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SOME PROPERTIES OF GENERALIZED EULER NUMBERS

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1. Introduction. We define infinitely many sequences of integers $\{E_n^{(k)}\}_{n=0}^{\infty}$, one sequence for each positive integer $k \geqq 2$ by

$$(1.1)\quad \begin{cases} E_0^{(k)} = 1 \\ \sum_{j=1}^{k} (E^{(k)} + \omega_j^{(k)})^n = \begin{cases} k, & n = 0 \\ 0, & n > 0 \end{cases} \end{cases}$$

where $\{\omega_j^{(k)}\}_{j=1}^{k}$ are the k-th roots of unity and $(E^{(k)})^n$ is replaced by $E_n^{(k)}$ after multiplying out. An immediate consequence of (1.1) is

$$(1.2)\quad E_n^{(k)} = 0 \quad n \not\equiv 0 (\mathrm{mod}\ k).$$

Therefore, we are interested in numbers of the form $E_{sk}^{(k)}$ ($s = 0, 1, 2, \ldots; k = 2, 3, \ldots$).

Some special cases have been considered in the literature. For $k = 2$, we obtain the Euler numbers (see e.g. [**8**]). The case $k = 3$ is considered briefly by D. H. Lehmer [**7**], and the case $k = 4$ by Leeming [**6**] and Carlitz ([**1**] and [**2**]).

2. General properties of the number sequences $\{E_{sk}^{(k)}\}_{s=0}^{\infty}$. There are some interesting properties shared by all sequences $\{E_{sk}^{(k)}\}$ defined by (1.1) which we present as a series of theorems.

THEOREM 2.1. *For $k = 2, 3, \ldots; n = 0, 1, \ldots$ we have*

$$(2.1)\quad \sum_{s=0}^{n} \binom{nk}{sk} E_{sk}^{(k)} = \begin{cases} 1, & n = 0 \\ 0, & n > 0. \end{cases}$$

Proof. We have defined $E_{sk}^{(k)}$ ($k = 2, 3, \ldots; s = 0, 1, \ldots$) by (1.1). Since $E_0^{(k)} = 1$, the result is true for $n = 0$. For the case $n > 0$,

$$\sum_{j=1}^{k} \sum_{s=0}^{nk} \binom{nk}{s} (E^{(k)})^s (\omega_j^{(k)})^{nk-s} = 0.$$

Using (1.2), this reduces to

$$\sum_{j=1}^{k} \sum_{s=0}^{n} \binom{nk}{sk} E_{sk}^{(k)} (\omega_j^{(k)})^{nk-sk} = 0.$$

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Since $(\omega_j^{(k)})^{nk-sk} = 1$ $(j = 1, 2, \ldots, k; s = 0, 1, \ldots)$, we have

$$\sum_{s=0}^{n} \binom{nk}{sk} E_{sk}^{(k)} (\omega_j^{(k)})^{nk-sk} = \sum_{s=0}^{n} \binom{nk}{sk} E_{sk}^{(k)}, \quad (n > 0)$$

independent of j. Therefore,

$$\sum_{j=1}^{k} \sum_{s=0}^{n} \binom{nk}{sk} E_{sk}^{(k)} = k \sum_{s=0}^{n} \binom{nk}{sk} E_{sk}^{(k)} = 0,$$

and (2.1) follows.

THEOREM 2.2. $E_{sk}^{(k)} \equiv 1 (\mathrm{mod}\ 2)$, $k = 2, 3, \ldots; s = 0, 1, \ldots$.

Proof. From (2.1) we have

$$E_0^{(k)} = 1, \ E_k^{(k)} = -1, \ E_{2k}^{(k)} = \binom{2k}{k} - 1,$$

and since

$$\binom{2k}{k} = 2\binom{2k-1}{k},$$

$E_{2k}^{(k)}$ is odd. Proceeding by induction, we assume $E_{(n-1)k}^{(k)}$ is odd for some fixed value of k. Using (2.1) we have

$$(2.2) \quad E_{nk}^{(k)} = -1 - \binom{nk}{k}\left(E_k^{(k)} + E_{(n-1)k}^{(k)}\right) - \sum_{s=2}^{n-2} \binom{nk}{sk} E_{sk}^{(k)}.$$

Now $E_k^{(k)} = -1$ and by our inductive assumption $E_{(n-1)k}^{(k)}$ is odd so the second term in the right hand member of (2.2) is even. Now we need only show that

$$\sum_{s=2}^{n-2} \binom{nk}{sk} E_{sk}^{(k)}$$

is even. There are two cases to consider.

Case 1. (n odd). Then we have

$$\sum_{s=2}^{n-2} \binom{nk}{sk} E_{sk}^{(k)} = \sum_{s=2}^{(n/2)-1} \binom{nk}{sk} (E_{sk}^{(k)} + E_{nk-sk}^{(k)}),$$

which is even by our inductive assumption.

Case 2. (n even). We remove the term with $s = n/2$ and pair the remaining $n - 4$ terms to obtain

$$(2.3) \quad \sum_{s=2}^{n-2} \binom{nk}{sk} E_{sk}^{(k)} = \binom{nk}{\frac{n}{2}k} E_{(n/2)k}^{(k)} + \sum_{s=2}^{(n/2)-1} \binom{nk}{sk} (E_{sk}^{(k)} + E_{nk-sk}^{(k)}).$$

Now

$$\binom{nk}{\frac{n}{2}k}$$

is even, and the second member on the right in (2.3) is even by the inductive assumption. Therefore, the left hand member of (2.3) is even.
This completes the proof of Theorem 2.2.

THEOREM 2.3. *For* $k = 2, 3, \ldots; s = 0, 1, \ldots$

$$E_{sk}^{(k)} = \sum_{\substack{n_1+\ldots+n_t=s \\ n_i>0}} (-1)^t \binom{sk}{n_1k, \ldots, n_tk}. \tag{2.5}$$

Proof. Define

$$A_{q,j}^{(k)} = \sum_{\substack{l_1+\ldots+l_j=q \\ l_i>0}} \frac{(qk)!}{(l_1k)!\ldots(l_jk)!} = \sum_{\substack{l_1+\ldots+l_j=q \\ l_i>0}} \binom{qk}{l_1k, \ldots, l_jk}.$$

Then,

$$A_{q,j+1}^{(k)} = \sum_{r=j}^{q-1} \sum_{\substack{l_1+\ldots+l_j=q \\ l_i>0}} \frac{(qk)!}{((q-r)k)!(l_1k)!\ldots(l_jk)!}.$$

We now show by induction that

$$E_{sk}^{(k)} = \sum_{j=1}^{s} (-1)^j A_{s,j}^{(k)}. \tag{2.6}$$

Let k be fixed ($k \geqq 2$). In the case $s = 1$

$$E_k^{(k)} = -A_{1,1}^{(k)} = -\frac{k!}{k!} = -1.$$

Proceeding by induction, assume (2.6) is true for some positive integer s. Now from (2.1) we have

$$\sum_{r=0}^{s+1} \binom{(s+1)k}{rk} E_{rk}^{(k)} = 0;$$

so, since $E_0^{(k)} = 1$,

$$\begin{aligned}
E_{(s+1)k}^{(k)} &= -\sum_{r=1}^{s} \binom{(s+1)k}{rk} E_{rk}^{(k)} - 1 \\
&= -\sum_{r=1}^{s} \binom{(s+1)k}{rk} \sum_{j=1}^{r} (-1)^j A_{r,j}^{(k)} - 1 \\
&= -\sum_{j=1}^{s} (-1)^j \sum_{r=j}^{s} \binom{(s+1)k}{rk} \sum_{\substack{l_1+\ldots+l_j=r \\ l_i>0}} \frac{(rk)!}{(l_1k)!\ldots(l_jk)!} - 1 \\
&= -\sum_{j=1}^{s} (-1)^j \sum_{\substack{r=j \\ l_1+\ldots+l_j=r \\ l_i>0}}^{s} \frac{((s+1)k)!}{[(s+1-r)k]!(l_1k)!\ldots(l_jk)!} - 1 \\
&= -\sum_{j=1}^{s} (-1)^j \sum_{r=j}^{s} A_{s+1,j+1}^{(k)} - 1.
\end{aligned}$$

Setting $\nu = j + 1$ yields

$$E_{(s+1)k}^{(k)} = -\sum_{\nu=2}^{s+1} (-1)^{\nu-1} A_{s+1,\nu}^{(k)} - 1 = \sum_{\nu=2}^{s+1} (-1)^{\nu} A_{s+1,\nu}^{(k)} - 1$$
$$= \sum_{\nu=1}^{s+1} (-1)^{\nu} A_{s+1,\nu}^{(k)}.$$

3. A generating function for the number sequences $\{E_{sk}{}^{(k)}\}_{s=0}^{\infty}$. Suppose $f(x)$ has the (formal) Maclaurin expansion

(3.1) $\quad f(x) = \sum_{m=0}^{\infty} a_m x^m.$

Then we have the following result.

THEOREM 3.1. *For* $k = 2, 3, \ldots$

(3.2) $\quad \sum_{j=1}^{k} f(E^{(k)} + x + \omega_j{}^{(k)}) = kf(x).$

Proof.

$$\sum_{j=1}^{k} f(E^{(k)} + x + \omega_j{}^{(k)}) = \sum_{j=1}^{k}\sum_{m=0}^{\infty} a_m (E^{(k)} + x + \omega_j{}^{(k)})^m$$
$$= \sum_{m=0}^{\infty} a_m \sum_{n=0}^{m} \left\{ \sum_{j=1}^{k} (E^{(k)} + \omega_j{}^{(k)})^n \right\} \binom{m}{n} x^{m-n}$$
$$= k\, a_0 + \sum_{m=1}^{\infty} a_m \left\{ kx^m + \sum_{n=1}^{m} \left(\sum_{j=1}^{k} (E^{(k)} + \omega_j{}^{(k)})^n \right) \binom{m}{n} x^{m-n} \right\}$$
$$= k\, a_0 + k \sum_{m=1}^{\infty} a_m x^m \quad \text{by (1.1)} \quad = kf(x).$$

COROLLARY 3.1. *For* $k = 2, 3, \ldots$ *we have*

(3.3) $\quad \sum_{j=1}^{k} \{E^{(k)} + (x + \omega_j{}^{(k)})\}^n = kx^n$

Note. If $n = x = 0$, the right hand side is to be read as k.

We now obtain a generating function for each number sequence $\{E_{sk}{}^{(k)}\}_{s=0}^{\infty}$ $(k = 2, 3, \ldots)$.

THEOREM 3.2. *For* $k = 2, 3, \ldots$

$$e^{E^{(k)}z} = \frac{1}{Q_k(z)} \quad \textit{where} \quad Q_k(z) = \sum_{s=0}^{\infty} \frac{z^{sk}}{(sk)!}.$$

Proof. Set $x = 0$ and $f(t) = e^{zt}$ in (3.2). Then $f(0) = a_0 = 1$, and we obtain (setting $\omega = \omega^{(k)}$)

$$e^{(E^{(k)}+1)z} + e^{(E^{(k)}+\omega)z} + \ldots + e^{(E^{(k)}+\omega^{k-1})z} = k$$

i.e.,

$$e^{E^{(k)}z}(e^z + e^{\omega z} + \ldots + e^{\omega^{k-1}z}) = k,$$

$$\frac{k}{e + e^{\omega z} + \ldots + e^{\omega^{k-1}z}} = e^{E^{(k)}z}.$$

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Using the property of the k-th roots of unity

$$1 + \omega^{l} + \omega^{2l} + \ldots + \omega^{(k-1)l} = \begin{cases} k, & l \equiv 0 \pmod{k} \\ 0, & l \not\equiv 0 \pmod{k} \end{cases}$$

it is easily shown that

$$e^{z} + e^{\omega z} + \ldots + e^{\omega^{k-1} z} = k \sum_{s=0}^{\infty} \frac{z^{ks}}{(ks)!}$$

and the result follows.

4. Some congruence relations for the number sequences $\{E_{sk}^{(k)}\}_{s=0}^{\infty}$. (Note: In this section, we shall drop the superscript (k).) Frobenius [**4**, p. 477] proved that

$$E_{2n} \equiv 1 - 2n + 8\binom{n}{2} \pmod{16},$$

as well as more precise results. Carlitz [**1**] proved that

$$E_{4n} \equiv 1 - 2n + 8\binom{n}{2} \pmod{16}.$$

We have proven a number of similar results for higher-order sequences, and are able to conjecture several more. These are stated in Theorems 4.1 and 4.2 and Conjecture 4.1.

THEOREM 4.1. *We have the following congruences* mod 16.

(i) $E_{3n} \equiv 3 - 4n + 8\binom{n+1}{3}$

(ii) $E_{6n} \equiv 3 - 4n$

(iii) $E_{8n} \equiv 1 - 2n + 8\binom{n}{2}$

(iv) $E_{16n} \equiv 1 - 2n + 8\binom{n}{2}$

(v) $E_{12n} \equiv 3 - 4n$.

THEOREM 4.2. *In addition, we have the following congruences*

(i) $E_{3n} \equiv (-1)^{n} \pmod{18}$.

(ii) $E_{5n} \equiv (-1)^{n} \pmod{250}$ $(250 = 2 \cdot 5^{3})$

(iii) $E_{6n} \equiv -1 \pmod{12}$.

CONJECTURE 4.1. *The following congruences appear to be valid, on the basis of considerable numerical evidence.*

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(i) $E_{7n} \equiv 7 - 8n \pmod{16}$
(ii) $E_{11n} \equiv (-1)^n \pmod{2662}$ $(2662 = 2 \cdot 11^3)$
(iii) $E_{13n} \equiv (-1)^n \pmod{4394}$ $(4394 = 2 \cdot 13^3)$
(iv) $E_{14n} \equiv E_{7n} \pmod{16}$ $(\equiv 7 - 8\text{n}?)$
(v) $E_{15n} \equiv 15 \pmod{16}$
(vi) $E_{2^k n} \equiv 1 - 2n + 8\binom{n}{2} \pmod{16}, k = 1, 2, \ldots.$

Notes. 1. Parts (i) and (ii) of Theorem 4.1 would disprove a conjecture that $E_{(2k)n} \equiv E_{kn} \pmod{16}$, $k = 2, 3, \ldots$.

2. Theorem 4.2, parts (i) and (ii), and Conjecture 4.1, parts (iii) and (iv) might suggest a conjecture of the type $E_{pn} \equiv (-1)^n \pmod{2p^\alpha}$ for some positive α. But $E_{7n} \equiv 4 \pmod{686}$ for $n = 11$ and 13.

3. There are other conjectures which could be made on the basis of numerical studies, but they don't seem to have quite such appealing formulas. For example, mod 16, E_{5n} reproduces the following set of twelve residues, repeating:

15, 11, 1, 3, 11, 9, 7, 3, 1, 11, 3, 9;

while E_{9n} reproduces the following set of 28 residues, repeating:

15, 11, 5, 3, 1, 9, 3, 3, 3, 1, 11, 9, 13, 11, 7, 3, 13, 3, 9, 9, 3,
11, 11, 9, 11, 1, 13, 11.

In order to prove the two theorems we shall establish several preliminary results.

LEMMA 4.1.1. *In order to show*

(4.1) $\quad E_{tn} \equiv f(n) \pmod{g(t)}, \ n = 1, 2, \ldots$

it suffices to prove that this is true for $n = 1$ *and that*

$$\sum_{k=0}^{n} \binom{tn}{tk} f(k) \equiv -1 + f(0) \pmod{g(t)}, \quad n = 2, 3, \ldots. \tag{4.2}$$

Proof. From (4.2) and (2.1) we would have

$$\sum_{k=1}^{n} \binom{tn}{tk} \{f(k) - E_{tk}\} \equiv 0 \bmod (g(t)).$$

Thus we would have

$$\sum_{k=1}^{n-1} \binom{tn}{tk} \{f(k) - E_{tk}\} + f(n) - E_{tn} \equiv 0 \bmod (g(t)),$$

and (4.1) is now a simple induction on n.

LEMMA 4.1.2.

(i) $$\sum_{k=0}^{[(n-l)/j]} \binom{n}{l + jk} x^{l+kj} = \frac{1}{j} \sum_{r=1}^{j} (\omega_j{}^r)^{-l}(1 + x\omega_j{}^r)^n,$$

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where $\omega_j = e^{2\pi i/j}$, l *a non-negative integer, and* $j - 1 \geqq l$. *In particular*

$$\text{(ii)} \sum_{k=0}^{[(n-l)/j]} \binom{n}{l+jk} = \frac{1}{j}\sum_{r=1}^{j}\left(2\cos\frac{\pi r}{j}\right)^n \cos\frac{(n-2l)r\pi}{j}$$

$$\text{(iii)} \sum_{k=0}^{n} \binom{jn}{jk} = \frac{2^{jn}}{j}\sum_{r=0}^{j-1}\left(\cos\frac{\pi r}{j}\right)^{jn}(-1)^{rn}$$

$$\text{(iv)} \sum_{k=0}^{n} \binom{jn-2}{jk} = \frac{2^{jn-1}}{j}\sum_{r=0}^{j-1}\left(\cos\frac{\pi r}{j}\right)^{jn}(-1)^{rn} - \frac{2^{jn-2}}{j}\sum_{r=0}^{j-1}\left(\cos\frac{\pi r}{j}\right)^{jn-2}(-1)^{rn}.$$

Proof. The first result appears as relation 1.53 in [**5**], and is easily proven from the binomial theorem and properties of primitive roots of unity. The terms for $r = 0$ and $r = j$ are the same.

LEMMA 4.1.3. *For* $j = 1, 2, \ldots$ *we have*

$$\text{(i)} \sum_{k=0}^{n} k\binom{nj}{kj} = \frac{n}{2}\sum_{k=0}^{n}\binom{nj}{kj}$$

$$\text{(ii)} \sum_{k=0}^{n} k^2\binom{nj}{kj} = \frac{n(nj-1)}{j}\sum_{k=0}^{n}\binom{nj-2}{kj} + \frac{n}{2j}\sum_{k=0}^{n}\binom{nj}{kj}$$

$$\text{(iii)} \sum_{k=0}^{n} k^3\binom{nj}{kj} = \frac{n^2}{4}\left(\frac{3}{j} - n\right)\sum_{k=0}^{n}\binom{nj}{kj} + \frac{3n^2(nj-1)}{2j}\sum_{k=0}^{n}\binom{nj-2}{kj}.$$

Proof. Relation (i) is most readily proved by noting that

$$\sum_{k=0}^{n} k\binom{nj}{kj} = \sum_{k=0}^{n} k\binom{nj}{(n-k)j} = \sum_{k=0}^{n}(n-k)\binom{nj}{kj}$$

so that we have

$$2\sum_{k=0}^{n} k\binom{nj}{kj} = \sum_{k=0}^{n}[k + (n-k)]\binom{nj}{kj} = n\sum_{k=0}^{n}\binom{nj}{kj}.$$

In order to prove relation (ii) we note that

$$k^2 = \frac{1}{j}k(kj-1) + \frac{1}{j}k,$$

so that we have

$$\sum_{k=0}^{n} k^2\binom{nj}{kj} = \frac{1}{j}n(nj-1)\sum_{k=0}^{n}\binom{nj-2}{j(n-k)} + \frac{1}{j}\sum_{k=0}^{n} k\binom{nj}{kj},$$

and relation (ii) follows from relation (i).

In proving relation (iii), we note that

$$\sum_{k=0}^{n} k^3\binom{nj}{kj} = \sum_{k=0}^{n}(n-k)^3\binom{nj}{kj} = n^3\sum_{k=0}^{n}\binom{nj}{kj} - 3n^2\sum_{k=0}^{n} k\binom{nj}{kj} + 3n\sum_{k=0}^{n} k^2\binom{nj}{kj} - \sum_{k=0}^{n} k^3\binom{nj}{kj},$$

so that from relation (i) we have

$$\sum_{k=0}^{n} k^3 \binom{nj}{kj} = -\frac{1}{4} n^3 \sum_{k=0}^{n} \binom{nj}{kj} + \frac{3}{2} n \sum_{k=0}^{n} k^2 \binom{nj}{kj},$$

and relation (iii) follows from relation (ii).

LEMMA 4.1.4.

$$\text{(i)} \sum_{k=0}^{n} \binom{3n}{3k} = \frac{1}{3}\{2^{3n} + (-1)^n 2\}$$

$$\text{(ii)} \sum_{k=0}^{n-1} \binom{3n-2}{3k} = \frac{1}{3}\{2^{3n-2} + (-1)^{n-1}\}$$

$$\text{(iii)} \sum_{k=0}^{n} \binom{6n}{6k} = \frac{1}{3}\{2^{6n-1} + (-1)^n 3^{3n} + 1\}$$

$$\text{(iv)} \sum_{k=0}^{n-1} \binom{6n}{6k+3} = \frac{1}{3}\{2^{6n-1} + (-1)^{n-1} 3^{3n} + 1\}$$

$$\text{(v)} \sum_{k=0}^{n} \binom{8n}{8k} = (-1)^n 2^{2n-1} \sum_{k=0}^{2n} 2^k \binom{4n}{2k} + 2^{4n-2} + 2^{8n-3}$$

$$\text{(vi)} \sum_{k=0}^{n} \binom{8n-2}{8k} = (-1)^n 2^{2n-2} \sum_{k=0}^{2n-1} 2^k \binom{4n-1}{2k} + 2^{8n-5}$$

$$\text{(vii)} \sum_{k=0}^{n} \binom{10n}{10k} = \frac{1}{5}\Bigg\{2^{10n-1} + (-1)^n \frac{1}{2^{5n-1}} \sum_{l=[(5n)/2]}^{5n} \binom{5n}{2l-5n} 5^l + \frac{1}{2^{10n-1}} \sum_{k=0}^{5n} \binom{10n}{2k} 5^k\Bigg\}$$

$$\text{(viii)} \sum_{k=0}^{n-1} \binom{10n}{10k+5} = \frac{1}{5}\Bigg\{2^{10n-1} + (-1)^{n-1} \frac{1}{2^{5n-1}} \sum_{l=[(5n)/2]}^{5n} \binom{5n}{2l-5n} 5^l + \frac{1}{2^{10n-1}} \sum_{k=0}^{5n} \binom{10n}{2k} 5^k\Bigg\}$$

$$\text{(ix)} \sum_{k=0}^{n} \binom{12n}{12k} = (-1)^n \frac{1}{3} \sum_{k=0}^{3n} 2^k 3^{3n-k} \binom{6n}{2k} + \frac{1}{6}\{3^{6n} + 1 + (-1)^n 2^{6n} + 2^{12n-1}\}$$

$$\text{(x)} \sum_{k=0}^{n} \binom{16n}{16k} = (-1)^n 2^{2n-1} \sum_{k=0}^{4n} \sum_{t=[(k+1)/2]}^{2n} 2^{k+t} \binom{8n}{2k} \binom{4n-k}{4n-2t} + 2^{4n-2} \sum_{k=0}^{4n} 2^k \binom{8n}{2k} + 2^{8n-3} + 2^{16n-4}$$

$$\text{(xi)} \sum_{k=0}^{n-1} \binom{16n-2}{16k} = (-1)^n 2^{2n-2} \sum_{k=0}^{4n-1} \sum_{t=[(k+1)/2]}^{2n} 2^{k+t} \binom{8n-1}{2k} \times \binom{4n-k}{4n-2t} + 2^{4n-3} \sum_{k=0}^{4n-1} 2^k \binom{8n-2}{2k} + 2^{16n-6}.$$

Proof. These results are all based on Lemma 4.1.2, together with evaluations of the appropriate cosines. As examples, we indicate the proofs of (iv), (v), (vii), and (x):

(iv) $\sum_{k=0}^{n-1} \binom{6n}{6k+3}$

$$= \frac{1}{6}\{2^{6n} + (\sqrt{3})^{6n}(-1)^{n-1} + 1 + 0 + 1 + (\sqrt{3})^{6n}(-1)^{n-1}\};$$

(v) $$\sum_{k=0}^{n} \binom{8n}{8k} = \frac{1}{8} 2^{8n} \sum_{r=0}^{7} \left(\cos\frac{\pi r}{8}\right)^{8n} (-1)^{rn}$$

$$= 2^{8n-3} + 2^{2n-2} \sum_{r=1}^{3} \left(\cos\frac{\pi r}{8}\right)^{8n} (-1)^{rn},$$

where

$$\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}, \quad \cos\frac{3\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2};$$

(vii) if $\theta = \pi/5$, then $2\theta = \pi - 3\theta$, so that $\sin 2\theta = \sin 3\theta$, whence

$$\cos\theta = \frac{1+\sqrt{5}}{4}, \cos\frac{\theta}{2} = \sqrt{\frac{5+\sqrt{5}}{8}}, \cos\frac{3\theta}{2} = \sqrt{\frac{5-\sqrt{5}}{8}},$$

$$\cos 2\theta = \frac{\sqrt{5}-1}{4};$$

(x) $$\cos\frac{\pi}{16} = \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}, \cos\frac{3\pi}{16} = \frac{\sqrt{2+\sqrt{2-\sqrt{2}}}}{2},$$

$$\cos\frac{5\pi}{16} = \frac{\sqrt{2-\sqrt{2-\sqrt{2}}}}{2}, \cos\frac{7\pi}{16} = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}.$$

Proof of Theorem 4.1. (i) From Lemma 4.1.1, it suffices to prove

$$\sum_{k=0}^{n} \binom{3n}{3k}\left\{3 - 4k + 8\binom{k+1}{3}\right\} \equiv 2 (\mathrm{mod}\ 16).$$

Since we have

$$3 - 4k + 8\binom{k+1}{3} = 3 - \frac{16}{3}k + \frac{4}{3}k^3,$$

we must show

$$3\sum_{k=0}^{n}\binom{3n}{3k} - \frac{16}{3}\sum_{k=0}^{n} k\binom{3n}{3k} + \frac{4}{3}\sum_{k=0}^{n} k^3\binom{3n}{3k} \equiv 2 (\mathrm{mod}\ 16).$$

From Lemma 4.1.3, this is equivalent to showing

$$\frac{9 - 8n + n^2 - n^3}{3}\sum_{k=0}^{n}\binom{3n}{3k} + \frac{6n^3 - 2n^2}{3}\sum_{k=0}^{n}\binom{3n-2}{3k}$$

$$\equiv 2 (\mathrm{mod}\ 16).$$

By Lemma 4.1.4, parts (i) and (ii), this is equivalent to showing

$$\frac{9-8n+n^2-n^3}{9}\{2^{3n}+(-1)^n 2\}+\frac{6n^3-2n^2}{9}\{2^{3n-2}+(-1)^{n-1}\} \equiv 2 (\mathrm{mod}\ 16),$$

or equivalently

$$\frac{2^{3n-1}}{9}(n^3+n^2-16n-18)+\frac{(-1)^n}{9}(-8n^3+4n^2-16n+18) \equiv 2 (\mathrm{mod}\ 16),$$

or equivalently

$$2^{3n-1}(n^3+n^2-16n-18)+(-1)^n(-8n^3+4n^2-16n+18) \equiv 18 (\mathrm{mod}\ 16).$$

For $n = 1$, the result is certainly true, while for $n \geqq 2$, it reduces to

$$(-1)^n(-8n^3+4n^2+2) \equiv 2 (\mathrm{mod}\ 16)$$

or

$$(-1)^n(-4n^3+2n^2+1)-1 \equiv 0 (\mathrm{mod}\ 8).$$

For $n = 2k$, the left side is $8(-4k^3+k^2)$, while for $n = 2k+1$, it is $8(4k^3+k^2)$, and we are done.

(ii). Proceeding as in part (i), we see that it suffices to show that, for integer $n \geqq 1$,

$$\frac{3-2n}{3}(2^{6n-1}+1+(-1)^n 3^{3n}) \equiv 2 (\mathrm{mod}\ 16)$$

or

$$(3-2n)(1+(-1)^n 3^{3n})-6 \equiv 0 (\mathrm{mod}\ 16)$$

or

$$(3-2n)(1+5^n)-6 \equiv 0 (\mathrm{mod}\ 16).$$

A simple induction argument shows that 16 divides $-10(5^n-1)+8n$, and this yields an induction argument to prove the required result.

(iii), (iv). These are easily checked for $n = 1, 2, 3$, while for $n \geqq 4$ the results follow from Lemma 4.1.3 and the appropriate parts of Lemma 4.1.4.

(v). Proceeding as in part (i), we see that it suffices to show that, for integer $n \geqq 1$,

$$\frac{3-2n}{3}(-1)^n \sum_{k=0}^{3n} 2^{2k} 3^{3n-k}\binom{6n}{2k} + \frac{3-2n}{6}(3^{6n}+(-1)^n 2^{6n}+1+2^{12n-1}) \equiv 2 (\mathrm{mod}\ 16)$$

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or equivalently

$$(3-2n)\frac{(-1)^n}{3}\left\{3^{3n}+4(3^{3n-1})\binom{6n}{2}+\sum_{k=1}^{3n-1}2^{2k}3^{3n-k}\binom{6n}{2k}+2^{6n}\right\}$$
$$+(3-2n)\left\{\frac{3^{6n-1}}{2}+\frac{(-1)^n2^{6n-1}}{3}+\frac{1}{6}+\frac{2^{12n-2}}{3}\right\}\equiv 2 \pmod{16}$$

or equivalently

$$(3-2n)(-1)^n\{3^{3n-1}+4(3^{3n-2})3n(6n-1)+2^{6n-1}\}$$
$$+(3-2n)\left\{\frac{3^{6n-1}}{2}+\frac{1}{6}+\frac{2^{12n-2}}{3}\right\}\equiv 2 \pmod{16}.$$

If we now replace $3^k = (4-1)^k$ by $(-1)^k + 4k(-1)^{k-1} + 8k(k-1)(-1)^k \pmod{16}$, the result follows readily.

Proof of Theorem 4.2. (i). From Lemma 4.1.1, it suffices to show

$$\sum_{k=0}^{n}\binom{3n}{3k}(-1)^k \equiv 0 \pmod{18}.$$

For $n = 2m+1$,

$$\sum_{k=0}^{2m+1}\binom{6m+3}{3k}(-1)^k = \sum_{k=0}^{m}\binom{6m+3}{6k} - \sum_{k=0}^{m}\binom{6m+3}{6k+3} = 0.$$

For $n = 2m$,

$$\sum_{k=0}^{2m}\binom{6m}{3k}(-1)^k = \sum_{k=0}^{m}\binom{6m}{6k} - \sum_{k=0}^{m-1}\binom{6m}{6k+3} = (2)3^{3m-1}(-1)^m,$$

by Lemma 4.1.4, parts (iii) and (iv), and for $m \geqq 1$, 18 divides $(2)3^{3m-1}(-1)^m$.

(ii). As in part (i), it suffices to show

$$\sum_{k=0}^{n}\binom{5n}{5k}(-1)^k = \sum_{k=0}^{m}\binom{10m}{10k} - \sum_{k=0}^{m-1}\binom{10m}{10k+5} \equiv 0 \pmod{250}.$$

Since

$$\binom{5n}{5k} = \binom{5n}{5n-5k} \quad \text{and} \quad \binom{2l}{l} \text{ is even, clearly we have}$$

$$\sum_{k=0}^{n}\binom{5n}{5k}(-1)^k \equiv 0 \pmod{2}.$$

Also, by Lemma 4.1.4, parts (vii) and (viii),

$$\sum_{k=0}^{m}\binom{10m}{10k} - \sum_{k=0}^{m-1}\binom{10m}{10k+5} = \frac{(-1)^m}{2^{5m-2}}\sum_{l=[(5m)/2]}^{5m}\binom{5m}{2l-5m}5^{l-1}.$$

For $m = 1$, the left side is (-250), while for $m \geqq 2$, $l-1 \geqq 4$, and the result follows.

(iii). By Lemma 4.1.1, it suffices to show that

$$\sum_{k=0}^{n} \binom{6n}{6k}(-1) \equiv -2 \pmod{12}.$$

By Lemma 4.1.4, part (iii), this is equivalent to showing

$$\tfrac{1}{3}(2^{6n-1} + (-1)^n 3^{3n} - 5) \equiv 0 \pmod{12}.$$

For this, it suffices to show 4 divides $(-1)^n 3^{3n} - 5$ and 9 divides $2^{6n-1} - 5$, each of which follows by a simple induction argument.

THEOREM 4.3. $(-1)^n E_{kn} > 0$, $k = 2, 3, \ldots$.

Proof. In effect, this is result (10.3) in [**3**], in which $(-1)^n E_{kn}$ counts a certain class of permutations.

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