ON DIRICHLET SERIES ATTACHED TO CUSP FORMS AND THE SIEGEL-ZERO

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1. Introduction and statement of results. Let k be an even integer greater than or equal to 12 and f an nonzero cusp form of weight k on $\mathrm{SL}(2, \mathbb{Z})$. We assume, further, that f is an eigenfunction for all Hecke-Operators and has the Fourier expansion

$$f(z)=\sum_{n=1}^{\infty} a(n)e^{2\pi i n z}, \quad \text{where } a(1)=1.$$

For every Dirichlet character $\chi \bmod q$ we define

$$L_f(s, \chi)=\sum_{n=1}^{\infty} \chi(n)a(n)n^{-s}. \tag{1.1}$$

Then the series in (1.1) is absolutely convergent for $\operatorname{Re} s>(k+1)/2$. $L_f(s, \chi)$ can be analytically continued as an entire function. For a primitive character χ, $L_f(s, \chi)$ satisfies the following functional equation (cf. [**2**])

$$\left(\frac{2\pi}{q}\right)^{-s}\Gamma(s)L_f(s, \chi)=\left(\frac{2\pi}{q}\right)^{-(k-s)}\Gamma(k-s)\varepsilon(\chi)L_f(k-s, \bar{\chi}), \tag{1.2}$$

where $|\varepsilon(\chi)|=1$. For a nonprimitive character χ the analytical continuation for $L_f(s, \chi)$ will be given later (Lemma 2).

We note, that throughout the paper A, a denote positive constants but not always the same, whereas $a_1, a_2, \ldots, b_1, b_2, \ldots, c_1, c_2, \ldots$ are the same positive constants. $\zeta(s)$ is Riemann's zeta-function and p always denotes a prime.

Rankin [**7**] proved that $L_f(s, \tilde{\chi}_0)\neq 0$ for $s=(k+1)/2+it$, except possibly for $t=0$, where $\tilde{\chi}_0$ is the principal character mod 1. Ogg [**4**] later proved that $L_f((k+1)/2, \tilde{\chi}_0)\neq 0$, if the Petersson conjecture holds. This conjecture has meanwhile been proved by Deligne [**1**]. We shall first generalize this result in the following way:

THEOREM 1. *Let $z\geqq 2$ and $L_f(s, \chi)$ as above. Then there is a c_1 independent of z, such that, for*

$$\sigma\geqq 1-\frac{c_1}{\log(z(|t|+2))}, \qquad L_f\left(s+\frac{k-1}{2}, \chi\right)\neq 0$$

for all $\chi \bmod q$ with $q\leqq z$, with the possible exception of those $L_f(s+(k-1)/2, \chi)$ with character χ equivalent to a unique primitive real character $\chi^* \bmod q^*(\chi^*=\chi^*(z), q^*=q^*(z))$. These exceptional $L_f(s+(k-1)/2, \chi)$, if they exist, all have the same zero $\hat{\sigma}$. This zero $\hat{\sigma}$ is real, simple and different from 1.

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Theorem 1 also holds for Dirichlet L-series (cf. [5], IV). The exceptional zero for the Dirichlet L-series is usually called the Siegel zero. Let us call $\hat{\sigma}$ the Siegel zero for the cusp form f. It is not known whether the Siegel zero exists or not, nor whether the Siegel zero for the cusp form f exists or not. The following theorem shows that both can not exist.

THEOREM 2. *Let $z \geqq 2$, $L_f(s, \chi)$ be as before and $L(s, \chi)$ a Dirichlet L-series. Then there is a c_2 independent of z, such that, for*

$$\sigma \geqq 1 - \frac{c_2}{\log(z(|t|+2))}, \qquad L(s, \chi) \neq 0 \quad \textit{and} \quad L_f\left(s + \frac{k-1}{2}, \chi\right) \neq 0$$

for all χ mod q with $q \leqq z$ with the following possible exception: for primitive characters χ at most one of the excepted functions $L_f(s+(k-1)/2, \chi)$ of Theorem 1 and the excepted functions $L(s, \chi)$ of [5], IV, Satz 6.9 can have a zero in

$$\sigma \geqq 1 - \frac{c_2}{\log(z(|t|+2))}.$$

As a special case of Theorem 2 we have the following result:

There is a c_2 independent of z such that either

$$L_f\left(s + \frac{k-1}{2}, \chi\right) \neq 0 \quad \text{for} \quad \sigma \geqq 1 - \frac{c_2}{\log(z(|t|+2))} \quad \text{for all } \chi \bmod q \text{ with } q \leqq z,$$

or

$$L(s, \chi) \neq 0 \quad \text{for} \quad \sigma \geqq 1 \frac{c_2}{\log(z(|t|+2))} \quad \text{for all } \chi \bmod q \text{ with } q \leqq z.$$

The zero-free domains of $L(s, \chi)$ and $L_f(s, \chi)$ allow us to prove the following results:

THEOREM 3. *Let $x \geqq 2$, $(l, q) = 1$ and B some constant. Then there is a constant c_3 such that either*

$$\text{(i)} \qquad \sum_{\substack{p \leqq x \\ p \equiv l \bmod q}} \log p - \frac{x}{\varphi(q)} = O(x \exp\{-c_3 \sqrt{\log x}\})$$

uniformly for $1 \leqq q \leqq \exp\{B\sqrt{\log x}\}$, or

$$\text{(ii)} \qquad \sum_{\substack{p \leqq x \\ p \equiv l \bmod q}} a(p) p^{-(k-1)/2} \log p = O(x \exp\{-c_3 \sqrt{\log x}\})$$

uniformly for $1 \leqq q \leqq \exp\{B\sqrt{\log x}\}$, or both (i) *and* (ii) *hold true.*

REMARKS.

(1) The particular alternative occurring in Theorem 3 may depend on the choice of x, because the exceptional character in Theorem 2 depends on z.

(2) Note that in both (i) and (ii) the conditions $q \leqq \exp\{B \log x\}$ may be dropped since otherwise the results become trivial.

THEOREM 4. *There is a constant* c_4 *such that*

$$\sum_{\substack{p \leqq x \\ p \equiv l \bmod q}} a(p)p^{-(k-1)/2} \log p = O(x \exp\{-c_4\sqrt{\log x}\}).$$

A proof for Theorem 4 in the case $q = 1$ was given by Moreno [**3**]. In §2 we prove several Lemmas from which we deduce the truth of Theorems 1 and 2. Theorems 3 and 4 are proved in §3. The paper contains part of the author's dissertation. He takes this opportunity to thank Professor Dr. H.-E. Richert for his advice and encouragement.

2. Proofs of Theorem 1 and Theorem 2.

LEMMA 1. *Let* χ *be a Dirichlet character* mod q. *Then*

$$L_f(s, \chi) \neq 0 \quad \text{for} \quad \sigma > \frac{k+1}{2}.$$

Proof. We have $a(p) \in \mathbb{R}$ and $|a(p)| \leqq 2p^{(k-1)/2}$ for all primes p (cf. [**1**]). Hence $\theta_p(a)$ is uniquely defined by

$$\cos \theta_p(a) = \tfrac{1}{2}a(p)p^{-(k-1)/2} \qquad (0 \leqq \theta_p(a) \leqq \pi).$$

For $L_f(s, \chi)$ the following Euler product representation holds:

$$L_f(s, \chi) = \prod_p \left\{\left(1 - \frac{\chi(p)e^{i\theta_p(a)}}{p^{s-(k-1)/2}}\right)\left(1 - \frac{\chi(p)e^{-i\theta_p(a)}}{p^{s-(k-1)/2}}\right)\right\}^{-1} \quad \left(\sigma > \frac{k+1}{2}\right). \tag{2.1}$$

Rankin proved this formula for $\chi = \tilde{\chi}_0 \bmod 1$ in [**7**]. For arbitrary character $\chi \bmod q$ the proof proceeds similarly, using the fact that $\chi(n)\chi(m) = \chi(nm)$ for all $n, m \in \mathbb{N}$. That $L_f(s, \chi) \neq 0$ for $\sigma > (k+1)/2$ now follows from (2.1).

LEMMA 2. *Let* χ *be* mod q, q^* *the conductor of* χ *and* χ^* mod q^* *the primitive character inducing* χ. *Then*

$$L_f(s, \chi) = L_f(s, \chi^*) \prod_{p \mid q} \left\{\left(1 - \frac{\chi^*(p)e^{i\theta_p(a)}}{p^{s-(k-1)/2}}\right)\left(1 - \frac{\chi^*(p)e^{-i\theta_p(a)}}{p^{s-(k-1)/2}}\right)\right\}; \tag{2.2}$$

$L_f(s, \chi)$ *and* $L_f(s, \chi^*)$ *have the same zeros for* $\sigma > (k-1)/2$.

Proof. For $\sigma > (k+1)/2$ (2.2) follows from (2.1) and for $\sigma > (k-1)/2$ by analytical continuation. Since the finite product in (2.2) is different from zero for $\sigma > (k-1)/2$, $L_f(s, \chi)$ and $L_f(s, \chi^*)$ have the same zeros for $\sigma > (k-1)/2$.

In order to prove zero-free regions for $L_f(s, \chi)$ it is now sufficient from now on to take χ primitive mod q.

LEMMA 3. *Let* $0 < \eta < \frac{1}{2}$, $(k-1)/2 - \eta \leqq \sigma \leqq (k+1)/2 + \eta$, $t_1 = |t| + 2$. *Then*

$$|L_f(s, \chi)| \leqq A\zeta(1+\eta)(q\,|1+s|)^{(k+1)/2-\sigma+\eta} \tag{2.3}$$

$$|L_f(s, \chi)| \leqq A \log(qt_1) \quad \textit{for} \quad \frac{k+1}{2} - \frac{a}{\log(qt_1)} \leqq \sigma \leqq \frac{k+3}{2}, \quad a \leqq 4. \tag{2.4}$$

Proof. We take $Q=(k-1)/2$ and $s=w-(k-1)/2$ in Lemma 3 of [**6**]. This gives

$$\left|\frac{\Gamma(k-w)}{\Gamma(w)}\right| \leqq |w+1|^{k-2u} \quad \text{for} \quad \frac{k}{2}-1 \leqq \operatorname{Re} w = u \leqq \frac{k}{2}. \tag{2.5}$$

For $\eta > 0$

$$\left|L_f\left(\frac{k+1}{2}+\eta+it, \chi\right)\right| \leqq \left\{\sum_{n=1}^{\infty} \frac{a^2(n)}{n^{k+\eta}}\right\}^{1/2} \zeta^{1/2}(1+\eta) \leqq A\zeta(1+\eta). \tag{2.6}$$

In the last step we used a result due to Rankin [**8**]. Then, by (1.2), (2.5) and (2.6),

$$\left|L_f\left(\frac{k-1}{2}-\eta+it, \chi\right)\right| \leqq Aq^{-1-2\eta}\left|\frac{k+1}{2}-\eta+it\right|^{1+2\eta} \zeta(1+\eta).$$

Choosing now $a=(k-1)/2-\eta$, $b=(k+1)/2+\eta$, $Q=1$ in Theorem 2 of [**6**] we obtain (2.3). We now take

$$\eta = \frac{1}{\log(qt_1+7)} \left(\leqq \tfrac{1}{2}\right), \qquad \sigma \geqq \frac{k+1}{2} - \frac{a}{\log(qt_1)}$$

in (2.3). Then (2.4) follows for $\sigma \leqq (k+1)/2 + 1/\log(qt_1+7)$. For $(k+1)/2 \geqq \sigma > (k+1)/2 - 1/\log(qt_1+7)$ (2.4) follows by (2.6) since $\zeta(1+\eta) \leqq (2/\eta)$ for $0<\eta\leqq\frac{1}{2}$.

LEMMA 4. *There is an* $a \leqq 1$, *such that*

$$L_f(s,\chi) \neq 0 \quad \textit{for} \quad \sigma \geqq \frac{k+1}{2} - \frac{a}{\log(q\,|t|)}, \quad |t| \geqq 3, \textit{ and all } \chi \bmod q. \tag{2.7}$$

Proof. Taking the logarithms and differentiating (2.1), we obtain

$$\frac{L_f'}{L_f}(s,\chi) = -\sum_{p,m\geqq 1} \frac{2\chi(p^m)\cos(m\theta_p(a))}{p^{m(s-(k-1)/2)}} \log p \quad \text{for} \quad \sigma > \frac{k+1}{2}. \tag{2.8}$$

Defining

$$\psi_f(s,\chi) = \prod_p \left(\left(1-\frac{\chi(p)e^{2i\theta_p(a)}}{p^{s-k+1}}\right)\left(1-\frac{\chi(p)e^{-2i\theta_p(a)}}{p^{s-k+1}}\right)\right)^{-1} \qquad (\sigma > k),$$

we obtain

$$\frac{\psi_f'}{\psi_f}\left(s+\frac{k-1}{2},\chi\right) = -\sum_{p,m\geqq 1} \frac{2\chi(p^m)\cos(2m\theta_p(a))}{p^{m(s-(k-1)/2)}} \log p \quad \left(\sigma > \frac{k+1}{2}\right). \tag{2.9}$$

For Dirichlet L-series we have

$$\frac{L'}{L}\left(s-\frac{k-1}{2},\chi\right) = -\sum_{p,m\geqq 1} \frac{\chi(p^m)}{p^{m(s-(k-1)/2)}} \log p \quad \left(\sigma > \frac{k+1}{2}\right). \tag{2.10}$$

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If χ_0 is the principal character mod q then (2.8), (2.9) and (2.10) give

$$\operatorname{Re}\left\{-2\frac{L'}{L}\left(\sigma-\frac{k-1}{2},\chi_0\right)-\frac{1}{2}\frac{\psi_f'}{\psi_f}\left(\sigma+\frac{k-1}{2},\chi_0\right)\right.$$
$$\left.-\frac{L'}{L}\left(\sigma-\frac{k-1}{2}+2it,\chi^2\right)-2\frac{L_f'}{L_f}(\sigma+it,\chi)\right\}$$
$$=\sum_{p,m\geqq 1}\frac{\log p}{p^{m(\sigma-(k-1)/2)}}\{2\chi_0(p^m)+\chi_0(p^m)\cos(2m\theta_p(a))+\operatorname{Re}(e^{-2imt\log p}\chi^2(p^m))$$
$$+4\operatorname{Re}(e^{-itm\log p}\chi(p^m))\cos(m\theta_p(a))\}\geqq 0\quad(\sigma>(h+1)/2),\tag{2.11}$$

for the expression in brackets on the right of (2.11) is not negative. Let now $\rho=\beta+i\gamma$ be a zero of $L_f(s,\chi)$ and put

$$\sigma_0=\frac{k+1}{2}+\frac{a_1}{\log(q\gamma)},$$

where $a_1\leqq 1$, $\gamma\geqq 3$ and $s_0=\sigma_0+i\gamma$. Then we have (cf. [5])

$$-\frac{L'}{L}\left(\sigma_0-\frac{k-1}{2},\chi_0\right)\leqq-\frac{\zeta'}{\zeta}\left(\sigma_0-\frac{k-1}{2}\right)\leqq\frac{5}{4a_1}\log(q\gamma)\tag{2.12}$$

and

$$-\operatorname{Re}\frac{L'}{L}\left(\sigma_0-\frac{k-1}{2}+2i\gamma,\chi^2\right)<A\left\{\log(q\gamma)+\log\frac{1}{a_1}\right\}\tag{2.13}$$

Let further $\mu_a(n)$ be a multiplicative function defined by

$$\mu_a(n)=\begin{cases}-a(p) & (n=p),\\ p^{k-1} & (n=p^2),\\ 0 & (n=p^\nu,\ \nu\geqq 3).\end{cases}$$

for all primes p. Then we have

$$\sum_{t\mid n}\mu_a(t)a\left(\frac{n}{t}\right)=\begin{cases}1 & (n=1),\\ 0 & (n>1),\end{cases}$$

and

$$|\mu_a(n)|\leqq d(n)n^{(k-1)/2},\quad\text{where}\quad d(n)=\sum_{t\mid n}1.$$

Hence

$$\frac{1}{|L_f(s_0,\chi)|}\leqq\sum_{n=1}^{\infty}\frac{|\mu_a(n)|}{n^{\sigma_0}}\leqq\zeta^2\left(1+\frac{a_1}{\log(q\gamma)}\right)\leqq\frac{4}{a_1^2}\log^2(q\gamma).$$

This together with the results of Lemma 3 then gives

$$\left|\frac{L_f(s,\chi)}{L_f(s_0,\chi)}\right|\leqq\frac{1}{a_1^2}(q\gamma)^A\quad\text{for}\quad|s-s_0|\leqq\tfrac{1}{2}.$$

If now $\rho=\beta+i\gamma$ is a zero for $L_f(s,\chi)$ with $|\rho-s_0|=\sigma_0-\beta\leqq\frac14$ we have ([**5**] Anhang)

$$-\mathrm{Re}\,\frac{L_f'}{L_f}(\sigma_0+i\gamma,\chi)\leqq A\left(\log(q\gamma)+\log\frac{1}{a_1}\right)-\frac{1}{\sigma_0-\beta}. \tag{2.14}$$

If χ_0 is the principal character mod q we further obtain

$$\psi_f\left(\sigma+\frac{k-1}{2},\chi_0\right)=\psi_f\left(\sigma+\frac{k-1}{2}\right)\prod_{p\mid q}\left\{\left(1-\frac{e^{2i\theta_p(a)}}{p^{\sigma-(k-1)/2}}\right)\left(1-\frac{e^{-2i\theta_p(a)}}{p^{\sigma-(k-1)/2}}\right)\right\}$$

where

$$\psi_f\left(\sigma+\frac{k-1}{2}\right)=\prod_{p}\left\{\left(1-\frac{e^{2i\theta_p(a)}}{p^{\sigma-(k-1)/2}}\right)\left(1-\frac{e^{-2i\theta_p(a)}}{p^{\sigma-(k-1)/2}}\right)\right\}^{-1}.$$

Taking logarithms and differentiating we deduce that

$$\frac{\psi_f'}{\psi_f}\left(\sigma+\frac{k-1}{2},\chi_0\right)=\frac{\psi_f'}{\psi_f}\left(\sigma+\frac{k-1}{2}\right)+\sum_{\substack{p\mid q\\ m\geqq 1}}\frac{2\cos\{2m\theta_p(a)\}}{p^{m(\sigma-(k-1)/2)}}\log p.$$

Using results of Rankin [**9**] concerning $\psi_f(s)$ and the inequality

$$\left|\sum_{\substack{p\mid q\\ m\geqq 1}}\frac{2\cos(2m\theta_p(a))}{p^{m(\sigma-(k-1)/2)}}\log p\right|\leqq A\log\log(q+2)\quad\text{for}\quad\sigma\geqq\frac{k+1}{2}$$

we obtain

$$-\frac12\frac{\psi_f'}{\psi_f}\left(\sigma_0+\frac{k+1}{2},\chi_0\right)\leqq\frac{\zeta'}{\zeta}\left(\sigma_0-\frac{k-1}{2}\right)+\frac{5}{8\left(\sigma_0-\frac{k-1}{2}\right)}, \tag{2.15}$$

for a_1 sufficiently small. Take $\sigma=\sigma_0$, $t=\gamma$ in (2.11); then (2.12), (2.13), (2.14) and (2.15) give

$$0\leqq\frac{15}{8a_1}\log(q\gamma)+A\left(\log(q\gamma)+\log\frac{1}{a_1}\right)-\frac{2}{\sigma_0-\beta}.$$

Hence

$$\beta\leqq\frac{k+1}{2}-\frac{a_2}{\log(q\gamma)}\quad\text{for}\quad\gamma\geqq 3$$

and a_2 sufficiently small. The inequality

$$\beta\leqq\frac{k+1}{2}-\frac{a_3}{\log(q\,|\gamma|)}\quad\text{for}\quad\gamma\leqq -3$$

follows since $L_f(\bar s,\chi)=\overline{L_f(s,\bar\chi)}$.

LEMMA 5. *Let $\chi^2\neq\chi_0$. Then we have*

$$L_f(s,\chi)\neq 0\quad\textit{for}\quad\sigma\geqq\frac{k+1}{2}-\frac{a}{\log q},\qquad |t|\leqq 3. \tag{2.16}$$

Proof. The proof runs as the proof of Lemma 4, taking now

$$\sigma_1 = \frac{k+1}{2} + \frac{a_3}{\log q}$$

instead of σ_0.

We have for $\chi_0 \bmod q$

$$L_f(s, \chi_0) \neq 0 \quad \text{for} \quad \sigma \geqq \frac{k+1}{2} - \frac{a_4}{\log(|t|+3)}, \qquad t \in \mathbb{R} \tag{2.17}$$

from (2.2) and

$$\zeta_f(s) = L_f(s, \tilde{\chi}_0) \neq 0 \quad \text{for} \quad \sigma \geqq \frac{k+1}{2} - \frac{a_4}{\log(|t|+3)}, \qquad t \in \mathbb{R}$$

where $\tilde{\chi}_0(n) = 1$ for all n (cf. [**3**]).

LEMMA 6. *Let* $\chi^2 = \chi_0$, $\chi \neq \chi_0$. *Then we have*

$$L_f(s, \chi) \neq 0 \quad \textit{for} \quad \sigma \geqq \frac{k+1}{2} - \frac{a}{\log q}, \qquad 0 < |t| \leqq 3. \tag{2.18}$$

Proof. We follow the proof of Lemma 4, taking $\sigma_2 = (k+1)/2 + a_5/\log q$ instead of σ_0. Then the estimates (2.12), (2.14) and (2.15) still hold with σ_2 instead of σ_0, a_5 instead of a_1 and q instead of $q\gamma$. If $\rho = \beta + i\gamma$ is a zero of $L_f(s, \chi)$ with $a/\log q \leqq \gamma \leqq 3$, then we have

$$-\text{Re}\, \frac{L'}{L}\left(\sigma_2 - \frac{k-1}{2} + 2i\gamma, \chi_0\right) \leqq A \log q.$$

As in the proof of Lemma 4 we finally obtain

$$\beta \leqq \frac{k+1}{2} - \frac{a}{\log q} \quad \text{where} \quad \frac{a}{\log q} \leqq |\gamma| \leqq 3. \tag{2.19}$$

If $\rho = \beta + i\gamma$ is a zero of $L_f(s, \chi)$ with

$$0 < \gamma < \frac{a_6}{\log q} \quad \text{and} \quad \frac{k+1}{2} \geqq \beta \geqq \frac{k+1}{2} - \frac{a_7}{\log q},$$

$\bar{\rho} = \beta - i\gamma$ is also a zero of $L_f(s, \chi)$, since $L_f(\bar{s}, \chi) = \overline{L_f(s, \chi)}$ and both zeros are in $|s - s_3| \leqq \frac{1}{4}$ for $s_3 = (k+1)/2 + a_8/\log q + i\gamma$, if a_6, a_7, a_8 are sufficiently small. Then we obtain (cf. [**5**])

$$-\text{Re}\, \frac{L_f'}{L_f}(\sigma_3 + i\gamma, \chi) < A\left(\log q + \log \frac{1}{a_8}\right) - \frac{1}{\sigma_3 - \beta} - \frac{\sigma_3 - \beta}{(\sigma_3 - \beta)^2 + 4\gamma^2}. \tag{2.20}$$

On the other hand we have

$$\text{Re}\, \frac{L_f'}{L_f}(\sigma_3 + i\gamma, \chi) \leqq \left\{ \sum_{p,m \geqq 1} \frac{4\cos^2(m\theta_p(a))}{p^{m(\sigma_3 - (k-1)/2)}} \log p \right\}^{1/2}$$

$$\times \left\{ \sum_{p,m \geqq 1} \frac{\log p}{p^{m(\sigma_3 - (k-1)/2)}} \right\}^{1/2} \leqq \frac{5}{4\left(\sigma_3 - \dfrac{k+1}{2}\right)} = \frac{5}{4a_8} \log q$$

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for a_8 sufficiently small (cf. [**9**]). This together with (2.20) gives

$$\frac{k+1}{2}-\beta+\frac{a_8}{\log q}<\left(\frac{1}{2a_8}+\frac{a_7}{a_8^2}\right)\{(a_7+a_8)^2+4a_6^2\}\frac{1}{\log q}. \tag{2.21}$$

If a_6 and a_7 are small enough, the right-side of (2.21) is smaller than $a_8/\log q$. Hence $\beta>(k+1)/2$, which contradicts Lemma 1. Hence

$$L_f(s,\chi) \text{ has no zero for } \frac{k+1}{2}\geqq\beta\geqq\frac{k+1}{2}-\frac{a_7}{\log q}, \qquad 0<|\gamma|<\frac{a_6}{\log q}.$$

This together with (2.19) proves (2.18).

LEMMA 7. *Let* $\chi^2=\chi_0, \chi\neq\chi_0$. *Then* $L_f(\sigma,\chi)$ *has at most a simple zero in*

$$\sigma\geqq\frac{k+1}{2}-\frac{a}{\log q}.$$

Proof. Let β, β' be two zeros of $L_f(\sigma,\chi)$ with $\beta'\geqq\beta$. Following the proof of Lemma 6 with $\sigma_4=(k+1)/2+a_9/\log q$ instead of σ_3, $\gamma=0$, we obtain for a_9 sufficiently small

$$\frac{2}{\sigma_4-\beta}<\frac{3}{2}\frac{\log q}{a_9}.$$

Hence

$$\beta<\frac{k+1}{2}-\frac{a}{\log q}.$$

LEMMA 8. *We have*

$$L_f\left(\frac{k+1}{2},\chi\right)\neq 0 \quad \textit{for all} \quad \chi \bmod q. \tag{2.22}$$

Proof. We use an idea due to Ogg [**4**]. In view of Lemma 5 and (2.17) we have to prove (2.22) only for $\chi^2=\chi_0$, $\chi\neq\chi_0$. Let

$$h(s)=\zeta(s)L_f^2\left(s+\frac{k-1}{2},\chi\right)\psi_f(s+k-1,\chi_0)L^2(s,\chi_0).$$

Now $\psi_f(s+k-1,\chi_0)\zeta(s)L(s,\chi_0)$ is a holomorphic function except at $s=1$ (cf. [**9**]). If we suppose $L_f((k+1)/2,\chi)=0$, then $h(s)$ is holomorphic in the whole plane. For $\sigma>1$ we have

$$\log h(s)=\sum_{p,\nu\geqq 1}\frac{\{1+\alpha^\nu(p)+\overline{\alpha(p)}^\nu}{\nu p^{\nu s}}=\sum_{n=2}^\infty\frac{c(n)}{n^s} \qquad (\sigma>1),$$

where $\alpha(p)=\chi(p)e^{i\theta_p(a)}$ and $c(n)$ is non-negative. If $\log h(s)$ were holomorphic in $[\sigma_0,1)$, $\sigma_0<1$, then

$$\log h(s)=\sum_{n=2}^\infty\frac{c(n)}{n^s} \quad \text{for} \quad \sigma>\sigma_0.$$

Let σ_0 be the largest real zero of $h(s)$. If it exists, $-\infty<\sigma_0\leqq 1$. In particular we have $\log|h(\sigma)|=\log h(\sigma)\geqq 0$ for $\sigma>\sigma_0$, which contradicts $h(\sigma_0)=0$. Hence $h(\sigma)\neq 0$. The functional equation (1.2) gives $L_f(0,\chi)=0$. Hence $s=-(k-1)/2$ is a zero of $h(s)$, which contradicts $h(\sigma)\neq 0$. This proves $L_f((k+1)/2,\chi)\neq 0$.

LEMMA 9. *Let χ_1^* and χ_2^* be two real primitive characters* mod q_1^* *and q_2^* respectively with $\chi_1^*\neq\chi_2^*$. Then there is a b_1 such that at most one of the functions $L_f(s,\chi_1^*)$, $L_f(s,\chi_2^*)$ has a simple zero for*

$$\sigma\geqq\frac{k+1}{2}-\frac{a}{\log(q_1^*q_2^*)},\qquad t=0.$$

Proof. First we may assume χ_1^* and χ_2^* to be different from χ_0 mod 1. Let χ_1 and χ_2 be the characters mod $q_1^*q_2^*$ induced by χ_1^* mod q_1^* and χ_2^* mod q_2^*, respectively. Then $\chi_1\neq\chi_0$, $\chi_2\neq\chi_0$, $\chi_1\neq\chi_2$, $\chi_1\chi_2\neq\chi_0$. We prove Lemma 9 with $L_f(s,\chi_1)$, $L_f(s,\chi_2)$ instead of $L_f(s,\chi_1^*)$, $L_f(s,\chi_2^*)$, which is sufficient by Lemma 2. We have

$$\begin{aligned}&-2\frac{L'}{L}\left(\sigma-\frac{k-1}{2},\chi_0\right)-\frac{\psi_f'}{\psi_f}\left(\sigma+\frac{k-1}{2}\right)-2\frac{\zeta'}{\zeta}\left(\sigma-\frac{k-1}{2}\right)-2\frac{L_f'}{L_f}(\sigma,\chi_1)-2\frac{L_f'}{L_f}(\sigma,\chi_2)\\&-2\frac{L'}{L}\left(\sigma-\frac{k-1}{2},\chi_1\chi_2\right)=\sum_{m\geqq 1,p}\frac{\log p}{p^{m(\sigma-(k-1)/2)}}\{2\chi_0(p^m)+4\cos^2(m\theta_p(a))+4[\chi_1(p^m)\\&+\chi_2(p^m)]\cos(m\theta_p(a))+2\chi_1\chi_2(p^m)\}\geqq 0\qquad\left(\sigma>\frac{k+1}{2}\right),\end{aligned}\tag{2.23}$$

since the expression in brackets is non-negative. Let

$$\rho_1=\beta_1=\frac{k+1}{2}-\frac{a_{10}}{\log q},\qquad\beta_2=\beta_2=\frac{k+1}{2}+\frac{a_{11}}{\log q}$$

be zeros of $L_f(s,\chi_1)$, $L_f(s,\chi_2)$, respectively and

$$\sigma_5=\frac{k+1}{2}+\frac{a_{12}}{\log q}.$$

Then

$$\begin{aligned}&\frac{L_f'}{L_f}(\alpha,\chi_1)>-A\left(\log q+\log\frac{1}{a_{12}}\right)+\frac{1}{\sigma_5-\beta_1}\\&\frac{L_f'}{L_f}(\sigma,\chi_2)>-A\left(\log q+\log\frac{1}{a_{12}}\right)+\frac{1}{\sigma_5-\beta_2}.\end{aligned}\tag{2.24}$$

We now take σ_5 instead of σ_0 in (2.12) and (2.15) and use the fact that

$$-\frac{L'}{L}\left(\sigma_5-\frac{k-1}{2},\chi_1\chi_2\right)<A\left(\log q+\log\frac{1}{a_{12}}\right).$$

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This together with (2.23) and (2.24) gives for sufficiently small a_{12}

$$0 \leqq \left\{\frac{15}{4a_{12}} - \frac{4}{a_{12}+\max(a_{10}, a_{11})}\right\} \log q + A\left(\log q + \log \frac{1}{a_{12}}\right). \tag{2.25}$$

But for a_{10}, a_{11}, a_{12} sufficiently small (2.25) cannot hold. Hence Lemma 9 follows.

LEMMA 10. *Let χ_1^* and χ_3^* be two real primitive characters* mod q_1^* *and q_3^*, respectively, where $q_1^* q_3^* > 1$. Then there is a b_2, such that at most one of the functions $L_f(s, \chi_1^*)$, $L(s-(k-1)/2, \chi_3^*)$ has a zero in*

$$\sigma \geqq \frac{k+1}{2} - \frac{b_2}{\log(q_1^* q_3^*)}, \qquad t = 0.$$

Proof. We may assume $\chi_1^* \neq \chi_0^*$, $\chi_3 \neq \chi_0^*$ (mod 1). Let χ_1 and χ_3 be the characters mod $q = q_1^* q_3^*$ induced by χ_1^* mod q_1^* and χ_3^* mod q_3^* respectively. We prove Lemma 10 with $L_f(s, \chi_1)$, $L(s-(k-1)/2, \chi_3)$ instead of $L_f(s, \chi_1^*)$, $L(s-(k-1)/2, \chi_3^*)$ which is sufficient by Lemma 2 and the analogue for L-series. Let

$$\rho_3 = \beta_3 = \frac{k+1}{2} + \frac{a_{13}}{\log q} \text{ be a zero of } L_f(s, \chi_1),$$

$$\rho_4 = \beta_4 = \frac{k+1}{2} + \frac{a_{14}}{\log q} \text{ be a zero of } L\left(s - \frac{k-1}{2}, \chi_3\right)$$

and

$$\sigma_6 = \frac{k+1}{2} + \frac{a_{15}}{\log q}.$$

Then

$$-\frac{L_f'}{L_f}(\sigma_6, \chi_1\chi_3) < A\left(\log q + \log \frac{1}{a_{15}}\right) \quad \text{for all } \chi_1\chi_3.$$

Taking $\sigma = \sigma_6$, $\chi_2 = \chi_1\chi_3$ in (2.23) we obtain as in the proof of Lemma 9 for sufficiently small a_{15}

$$0 \leqq \left\{\frac{15}{4a_{15}} - \frac{4}{a_{15}+\max(a_{13}, a_{14})}\right\} \log q + A\left(\log q + \log \frac{1}{a_{15}}\right). \tag{2.26}$$

But for a_{13}, a_{14}, a_{15} small enough, (2.26) cannot hold. Hence Lemma 10 follows. We now turn to the proofs of Theorem 1 and Theorem 2.

Proof of Theorem 1. In view of the above lemmas it is sufficient to prove the following: for $z \geqq 2$ there is a c_1 independent of z, such that $L_f(s, \chi) \neq 0$ for $\sigma \geqq (k+1)/2 - c_1/\log z$, $t = 0$, for all real primitive χ^* mod q^* where $q^* \leqq z$ with at most one exception. If χ_1^* mod q_1^* and χ_2^* mod q_2^* were two such characters, then Lemma 9 shows that at most one of the functions $L_f(s, \chi_1^*)$, $L_f(s, \chi_2^*)$ has a zero for

$$\sigma \geqq \frac{k+1}{2} - \frac{b_1}{2\log z} \geqq \frac{k+1}{2} - \frac{b_1}{\log(q_1^* q_2^*)}, \qquad t = 0.$$

Proof of Theorem 2. In view of Theorem 1 and [**5**], IV, Satz 6.9, it is sufficient, to prove the following: for $z \geqq 2$ there is a c_2 independent of z, such that at most one of the functions $L_f(s, \chi^*)$ and $L(s-(k-1)/2, \hat{\chi}^*)$ with real primitive characters $\chi^* \bmod q^*$, $\hat{\chi}^* \bmod \hat{q}^*$ with $q^* \leqq z$, $\hat{q}^* \leqq z$ has a zero in $\sigma \geqq (k+1)/2 - a/\log z$, $t=0$. But this follows from Lemma 10.

3. Proofs of Theorem 3 and Theorem 4.

Proof of Theorem 3. We may assume that the Siegel zero exists, otherwise part (i) of Theorem 3 holds (cf. [**5**]). Let

$$\Lambda_a(n) = \begin{cases} 2\cos(m\theta_p(a))\log p & (n = p^m,\ m \in \mathbb{N}) \\ 0 & (\text{otherwise}), \end{cases}$$

$$\psi_a(x, \chi) = \sum_{n \leqq x} \chi(n)\Lambda_a(n),$$

$$\psi_a(x, q, l) = \sum_{\substack{n \leqq x \\ n \equiv l(q)}} \Lambda_a(n) \quad \text{for} \quad (q, l) = 1.$$

Then

$$\psi_a(x, q, l) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \bar{\chi}(l)\psi_a(x, \chi). \tag{3.1}$$

and by Perron's formula

$$\sum_{n \leqq x} \chi(n)\Lambda_a(n) = -\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{L_f'}{L_f}(s, \chi) \frac{x^s - (k-1)/2}{s-(k-1)/2} \, ds + O\left(\frac{x}{T}\log^2 x\right), \tag{3.2}$$

where

$$c = \tfrac{1}{2}(k+1) + \frac{1}{\log x} \quad \text{for} \quad T \leqq 2 \quad \text{and} \quad x = [x] + \tfrac{1}{2}.$$

Let Γ_1, Γ_2, Γ_3 be the following curves in the complex plane.

$$\Gamma_1 : s = \frac{k+1}{2} + \sigma + iT, \qquad -\frac{b_3}{3\log(q(T+2))} \leqq \sigma \leqq \frac{1}{\log x}.$$

$$\Gamma_2 : s = \frac{k+1}{2} - \frac{b_3}{3\log(q(|t|+2))} - it, \qquad -T \leqq t \leqq T.$$

$$\Gamma_3 : s = \frac{k+1}{2} + \sigma - iT, \qquad -\frac{b_3}{3\log(q(T+2))} \leqq \sigma \leqq \frac{1}{\log x}.$$

Applying Satz 4.6 (Anhang) of [**5**] with

$$r = \tfrac{1}{2}, \qquad r_1 = \frac{2b_3}{3\log(q(|t|+2))}, \qquad s_0 = \frac{k+1}{2} - \frac{b_3}{3\log(q(|t|+2)} + it$$

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for sufficiently small b_3, we obtain

$$\frac{L_f'}{L_f}(s,\chi)=O(\log(q(|t|+2))) \tag{3.3}$$

for

$$\frac{k+1}{2}-\frac{b_3}{3\log(q(|t|+2))}\leqq\sigma\leqq\frac{k+1}{2}+\frac{b_3}{\log(q(|t|+2))},\qquad t\in\mathbb{R}.$$

For

$$\sigma>\frac{k+1}{2}+\frac{b_3}{\log(q(|t|+2))}$$

(3.3) follows since

$$\left|\frac{L_f'}{L_f}(s,\chi)\right|\leqq\sum_{n=1}^{\infty}\frac{\Lambda_a(n)}{n^{\sigma-(k-1)/2}}\leqq-2\frac{\zeta'}{\zeta}\left(\sigma-\frac{k-1}{2}\right)\leqq\frac{4}{\sigma-\dfrac{k+1}{2}}\qquad\left(\sigma>\frac{k+1}{2}\right).$$

(3.2) and Cauchy's theorem now gives

$$\sum_{n\leqq x}\chi(n)\Lambda_a(n)=O\left(\left|\int_{\Gamma_1-\Gamma_2-\Gamma_3}\frac{L_f'}{L_f}(s,\chi)\frac{x^{s-(k-1)/2}}{s-(k-1)/2}\,ds\right|\right)+O\left(\frac{x}{T}\log^2 x\right) \tag{3.4}$$

Then, by (3.3),

$$\int_{\Gamma_1-\Gamma_3}\frac{L_f'}{L_f}(s,\chi)\frac{x^{s-(k-1)/2}}{s-\dfrac{k-1}{2}}\,ds=O\left\{\frac{x}{T}\log(qT)\right\},$$

$$\int_{\Gamma_2}\frac{L_f'}{L_f}(s,\chi)\frac{x^{s-(k-1)/2}}{s-\dfrac{k-1}{2}}\,ds=O\left(\exp\left\{\left(1-\frac{b_3}{3\log(q(T+2))}\right)\log x\right\}.\log^2(qT)\right)$$

Here now take $\log T=\sqrt{\log x}$, (3.4) and the inequality $\log q\leqq B\sqrt{\log x}$ give

$$\sum_{n\leqq x}\chi(n)\Lambda_a(n)=O(x\exp\{-c_3\sqrt{\log x}\}).$$

Hence, by (3.1),

$$\psi_a(x,q,l)=O(x\exp\{-c_4\sqrt{\log x}\},\quad\text{for}\quad q\leqq\exp\{B\sqrt{\log x}\}\quad\text{and}\quad x\geqq 2.$$

We note that

$$\sum_{\substack{p\leqq x\\ p\equiv l(q)}}a(p)p^{-(k-1)/2}\log p=\psi_a(x,q,l)+O\Bigg(\sum_{\substack{\nu\\ p\leqq x\\ \nu\geqq 2}}\log p\Bigg)=O(x\exp\{-c_4\sqrt{\log x}\}).$$

This completes the proofs.

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Proof of Theorem 4. The proof proceeds as in the proof of Theorem 3. The condition

$$L_a(s,\chi)\neq 0 \quad \text{for} \quad \sigma \geqq \frac{k+1}{2}-\frac{a}{\log(q(|t|+2))}$$

(which in the proof of Theorem 3 follows from Theorem 2) holds now for fixed q by Lemmas 1, 4, 5, 6 and 8.

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