is a block matrix, each block of which has a unique eigenvalue. We also obtain a bound for the condition number. That bound enables us to establish norm estimates for functions of G, nonregular on the closed convex hull $\mathrm{co}(G)$ of the spectrum of G. The functions $G^{-\alpha}$ $(\alpha > 0)$ and $(\ln G)^{-1}$ are examples of such functions. In addition, in the appropriate situations we improve the previously published estimates for the resolvent and functions of G regular on $\mathrm{co}(G)$. Since differential operators with variable coefficients often can be considered as perturbations of operators with constant coefficients, the results mentioned above give us estimates for functions and bounds for the spectra of differential operators with variable coefficients.

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1. Introduction and statement of the main result

Throughout this paper $\mathcal{E}$ is a separable Hilbert space with a scalar product $\langle .\,,.\rangle_{\mathcal{E}}$ and norm $\|.\|_{\mathcal{E}} = \sqrt{\langle .\,,.\rangle_E}$, $\mathbb{C}^n$ is the n-dimensional complex Euclidean space with a scalar product $\langle .\,,.\rangle_n$, the Euclidean norm $\|.\|_n = \sqrt{\langle .\,,.\rangle_n}$ and $\mathcal{H} = \mathbb{C}^n \otimes \mathcal{E}$ is the tensor product of $\mathbb{C}^n$ and $\mathcal{E}$. The scalar product in $\mathcal{H}$ is defined by

$$\langle h \otimes y, h_1 \otimes y_1\rangle_{\mathcal{H}} = \langle y, y_1\rangle_{\mathcal{E}}\,\langle h, h_1\rangle_n \quad (y, y_1 \in \mathcal{E}; h, h_1 \in \mathbb{C}^n)$$

and the cross norm is $\|.\|_{\mathcal{H}} = \sqrt{\langle .\,,.\rangle_{\mathcal{H}}}$. $I_{\mathcal{H}}, I_{\mathcal{E}}$ and I_n are the unit operators in $\mathcal{H}, \mathcal{E}$ and $\mathbb{C}^n$, respectively. From the theory of tensor products we only need the basic definition and elementary facts which can be found in [5].

For an operator B, $\sigma(B)$ denotes the spectrum, $\|B\|$ is the operator norm, B^* is the adjoint operator, and $\mathrm{Dom}(B)$ is the domain.

Throughout this paper S is a normal operator in $\mathcal{E}$ and A is an $n \times n$ matrix. Our main object is the operator

$$G = I_n \otimes S + A \otimes I_{\mathcal{E}}. \tag{1.1}$$

By $\|A\|_F$ we denote the Frobenius norm of A: $\|A\|_F := (\mathrm{Trace}\, A^*A)^{1/2}$; $\lambda_j(A)$ ($j = 1, \ldots, m$; $m \geq 2$) are the different eigenvalues of A; μ_j is the algebraic multiplicity of $\lambda_j(A)$. So

$$\delta := \min_{j,k=1,\ldots,m;\ k\neq j} |\lambda_j(A) - \lambda_k(A)| > 0 \tag{1.2}$$

and $\mu_1 + \cdots + \mu_m = n$. Numerous differential operators with constant matrix coefficients can be represented as in (1.1). Moreover, various matrix differential operators with variable coefficients can be considered as perturbations of G (see the example below).

Two operators A and B acting in $\mathcal{H}$ are said to be similar if there exists a boundedly invertible bounded operator T such that $A = T^{-1}BT$. The constant $\kappa_T = \|T^{-1}\|\,\|T\|$ is called the condition number. The condition number is important in various applications. We refer the reader to [3], where condition number estimates are suggested for combined potential boundary integral operators in acoustic scattering, and [24], where condition numbers are estimated for second-order elliptic operators. Conditions that provide the similarity of various operators to normal and selfadjoint ones were considered by many mathematicians; see [2, 6, 16–19, 21–23] and references given therein. In many cases, the condition number must be numerically calculated; see, for example, [1, 22]. The interesting generalization of condition numbers of bounded linear operators in Banach spaces was explored in [15]. Bounds for condition numbers of unbounded operators with Hilbert–Schmidt and Shatten–von Neumann hermitian components have been established in [10] and [13]. The paper [12] deals with bounded perturbations of unbounded normal operators.

Let P_j be the orthogonal invariant projection of A corresponding to $\lambda_j(A)$:

$$0 = P_0\mathbb{C}^n \subset P_1\mathbb{C}^n \subset P_2\mathbb{C}^n \subset \cdots \subset P_m\mathbb{C}^n = \mathbb{C}^n$$

and $P_jAP_j = AP_j$ $(j = 1, \ldots, m)$. Put

$$\hat{D} = \sum_{j=1}^{m} \Delta P_j A \Delta P_j \quad (\Delta P_j = P_j - P_{j-1}),$$

that is, $\hat{D}$ is the block diagonal matrix each of whose blocks $A_{jj} := \Delta P_j A \Delta P_j$ has only one eigenvalue. In the present paper we show that G is similar to the operator

$$M = I_n \otimes S + \hat{D} \otimes I_{\mathcal{E}}$$

and obtain a bound for the condition number. That bound enables us to establish norm estimates for the functions of G, which are nonregular on the closed convex hull $\mathrm{co}(G)$

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of the spectrum of G. The functions $G^{-\alpha}$ $(\alpha > 0)$ and $(\ln G)^{-1}$ are examples of such functions. In addition, in the appropriate situations we improve the estimates for the resolvent and functions of G, regular on the $\mathrm{co}(G)$; see [14].

The following quantity (the departure from normality) plays an essential role hereafter:

$$g(A) := \Big[\|A\|_F^2 - \sum_{k=1}^m \mu_k |\lambda_k(A)|^2\Big]^{1/2}.$$

$g(A)$ enjoys the following properties:

$$g^2(A) \le 2\|A_I\|_F^2 \quad (A_I = (A - A^*)/2i) \quad \text{and} \quad g^2(A) \le \|A\|_F^2 - |\mathrm{trace}\, A^2|;$$

see [8, Section 2.1]. If A is normal, then $g(A) = 0$. In addition, denote

$$d_j := \sum_{k=0}^{j} \frac{j!}{((j-k)!k!)^{3/2}} \quad (j = 0, \dots, n-2),$$

$$\theta(A) := \sum_{k=0}^{n-2} \frac{d_k g^k(A)}{\delta^{k+1}} \quad \text{and} \quad \gamma(A) := \Big(1 + \frac{g(A)\theta(A)}{\sqrt{m-1}}\Big)^{2(m-1)}.$$

It is not hard to check that $d_j \le 2^j$. Now we are in a position to formulate our main result.

THEOREM 1.1. *Let the operator G be defined as in (1.1). Then there is a bounded and boundedly invertible operator T_0 acting in $\mathcal{H}$, such that the equality*

$$T_0^{-1} G T_0 = M \tag{1.3}$$

is valid. Moreover,

$$\kappa(T_0) := \|T_0\|\,\|T_0^{-1}\| \le \gamma(A). \tag{1.4}$$

This theorem is proved in the next two sections. Note that we do not assume that the spectrum consists of simple eigenvalues or M is normal. Theorem 1.1 is sharp: if A is normal, then we have $\gamma(A) = 1$.

2. Auxiliary results

Put $Q_k = I - P_k$, $B_k = Q_k A Q_k$, $A_{jk} = \Delta P_j A \Delta P_k$ and $C_k = \Delta P_k A Q_k$ $(j, k = 1, \dots, m)$. Represent A, B_j and C_j in block form as

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1m} \\ 0 & A_{22} & A_{23} & \cdots & A_{2m} \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & A_{mm} \end{pmatrix},$$

$$B_j = Q_j A Q_j = \begin{pmatrix} A_{j+1,j+1} & A_{j+1,j+2} & \cdots & A_{j+1,m} \\ 0 & A_{j+2,j+2} & \cdots & A_{j+2,m} \\ . & . & \cdots & . \\ 0 & 0 & . & A_{mm} \end{pmatrix}$$

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and

$$C_j = \Delta P_j A Q_j = (A_{j,j+1}\, A_{j,j+2} \cdots A_{j,m}) \quad (j \le m-1).$$

Since B_j is a block triangular matrix, it is not hard to see that

$$\sigma(B_j) = \bigcup_{k=j+1}^{m} \sigma(A_{kk}) = \bigcup_{k=j+1}^{m} \lambda_k(A) \quad (j = 1, \ldots, m-1);$$

see [9, Lemma 2.1]. So due to (1.2),

$$\sigma(B_j) \cap \sigma(A_{jj}) = \emptyset \quad (j = 1, \ldots, m-1). \tag{2.1}$$

Under this condition, the equation

$$A_{jj}X_j - X_j B_j = -C_j \quad (j = 1, \ldots, m-1) \tag{2.2}$$

has a unique solution; see, for example, [4]. The following result has been proved in [9, Lemma 2.2].

Lemma 2.1. *Let conditions (2.1) hold and X_j be a solution to (2.2). Then*

$$(I - X_{m-1})(I - X_{m-2}) \cdots (I - X_1)\, A\, (I + X_1)(I + X_2) \cdots (I + X_{m-1}) = \hat{D}. \tag{2.3}$$

Take

$$T = (I + X_1)(I + X_2) \cdots (I + X_{m-1}). \tag{2.4}$$

It is simple to see that the inverse to $I + X_j$ is the matrix $I - X_j$; see [9]. Thus,

$$T^{-1} = (I - X_{m-1})(I - X_{m-2}) \cdots (I - X_1)$$

and (2.3) can be written as

$$T^{-1}AT = \operatorname{diag}(A_{kk})_{k=1}^{m} = \hat{D}. \tag{2.5}$$

By the inequalities between the arithmetic and geometric means, we get

$$\|T\| \le \prod_{j=1}^{m-1}(1 + \|X_j\|) \le \left(1 + \frac{1}{m-1}\sum_{j=1}^{m-1}\|X_j\|\right)^{m-1} \tag{2.6}$$

and

$$\|T^{-1}\| \le \left(1 + \frac{1}{m-1}\sum_{k=1}^{m-1}\|X_k\|\right)^{m-1}. \tag{2.7}$$

3. Proof of Theorem 1.1

Lemma 3.1. *Let condition (1.2) be fulfilled. Then there is an invertible $n \times n$ matrix T, such that (2.5) holds with*

$$\kappa_T = \|T\|\,\|T^{-1}\| \le \gamma(A).$$

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PROOF. Consider the Sylvester equation

$$ZX - X\tilde{Z} = C, \tag{3.1}$$

where $Z \in \mathbb{C}^{n_1\times n_1}, \tilde{Z} \in \mathbb{C}^{n_2\times n_2}$ and $C \in \mathbb{C}^{n_1\times n_2}$ are given; $X \in \mathbb{C}^{n_1\times n_2}$ should be found. Assume that the eigenvalues $\lambda_k(Z)$ and $\lambda_j(\tilde{Z})$ of Z and $\tilde{Z}$, respectively, satisfy the condition.

$$\rho_0(Z,\tilde{Z}) := \operatorname{dist}(\sigma(Z),\sigma(\tilde{Z})) = \min_{j,k} |\lambda_k(Z) - \lambda_j(\tilde{Z})| > 0.$$

Then Equation (3.1) has a unique solution X [4]. Corollary 6.2 in [11] implies the inequality

$$\|X\|_F \le \|C\|_F \sum_{p=0}^{n_1+n_2-2} \frac{1}{\rho_0^{p+1}(Z,\tilde{Z})} \sum_{k=0}^{p} \binom{p}{k} \frac{g^k(\tilde{Z})g^{p-k}(Z)}{\sqrt{(p-k)!k!}}$$

and therefore

$$\|X\|_F \le \|C\|_F \sum_{p=0}^{n_1+n_2-2} \frac{d_p\hat{g}^p}{\rho_0^{p+1}(Z,\tilde{Z})}, \tag{3.2}$$

where $\hat{g} = \max\{g(Z), g(\tilde{Z})\}$.

Let us return to Equation (2.2). In this case $Z = A_{jj}$, $\tilde{Z} = B_j$, $C = -C_j$, $n_1 = \mu_j$, $n_2 = \dim\ Q_j\mathbb{C}^n$, and due to (1.2), $\rho_0(A_{jj}, B_j) \ge \delta$ $(j = 1,\dots,m)$. In addition, $\mu_j + \dim\ Q_j\mathbb{C}^n \le n$. Now (3.2) implies

$$\|X_j\|_F \le \|C_j\|_F \sum_{k=0}^{n-2} \frac{d_k\hat{g}_j^k}{\delta^{k+1}}, \tag{3.3}$$

where $\hat{g}_j = \max\{g(B_j), g(A_{jj})\}$. By Schur's theorem, for any operator A in $\mathbb{C}^n$, there is an orthogonal normal basis (Schur's basis) $\{e_k\}_{k=1}^n$, in which A is represented by a triangular matrix; see [7]. That is,

$$Ae_k = \sum_{j=1}^{k} a_{jk}e_j \quad \text{with } a_{jk} = (Ae_k, e_j)\ (j = 1,\dots,n).$$

So $A = D_A + V_A$ $(\sigma(A) = \sigma(D_A))$ with a normal (diagonal) matrix D_A defined by $D_Ae_j = a_{jj}e_j$ $(j = 1,\dots,n)$ and a nilpotent (strictly upper-triangular) matrix V_A defined by $V_Ae_k = a_{1k}e_1 + \cdots + a_{k-1,k}e_{k-1}$ $(k = 2,\dots,n)$, $V_Ae_1 = 0$. D_A and V_A will be called *the diagonal part and nilpotent part* of A, respectively. Note that Schur's basis is not unique. Besides, $g(A) = \|V_A\|_F$. In addition, simple calculations show that the nilpotent part V_j of A_{jj} is $\Delta P_jV_A\Delta P_j$ and the nilpotent part W_j of B_j is $Q_jV_AQ_j$. So V_j and W_j are mutually orthogonal, and

$$g(A_{jj}) = \|V_j\|_F \le \|V_A\|_F = g(A), \quad g(B_j) = \|W_j\|_F \le \|V_A\|_F^2 = g(A). \tag{3.4}$$

From (3.3) and (3.4) it follows that

$$\|X_j\|_F \le \|C_j\|_F \sum_{k=0}^{n-2} \frac{d_kg^k(A)}{\delta^{k+1}} = \|C_j\|_F\theta(A). \tag{3.5}$$

It can be directly checked that

$$\|C_j\|_F^2 = \sum_{k=j+1}^{m} \|A_{jk}\|_F^2$$

and

$$\sum_{i=1}^{m-1}\sum_{k=i+1}^{m} \|A_{ik}\|_F^2 = \sum_{i=1}^{m}\sum_{k=i}^{m} \|A_{ik}\|_F^2 - \sum_{k=1}^{m} \|A_{kk}\|_F^2 = \|A\|_F^2 - \sum_{k=1}^{m} \|A_{kk}\|_F^2.$$

Since $\|A_{kk}\|_F \geq \mu_k|\lambda_k(A)|$, we have

$$\sum_{j=1}^{m-1}\sum_{k=j+1}^{m} \|A_{jk}\|_F^2 \leq g^2(A),$$

and consequently,

$$\sum_{k=1}^{m-1} \|C_k\|_F^2 \leq g^2(A). \tag{3.6}$$

Furthermore, take T as in (2.4). Then (2.6), (2.7) and (3.5) imply

$$\|T\| \leq \left(1 + \frac{1}{m-1}\sum_{k=1}^{m-1} \|X_k\|_F\right)^{m-1} \leq \left(1 + \frac{\theta(A)}{m-1}\sum_{k=1}^{m-1} \|C_k\|_F\right)^{m-1}$$

and

$$\|T^{-1}\| \leq \left(1 + \frac{\theta(A)}{m-1}\sum_{k=1}^{m-1} \|C_k\|_F\right)^{m-1}.$$

But by the Schwarz inequality and (3.6),

$$\left(\sum_{j=1}^{m-1} \|C_j\|_F\right)^2 \leq (m-1)\sum_{j=1}^{m-1} \|C_j\|_F^2 \leq (m-1)g^2(A).$$

Thus,

$$\|T\|^2 \leq \left(1 + \frac{\theta(A)}{\sqrt{m-1}}g(A)\right)^{2(m-1)} = \gamma(A)$$

and $\|T^{-1}\|^2 \leq \gamma(A)$. Now (2.5) proves the lemma. □

LEMMA 3.2. *Let there be a matrix T such that (2.5) holds. Then (1.3) is valid with $T_0 = T \otimes I_{\mathcal{E}}$.*

PROOF. Indeed, we have

$$(T^{-1} \otimes I_{\mathcal{E}})(I_n \otimes S + A \otimes I_{\mathcal{E}})(T \otimes I_{\mathcal{E}}) = I_n \otimes S + (T^{-1}AT) \otimes I_{\mathcal{E}} = I_n \otimes S + \hat{D} \otimes I_{\mathcal{E}}.$$

This proves the result. □

The assertion of Theorem 1.1 follows from Lemmas 3.1 and 3.2. □

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4. Applications to operator functions

Let $E(s)$ be the orthogonal resolution of the identity of S defined on $\sigma(S)$, such that

$$S = \int_{\sigma(S)} s\, dE(s).$$

Then the operators G and M defined in Section 1 can be written as

$$G = \int_{\sigma(S)} (sI_n + A) \otimes dE(s) \quad \text{and} \quad M = \int_{\sigma(S)} (sI_n + \hat{D}) \otimes dE(s). \tag{4.1}$$

Let $f(z)$ be a scalar function, regular on $\sigma(G)$, and

$$\sup_{s\in\sigma(S)} \max_{j=1,\dots,m} |f^{(k)}(\lambda_j(A) + s)| < \infty \quad (k = 1, \dots, \mu_j - 1). \tag{4.2}$$

Define $f(A + sI_n)$ $(s \in \sigma(S))$ in the usual way via the Cauchy integral [4] and consider the operator function

$$f(G) = \int_{\sigma(S)} f(sI_n + A) \otimes dE(s).$$

Similarly,

$$f(M) = \int_{\sigma(S)} f(sI_n + \hat{D}) \otimes dE(s).$$

Note that the considerably more general operator functions on tensor products of spaces have been considered in [20] and references given therein.

By Lemma 3.1, $f(sI_n + A) = T^{-1} f(sI_n + \hat{D})T$ and therefore $f(G) = T_0^{-1} f(M)T_0$. So by (1.4) we have $\|f(G)\|_{\mathcal{H}} \le \kappa(T_0)\|f(M)\|_{\mathcal{H}} \le \gamma(A)\|f(M)\|_{\mathcal{H}}$. It is not hard to see that

$$\|f(M)\|_{\mathcal{H}} \le \sup_{s\in\sigma(S)} \|f(sI_n + \hat{D})\|_n.$$

Since A_{jj} are mutually orthogonal, we have

$$f(\hat{D} + sI_n) = \sum_{k=1}^{m} \Delta P_j f(A_{jj} + sI_n) \quad \text{and} \quad \|f(\hat{D} + sI_n)\|_n = \max_j \|\Delta P_j f(A_{jj} + sI_n)\|_n.$$

We thus arrive at the following theorem.

Theorem 4.1. *Let $f(z)$ be a scalar function, regular on a neighborhood of $\sigma(G)$, and let condition (4.2) hold. Then*

$$\|f(G)\| \le \kappa(T_0) \sup_{s\in\sigma(S)} \max_j \|\Delta P_j f(A_{jj} + sI_n)\|_n.$$

Due to [8, Corollary 2.7.2], we have

$$\|f(A_{jj} + sI_n)\|_{\mu_j} \le \sum_{k=0}^{\mu_j - 1} |f^{(k)}(\lambda_j(A) + s)| \frac{g^k(A_{jj})}{(k!)^{3/2}}.$$

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Taking into account (3.4), we get

$$\|f(A+sI_n)\|_n \le \gamma(A)\max_j \sum_{k=0}^{\mu_j-1} |f^{(k)}(\lambda_j(A)+s)| \frac{g^k(A)}{(k!)^{3/2}}.$$

Now Theorem 4.1 immediately implies our next result.

COROLLARY 4.2. *Under the hypothesis of Theorem 4.1,*

$$\|f(G)\| \le \gamma(A) \sup_{s\in\sigma(S)} \max_j \sum_{k=0}^{\mu_j-1} |f^{(k)}(\lambda_j(A)+s)| \frac{g^k(A)}{(k!)^{3/2}}.$$

In the following examples the operator G is defined by (1.1) (and therefore, by (4.1)).

EXAMPLE 4.3. Let $\alpha(S) := \sup_{s\in\sigma(S)} \Re\sigma(S) < \infty$. Then the semigroup e^{tG} of G is representable by

$$e^{tG} = \int_{\sigma(S)} e^{(sI_n+A)t} \otimes dE(s), \quad t \ge 0.$$

Now Corollary 4.2 implies

$$\|e^{tG}\| \le \gamma(A) e^{(\alpha(A)+\alpha(S))t} \sum_{k=0}^{\hat{\mu}-1} t^k \frac{g^k(A)}{(k!)^{3/2}} \quad (t \ge 0),$$

where $\alpha(A) = \max_k \Re\, \lambda_k(A)$ and $\hat{\mu} = \max_j \mu_j$.

EXAMPLE 4.4. Let

$$\xi_0 := \inf_{j=1,\ldots,m, s\in\sigma(S)} |\lambda_j(A)+s| > 0.$$

Define the fraction power by

$$G^{-\nu} = \int_{\sigma(S)} (sI+A)^{-\nu} \otimes dE(s) \quad (0<\nu<1).$$

Here the fraction power of the nonsingular matrix is defined in the standard way; see, for example, [7]. Since

$$\sup_{j=1,\ldots,m; z\in\sigma(S)} \left| \frac{d^k}{dz^k} \frac{1}{(z+\lambda_j(A))^\nu} \right| \le \frac{\nu(\nu+1)\cdot(\nu+k-1)}{\xi_0^{k+\nu}} \quad (j=1,\ldots,m; k=1,2,\ldots),$$

by Corollary 4.2,

$$\|G^{-\nu}\| \le \gamma(A) \sum_{k=0}^{\hat{\mu}-1} \frac{g^k(A)\nu(\nu+1)\cdot(\nu+k-1)}{\xi_0^{k+\nu}(k!)^{3/2}}.$$

Similarly, one can estimate the function

$$(\ln G)^{-1} = \int_{\sigma(S)} (\ln(sI_n+A))^{-1} \otimes dE(s)$$

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provided the condition $\xi_0 > 0$ holds. The matrix logarithm is defined in that standard way; see, for example, [4, Section V.1].

Furthermore, it is simple to see that the resolvent of G is representable by

$$(G - \lambda I_{\mathcal{H}})^{-1} = \int_{\sigma(S)} (sI_n + A - \lambda I_n)^{-1} \otimes dE(s) \quad (\lambda \notin \sigma(G)). \tag{4.3}$$

Theorem 4.1 yields

$$\|R_z(G)\|_{\mathcal{H}} \le \kappa(T_0) \sup_{s\in\sigma(S)} \max_j \|\Delta P_j(A_{jj} - (\lambda - sI_{\mu_j}))^{-1}\|_{\mu_j}.$$

Due to [11, Theorem 1.1],

$$\|(A_{jj} - (\lambda - sI_{\mu_j}))^{-1}\|_{\mu_j} \le \sum_{k=0}^{\mu_j - 1} \frac{g^k(A_{jj})}{\rho^{k+1}(A_{jj} + sI_{\mu_j}, \lambda)\sqrt{k!}},$$

where $\rho(B, \lambda)$ denotes the distance between $\lambda \in \mathbb{C}$ and the spectrum of an operator B. Clearly, $\rho(A_{jj} + sI_{\mu_j}, \lambda) = |\lambda_j(A) + s - \lambda| \ge \rho(G, \lambda)$ $(j = 1, \ldots, m;\ s \in \sigma(S))$. Now Corollary 4.2 implies our next result.

Corollary 4.5. *We have*

$$\|R_z(G)\|_{\mathcal{H}} \le \gamma(A) \sum_{k=0}^{\hat{\mu}-1} \frac{g^k(A)}{\rho^{k+1}(G, \lambda)\sqrt{k!}} \quad (\lambda \notin \sigma(G)).$$

5. Spectrum perturbations

Corollary 4.5 enables us to investigate spectrum perturbations. Let G and $\tilde{G}$ be linear operators in $\mathcal{H}$ with

$$\mathrm{Dom}(\tilde{G}) = \mathrm{Dom}(G) \quad \text{and} \quad q := \|G - \tilde{G}\|_{\mathcal{H}} < \infty. \tag{5.1}$$

Introduce the quantity

$$sv_G(\tilde{G}) := \sup_{s\in\sigma(\tilde{G})} \inf_{t\in\sigma(G)} |t - s|$$

(the spectral variation of $\tilde{G}$ with respect to G). We need the following technical lemma.

Lemma 5.1. *Let condition (5.1) hold and*

$$\|R_\lambda(G)\| \le \psi\left(\frac{1}{\rho(G, \lambda)}\right) \quad (\lambda \notin \sigma(G)),$$

where $\psi(x)$ is a monotonically increasing continuous function of a nonnegative variable x, such that $\psi(0) = 0$ and $\psi(\infty) = \infty$. Then $sv_G(\tilde{G}) \le z(\psi, q)$, where $z(\psi, q)$ is the unique positive root of the equation $q\psi(1/z) = 1$.

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For the proof see [8, Lemma 8.4.2]. Now let G be defined by (1.1). Then Corollary 4.5 and Lemma 5.1 imply $sv_G(\tilde{G}) \le z(A,q)$, where $z(A,q)$ is the unique positive root of the equation

$$q\gamma(A)\sum_{k=0}^{\hat{\mu}-1}\frac{g^k(A)}{z^{k+1}\sqrt{k!}} = 1.$$

This equation is equivalent to the algebraic one

$$z^{\hat{\mu}} = q\gamma(A)\sum_{k=0}^{\hat{\mu}-1}\frac{g^k(A)z^{\hat{\mu}-k-1}}{\sqrt{k!}}.$$

Various estimates for the roots of algebraic equations are well known. For example, if

$$p(A,q) := q\gamma(A)\sum_{k=0}^{\hat{\mu}-1}\frac{g^k(A)}{\sqrt{k!}} < 1, \tag{5.2}$$

then, due to [8, Lemma 1.6.1], we have $z^{\hat{\mu}}(A,q) \le p(A,q)$. So we arrive at the following result.

Corollary 5.2. *Let G be defined by (1.1) and let condition (5.1) hold. Then $sv_G(\tilde{G}) \le z(A,q)$. In particular, if condition (5.2) is fulfilled, then $sv_G^{\hat{\mu}}(\tilde{G}) \le p(A,q)$.*

6. Differential operators with matrix coefficients

In this section we apply our results to matrix differential operators. Here $\mathcal{E} = L^2(0,1)$ (the space of scalar square integrable functions defined on $[0,1]$) and $\mathcal{H} = \mathbb{C}^n \otimes \mathcal{E} = L^2([0,1],\mathbb{C}^n)$ (the space of square integrable functions defined on $[0,1]$ with values in $\mathbb{C}^n$). On the domain

$$\mathrm{Dom}(\tilde{G}_0) = \{u \in \mathcal{H} : u'' \in \mathcal{H};\ u(0) = u(1) = 0\}$$

consider the operator

$$\tilde{G}_0 = -\frac{d}{dx}a(x)\frac{d}{dx} + C(x) \quad (x \in (0,1)),$$

where $a(x)$ is a scalar positive function having a continuous derivative and $C(x)$ is a variable bounded measurable $n \times n$ matrix. Take $S = -(d/dx)a(x)(d/dx)$ with

$$\mathrm{Dom}(S) = \{u \in L^2(0,1) : u'' \in L^2(0,1);\ u(0) = u(1) = 0\}.$$

In addition, $G_0 = I_n \otimes S + I_{\mathcal{E}} \otimes A$ with some constant $n \times n$ matrix A and $\mathrm{Dom}(\tilde{G}_0) = \mathrm{Dom}(G_0) = \mathrm{Dom}(S \otimes I_n)$. Then $q = \|G_0 - \tilde{G}_0\|_{\mathcal{H}} \le \sup_x \|C(x) - A\|_n$ and

$$\sigma(G_0) = \{\lambda_k(S) + \lambda_j(A) : k = 1,2,\ldots;\ j = 1,\ldots,m\}.$$

Due to Corollary 5.2, $sv_{G_0}(\tilde{G}_0) \le z(A,q)$ and $z(A,q)$ can be estimated as in that corollary. Thus, the spectrum of $\tilde{G}_0$ lies in the sets

$$\{\mu \in \mathbb{C} : |\mu - (\lambda_k(S) + \lambda_j(A))| \le z(A,q) : k = 1,2,\ldots;\ j = 1,\ldots,m\}.$$

For example, if $a(x) = 1$ identically, then $\lambda_k(S) = \pi k^2$.

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Furthermore, Example 4.3 gives us the norm estimate for the semigroup of $-G_0$, and the formula

$$e^{-G_0 t} - e^{-\tilde{G}_0 t} = \int_0^t e^{-G_0(t-t_1)}(G_0 - \tilde{G}_0)e^{-\tilde{G}_0 t_1}\,dt_1$$

enables us to estimate the semigroup of $-\tilde{G}_0$. Moreover, Example 4.4 gives us a norm estimate for the fractional powers of G_0 and the relations

$$\begin{aligned} G_0^{-\nu} - \tilde{G}_0^{-\nu} &= \frac{\sin(\pi\nu)}{\pi}\int_0^\infty t^{-\nu}((G_0 + tI_{\mathcal{H}})^{-1} - (\tilde{G}_0 + tI_{\mathcal{H}})^{-1})\,dt \\ &= \frac{\sin(\pi\nu)}{\pi}\int_0^\infty t^{-\nu}(G_0 + tI_{\mathcal{H}})^{-1}(G_0 - \tilde{G}_0)(\tilde{G}_0 + tI_{\mathcal{H}})^{-1}\,dt \\ &\qquad (\inf \Re\sigma(G_0) > 0, \inf \Re\sigma(\tilde{G}_0) > 0) \end{aligned}$$

enable us to estimate the fractional powers of $\tilde{G}_0$.

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