2

Spaces and Manifolds of Smooth Maps

In this chapter, we consider spaces of differentiable mappings as infinite-dimensional spaces. These spaces will then serve as the model spaces for manifolds of mappings, that is, manifolds of differentiable mappings between manifolds.

2.1 Topological Structure of Spaces of Differentiable Mappings

In this section, we denote by M, N (possibly infinite-dimensional) manifolds.

2.1 Definition Endow the space $C^\infty(M,N)$ with the initial topology with respect to the map

$$\Phi\colon C^\infty(M,N) \to \prod_{k\in\mathbb{N}_0} C(T^kM, T^kN)_{\text{c.o.}}, \quad f \mapsto (T^kf)_{k\in\mathbb{N}_0},$$

where the spaces on the right-hand side carry the compact open topology (see §B.2). The resulting topology on $C^\infty(M,N)$ is called the *compact open C^∞-topology*.

2.2 Remark (a) The map Φ is clearly injective (as $T^0f := f$). Therefore, Φ is a homeomorphism onto its image.

(b) Note that the compact open C^∞-topology is also the initial topology with respect to the mappings

$$T^k : C^\infty(M,N) \to C(T^kM, T^kN)_{\text{c.o}}, \quad f \mapsto T^kf, \quad k \in \mathbb{N}_0.$$

(c) If $M \Subset E, N \Subset F$ for E, F locally convex spaces, the compact open C^∞-topology is the initial topology with respect to the mappings

$$d^k : C^\infty(M,N) \to C(M \times E^k, N)_{\text{c.o.}}, \quad f \mapsto d^kf, \quad k \in N_0.$$

https://doi.org/10.1017/9781009091251.003 Published online by Cambridge University Press

(d) By construction the compact open C^∞-topology is finer than the compact open topology (i.e. the topology induced by the inclusion $C^\infty(M,N) \to C(M,N)$). In particular, if M is locally compact (i.e. M is finite dimensional), the evaluation $\mathrm{ev}\colon C^\infty(M,N) \times M \to N$, $(f,x) \mapsto f(x)$ is continuous by Lemma B.10.

2.3 Lemma *Let L, O be manifolds and $h\colon L \to M$ and $f\colon N \to O$ be smooth. Then the pushforward and the pullback*

$$f_*\colon C^\infty(M,N) \to C^\infty(M,O), \quad g \mapsto f \circ g,$$
$$h^*\colon C^\infty(M,N) \to C^\infty(L,N), \quad g \mapsto g \circ h$$

are continuous.

Proof Since the compact open C^∞-topology is initial with respect to the family $(T^k)_{k \in \mathbb{N}_0}$, it suffices to prove that $T^k \circ f_*$ and $T^k \circ h^*$ are continuous for each $k \in \mathbb{N}_0$. However, the chain rule yields for each k commutative diagram

$$\begin{array}{ccc} C^\infty(M,N) & \xrightarrow{f_*} & C^\infty(M,O) \\ \downarrow{\scriptstyle T^k} & & \downarrow{\scriptstyle T^k} \\ C(T^kM, T^kN)_{\text{c.o.}} & \xrightarrow{(T^kf)_*} & C(M,O)_{\text{c.o.}}, \end{array} \qquad \begin{array}{ccc} C^\infty(M,N) & \xrightarrow{h^*} & C^\infty(L,N) \\ \downarrow{\scriptstyle T^k} & & \downarrow{\scriptstyle T^k} \\ C(T^kM, T^kN)_{\text{c.o.}} & \xrightarrow{(T^kh)^*} & C(T^kL, T^kN)_{\text{c.o.}} \end{array}$$

The pushforward and the pullback in the lower row are continuous by Lemma B.8. We conclude that f_* and h^* are continuous. □

2.4 Proposition *Let E be a locally convex space. Then the compact open C^∞-topology turns $C^\infty(M,E)$ with the pointwise operations into a locally convex space.*

Proof The compact open C^∞-topology is initial with respect to the map

$$\Phi\colon C^\infty(M,E) \to \prod_{k \in \mathbb{N}_0} \underbrace{C(T^kM, T^kE)_{\text{c.o.}}}_{\cong C(T^kM, E^{2^k})_{\text{c.o.}}}, \quad f \mapsto (T^kf)_{k \in \mathbb{N}_0}.$$

Now the spaces $C(T^kM, E^{2^k})_{\text{c.o.}}$ are locally convex spaces by Lemma B.7 since E^{2^k} is a locally convex space. The product of locally convex spaces is again a locally convex space. Thus every linear subspace of the product becomes a locally convex space. Now it is easy to see that Φ is linear with respect to pointwise addition and scalar multiplication. Thus the image of Φ is a linear subspace and Φ is an isomorphism of locally convex spaces identifying $C^\infty(M,E)$ as a locally convex subspace of $\prod_{k \in \mathbb{N}_0} C(T^kM, E^{2^k})$. □

https://doi.org/10.1017/9781009091251.003 Published online by Cambridge University Press

Interlude: Certain Open Sets in the Compact Open C^∞-topology

In this section, we recall the classic arguments (see e.g. Hirsch, 1994) that certain subsets of mappings are open in the compact open C^∞-topology. Observe that one needs here (and we shall require it in the whole section) for the source manifold to be compact. For non-compact source manifolds, the sets discussed here will, in general, not be open in the compact open C^∞-topology. We will need the results collected in this subsection in Chapter 3 when discussing the group of diffeomorphisms.

2.5 Lemma *Let M,N be manifolds, M compact and a finite open cover $U_1,\ldots,U_n$ of M. Then the map*

$$\Psi\colon C^\infty(M,N) \to \prod_{i=1}^n C^\infty(U_i,N), \qquad f \mapsto (f|_{U_i})_{i\in I}$$

is a homeomorphism onto

$$\mathcal{I} := \{(f_i) \in \prod_{i=1}^n C^\infty(U_i,N) \mid f_i|_{U_i\cap U_j} \equiv f_j|_{U_i\cap U_j}, \text{ for all } i,j \in I\}.$$

Moreover, $\mathcal{I}$ is closed in $\prod_{i=1}^n C^\infty(U_i,N)$.

Proof To see that $\mathcal{I}$ is closed, we introduce, for every $i,j \in I := \{1,\ldots,n\}$ and $x \in U_i \cap U_j$, the map $\mathrm{ev}_{x,i,j}\colon \prod_{i=1}^n C^\infty(U_i,N) \to N\times N$, $(\gamma_k)_{1\le k\le n} \mapsto (\gamma_i(x),\gamma_j(x))$. These are continuous since projections onto components in a product and the point evaluations are continuous (see Remark B.5). Now we denote by $\Delta N \subseteq N\times N$ the diagonal (i.e. all elements of the form (n,n)), which is closed in $N\times N$ due to N being Hausdorff. Then $\mathcal{I}$ is closed as the preimage:

$$\mathcal{I} = \bigcap_{x\in\sqcup_{i,j\in I} U_i\cap U_j} \mathrm{ev}_{x,i,j}^{-1}(\Delta N).$$

Note that we can write the restriction $f \mapsto f|_{U_i}$ as the pullback $f|_{U_i} = f\circ \iota_{U_i} = (\iota_{U_i})^*(f)$ with the inclusion of U_i into M. Hence the restriction map is continuous by Lemma 2.3, and as a consequence Ψ is continuous. Moreover, Ψ is clearly injective and we only have to prove that Ψ is an open mapping onto its image. To see this, we need to check that finite intersections of sets of the form

$$\lfloor K,U,k\rfloor := \{f \in C^\infty(M,N) \mid T^k f(K) \subseteq U\},$$
$$K \subseteq T^k M \text{ compact}, \quad U \Subset T^k N, \quad k\in\mathbb{N}$$

are mapped to open sets by Ψ (recall that the topology is initial with respect to the T^k). Now since M is compact, Lang (1999, II, §3 Proposition 3.2) implies

https://doi.org/10.1017/9781009091251.003 Published online by Cambridge University Press

that there are open sets $V_i \subseteq \overline{V}_i \subseteq U_i, 1 \leq i \leq n$ and $M \subseteq \bigcup_{1 \leq i \leq n} V_i$. Consider now $f \in \bigcap_{1 \leq r \leq \ell} \lfloor K_r, U_r, k_r \rfloor$ and define $T^{k_r}\overline{V}_i := \pi_{k_r}^{-1}(\overline{V}_i)$, where $\pi_{k_r} : T^{k_r}U_i \to U_i$ is the bundle projection. Note that $T^{k_r}\overline{V}_i$ is closed in $T^{k_r}U_i$ for every $1 \leq i \leq n$. Then clearly

$$\gamma|_{U_i} \in \bigcap_{1 \leq r \leq \ell} \lfloor K_r \cap T^{k_r}\overline{V}_i, O_r, k_r \rfloor \quad \text{for all } 1 \leq i \leq n. \tag{2.1}$$

Now let $(g_i)_{1 \leq i \leq n} \in \mathcal{I}$ such that every g_i satisfies (2.1) for i. Since the $T^{k_r}V_i$ cover K_r, the unique map g defined by $g|_{U_i} = g_i$ satisfies $g \in \bigcap_{1 \leq r \leq \ell} \lfloor K_r, U_r, k_r \rfloor$. We conclude

$$\prod_{i=1}^{n} \bigcap_{1 \leq r \leq \ell} \lfloor K_r \cap T^{k_r}\overline{V}_i, O_r, k_r \rfloor \subseteq \Psi\left(\bigcap_{1 \leq r \leq \ell} \lfloor K_r, U_r, k_r \rfloor \right)$$

and thus Ψ is open onto its image. □

2.6 Lemma *Let M be a compact manifold and N a finite-dimensional manifold. Then the sets*

$$\operatorname{Imm}(M,N) = \{f \in C^\infty(M,N) \mid f \textit{ is an immersion}\},$$
$$\operatorname{Sub}(M,N) = \{f \in C^\infty(M,N) \mid f \textit{ is a submersion}\}$$

are open in the compact open C^∞-topology.

Proof Since M is compact (hence finite dimensional), a map is an immersion (submersion) if and only if it is infinitesimally injective (surjective). We need to check that these properties define an open set in $C(TM,TN)$, whence they induce an open set in the compact open C^∞-topology.

Consider a map $f \in C^\infty(M,N)$ and pick a pair of charts (U_φ, φ) of M and (U_κ, κ) of N such that $f(U_\varphi) \subseteq U_\kappa$. In addition, we pick a compact set $K_\varphi \subseteq U_\varphi$ with non-empty interior. By compactness of M, we can choose a finite set of pairs of charts and compact sets such that the interior of the K_φ cover M. Apply Lemma 2.5 to obtain an embedding $C^\infty(M,N) \to \prod_{i=1}^n C^\infty(U_{\varphi_i}, N)$. We will now construct for each K_{φ_i} an open neighbourhood in $C^\infty(U_{\varphi_i}, N)$ which consists only of immersions (submersions) if f has this property. Pulling back the product of these neighbourhoods with the embedding then yields the desired neighbourhood of f in $C^\infty(M,N)$.

To this end, we consider the smooth map $g := \kappa \circ f \circ \varphi^{-1} \in C^\infty(V_\varphi, V_\kappa)$ where $V_\varphi \Subset \mathbb{R}^d$ and $V_\kappa \Subset \mathbb{R}^n$. Set $L := \varphi(K_\varphi)$ and observe that g is an immersion (submersion) if and only if f is an immersion (submersion). Recall that on open subsets of vector spaces we have $Tg = (g, dg) \in C(TV_\varphi, TV_\kappa) = C(V_\varphi \times \mathbb{R}^d, V_\kappa \times \mathbb{R}^n)$. Let $e_1, \ldots, e_d$ be the standard basis of $\mathbb{R}^d$. For $x \in L$ we represent the Jacobian as $J_x(g) = [df(x; e_1), \ldots, df(x, e_d)]$. If g is

https://doi.org/10.1017/9781009091251.003 Published online by Cambridge University Press

an immersion (submersion) then the Jacobi matrix has for every x maximal rank, that is, if g is an immersion, the differential is injective if and only if $\mathrm{rk} J_x(g) = d \leq n$. In particular, the rank of the Jacobi matrix is constant, say, $\mathrm{rk} J_x(g) = N$ for all $x \in L$. We can thus pick for every $x \in L$ a subset $I_x \subseteq \{1, \ldots, d\}$ of N elements such that $\{dg(x; e_j)\}_{j \in I_x}$ is linearly independent (note that the indices will, in general, depend on x!). If $\{df(x; e_j)\}_{j \in I_x}$ is linearly independent, then there exists $\varepsilon_x > 0$ such that every tuple $(x_1, \ldots, x_N) \in \prod_{j \in I_x} B_\varepsilon(dg(x; e_j))$ is linearly independent (where $B_\varepsilon(z)$ is the ε-ball in $\mathbb{R}^d$; see Margalef-Roig and Domínguez, 1992, Lemma 1.6.7). By continuity of dg, there is for every $x \in L$ a compact neighbourhood N_x of x such that $(dg(y; e_{j_1}), \ldots, dg(y; e_{j_N})) \in \prod_{j \in I_x} B_\varepsilon(dg(x; e_j)$ for all $y \in \overline{N}_x$. Thus

$$dg \in \Omega(g, x) := \bigcap_{j \in I_x} \lfloor N_x \times \{e_j\}, B_{\varepsilon_x}(df(x; e_j)) \rfloor. \tag{2.2}$$

By construction, every $h \in C^\infty(V_\varphi, V_\kappa)$ with $dh \in \Omega(g, x)$ has a Jacobian of rank N at every point in N_x. In other words every such h is an immersion (submersion) on N_x if g is such a map. In Exercise 2.1.4 we shall now construct from $\Omega(g, x)$ an open neighbourhood of f in $C^\infty(U_\varphi, N)$ consisting only of maps which restrict to immersions (submersions) on K_φ if f is an immersion (submersions). We conclude that $\mathrm{Imm}(M, N)$ and $\mathrm{Sub}(M, N)$ are neighbourhoods of their points, hence open. □

2.7 Proposition *Let M be a compact manifold and N a finite-dimensional manifold. Then the set of embeddings*

$$\mathrm{Emb}(M, N) := \{f \in \mathrm{Imm}(M, N) \mid f \text{ is a topological embedding}\}$$

is open in the compact open C^∞-topology.

Proof Let $f \in \mathrm{Emb}(M, N)$ and fix a finite family of charts

(a) (U_i, φ_i) of M and (V_i, ψ_i) of N such that $f(U_i) \subseteq V_i$, and such that
(b) for every i there is a compact set $K_i \subseteq U_i$ and the interiors of the K_i cover M.

Recall that an embedding is, in particular, an injective immersion. Hence Lemma 2.6 allows us to choose an open neighbourhood $O_f \Subset C^\infty(M, N)$ of f consisting only of immersions which satisfy also property (a). We shall now show that we can shrink O_f to obtain an open neighbourhood of f consisting only of immersions.

We have already seen in Lemma 1.50 that every immersion in O_f restricts locally to an embedding. However, since M, N are finite-dimensional manifolds, we can use the quantitative version of the inverse function theorem (see

https://doi.org/10.1017/9781009091251.003 Published online by Cambridge University Press

Glöckner, 2016, 1.1 and the references there) to obtain a uniform estimate on the size of these neighbourhoods: Shrinking O_f, we may assume that every $g \in O_f$ satisfies

(c) $g|_{U_i}$ is an embedding for every i

(an alternative proof of this fact using uniform estimates can be found in Hirsch, 1994, 2. Lemma 1.3). Now since f is an embedding, we see that for every i the compact sets $f(K_i)$ and $f(M \setminus U_i)$ are disjoint and we can thus find disjoint $A_i, B_i \Subset N$ such that $f(K_i) \subseteq A_i$ and $f(M \setminus U_i) \subseteq B_i$. As there are only finitely many i, we can shrink O_f further such that every $g \in O_f$ satisfies

(d) $g(K_i) \subseteq A_i$ and $g(M \setminus U_i) \subseteq B_i$ for all i.

We shall now show that $g \in O_f$ is injective. Let $x, y \in M$ be distinct points and $x \in K_i$. If $y \in U_i$, then $g(x) \neq g(y)$ by (c). If $y \in M \setminus U_i$, then $g(x) \in A_i$ and $g(y) \in B_i$ by (d), so again $g(x) \neq g(y)$. We conclude that g is injective.

Summing up, O_f is an open set consisting entirely of injective immersions. However, since M is compact, every injective immersion is an embedding (see e.g. Margalef-Roig and Domínguez, 1992, Proposition 3.3.4). We conclude that O_f consists only of embeddings, whence $\operatorname{Emb}(M, N)$ is open. □

2.8 Corollary *If M is a compact manifold, the set of diffeomorphisms*

$$\operatorname{Diff}(M) := \{f \in C^\infty(M, M) \mid \text{ there exists } g \in C^\infty(M, M) \text{ with } g \circ f = \operatorname{id}_M\}$$

is open in $C^\infty(M, M)$ with the compact open C^∞-topology.

Proof A diffeomorphism ϕ permutes the connected components of M and induces on every component a diffeomorphism onto another component. Since the components are compact, there is an open ϕ-neighbourhood in $C^\infty(M, M)$ whose elements map every component to the same component as ϕ. Thus we may assume that M is connected.

A diffeomorphism $\phi \colon M \to M$ is, in particular, a map which is an embedding and a submersion. Assume conversely that $\psi \colon M \to M$ is a mapping which is a submersion and an embedding. Since the image of a submersion is open (Exercise 1.7.5), the set $\phi(M)$ is open and closed in M, whence $\phi(M) = M$ by connectedness of M. Hence ϕ is a bijective map. Its inverse is smooth by Exercise 1.7.6 as ϕ is a submersion and $\operatorname{id}_M = \phi^{-1} \circ \phi$ (alternatively a bijective embedding) is a diffeomorphism by Lemma 1.61. □

2.9 Remark As was already mentioned for non-compact M, the subsets considered in this subsection will, in general, not form open subsets of $C^\infty(M, N)$ with respect to the compact open C^∞-topology. The reason for this is that the

https://doi.org/10.1017/9781009091251.003 Published online by Cambridge University Press

compact open topology can only control a function's behaviour on compact sets. On a non-compact manifold, this topology is too weak to control the behaviour of a function on all of M. For this reason, one has to refine the topology if M is non-compact. The Whitney-type topologies are a common choice; see Hjelle and Schmeding (2017). However, many results presented in the next sections do not hold (at least not in the generality stated) for the Whitney-type topologies. A few examples of this behaviour for M non-compact are:

- the pullback h^* is, in general, discontinuous for the Whitney-topologies (whereas it is continuous in the compact open topology by Lemma 2.3);
- the exponential law, Theorem 2.12, is wrong.

While one can develop a general theory for function spaces on non-compact manifolds (see e.g. Michor, 1980), these examples show already that the resulting theory will require a much higher technical investment. We refrain from a discussion in the context of this book and refer the interested reader instead to the literature (Hjelle and Schmeding, 2017; Michor, 1980).

Exercises

2.1.1 Fill in the details for Remark 2.2.

(a) Show that the compact open C^∞-topology is the initial topology with respect to $(T^k)_{k\in\mathbb{N}}$.
Hint: A mapping into a product is continuous if and only if each component is continuous.

(b) If M, N are open subsets of locally convex spaces, show that the initial topologies with respect to the families $(T^k)_{k\in\mathbb{N}_0}$ and $(d^k)_{k\in\mathbb{N}_0}$ coincide.
Hint: Exercise 1.6.3 yields one inclusion of topologies. For the converse show inductively that $d^k \circ f$, for all $k \in \mathbb{N}_0$ continuous, implies $T^k \circ f$ continuous for all $f\colon Z \to C^\infty(M,N)$.

2.1.2 Let $\varphi\colon M \to N$ be a smooth map between smooth manifolds and E a locally convex space. Show that the pullback $\varphi^*\colon C^\infty(N,E) \to C^\infty(M,E)$, $f \mapsto f\circ\varphi$ is continuous linear. Deduce that if φ is a diffeomorphism, then φ^* is an isomorphism of locally convex spaces.

2.1.3 Let K, L be compact manifolds and M be a manifold. Show that the composition map $\mathrm{Comp}\colon C^\infty(K,M) \times C^\infty(L,K) \to C^\infty(L,M)$, $(f,g) \mapsto f \circ g$ is continuous.

https://doi.org/10.1017/9781009091251.003 Published online by Cambridge University Press

2.1.4 Fill in the missing details in the proof of Lemma 2.6. Show, in particular, that, thanks to the compactness of L, the $\Omega(g,x)$ yield an open neighbourhood of g in $C^\infty(V_\varphi, V_\kappa)$ consisting only of mappings whose Jacobian has maximal rank on the compact set L. Moreover, construct a neighbourhood of $f \in C^\infty(M,N)$ consisting entirely of immersions (submersions) if f is an immersion (submersion).

2.2 The Exponential Law and Its Consequences

In this section, we prove a version of the exponential law, Theorem 2.12, for smooth mappings. Before we begin, let us observe a crucial fact about the compact open C^∞-topology.

Assume that M is a *compact* manifold and E a locally convex space. Since the compact open C^∞-topology is finer than the compact open topology, we see that for every

$$O \Subset E, \qquad C^\infty(M,O) := \{f \in C^\infty(M,E) \mid f(M) \subseteq O\}$$

is an open subset. Now $C^\infty(M,E)$ is a locally convex space by Proposition 2.4, whence $C^\infty(M,O)$ becomes a manifold and it makes sense to consider differentiable mappings with values in $C^\infty(M,O)$.

We now prepare the proof of the exponential law by providing several auxiliary results.

2.10 Lemma *Let E,F,H be locally convex spaces, $U \Subset E$ and $V \Subset F$. If $f\colon U \times V \to H$ is smooth, then so is $f^\vee\colon U \to C^\infty(V,H) f^\vee(u) := f(u,\cdot)$. Its derivative is given by*

$$df^\vee(x;\cdot) = (d_1 f)^\vee(x). \tag{2.3}$$

Proof

Step 1: $f^\vee$ is continuous.

It suffices to prove that $d^k \circ f^\vee\colon U \to C(V \times F^k, H)_{\text{c.o.}}$ is continuous for every $k \in \mathbb{N}_0$ (see Remark 2.2 and Exercise 2.1.1). For $k = 0$ this was proved in Proposition B.13. We prove by induction that

$$d^k \circ f^\vee(x) = d^k(f^\vee(x)) = (d_2^k f)^\vee(x) \quad \text{for all } x \in U. \tag{2.4}$$

The induction start for $k = 0$ is trivial. For the induction step let $k > 0$ and we compute

https://doi.org/10.1017/9781009091251.003 Published online by Cambridge University Press

$$\begin{aligned}
&d^k(f^\vee(x))(y;v_1,\ldots,v_k)\\
&\quad= \lim_{t\to 0} t^{-1}\Big(d^{k-1}(f^\vee(x))(y+tv_k;v_1,\ldots,v_{k-1})\\
&\qquad -d^{k-1}(f^\vee(x))(y;v_1,\ldots,v_{k-1})\Big)\\
&\quad= \lim_{t\to 0} t^{-1}\Big(d^{k-1}(f(x,y+tv_k;(0,v_1),\ldots,(0,v_{k-1}))\\
&\qquad -d^{k-1}(f(x,y;(0,v_1),\ldots,(0,v_{k-1}))\Big)\\
&\quad= d^k f(x,y;(0,v_1),\ldots,(0,v_k)).
\end{aligned}$$

Thus we have identified $d^k \circ f$ as $(d_2^k f)^\vee$ which is again continuous by Proposition B.13. We conclude that $f^\vee$ is continuous.

Step 2: $f^\vee$ is C^1 and the derivative satisfies (2.3).
Pick $x \in U$, $z \in E$ and $t \in \mathbb{R}$ small. We shall show that

$$\Delta(t,x,z) := t^{-1}(f^\vee(x+tz) - f^\vee(x)) \xrightarrow{t\to 0} (d_1 f)^\vee(x,\cdot;z)$$

in $C^\infty(V,H)$. Recall that the compact open C^∞-topology is initial with respect to the family $(d^k\colon C^\infty(V,H) \to C(V\times F^k,H)_{\mathrm{c.o}})_{k\in\mathbb{N}_0}$. Thus $\Delta(t,x,z)$ converges for $t\to 0$ if and only if $d^k\circ\Delta(t)$ converges. Therefore, we pick $k\in\mathbb{N}_0$ and a neighbourhood $\lfloor K,U\rfloor$ of $d^k((d_1 f)^\vee)$, i.e. $K\subseteq V\times F^k$ is compact such that $(d_1 f)^\vee(x,\cdot,z)(K)\subseteq U \Subset H$. Since higher differentials are symmetric by Schwarz' theorem (Exercise 1.3.3), we have

$$\begin{aligned}
d^k(d_1 f)^\vee(x;z)(y;v_1,\ldots,v_k) &= d^{k+1} f(x,y;(z,0),(0,v_1),\ldots,(0,v_k))\\
&= d^{k+1} f(x,y;(0,v_1),\ldots,(0,v_k),(z,0)).
\end{aligned}$$

Now, we saw in Lemma 1.21 that the difference quotient extends continuously to $t=0$ by the differential. We apply this to $d^k f$: For each $\overline{y} := (y_0,v_1,\ldots,v_k)\in K\subseteq V\times F^k$, there exists $\overline{y}\in N_{\overline{y}}\Subset V\times F^k$ and $\varepsilon_{\overline{y}}>0$ such that

$$\begin{aligned}
&N_{\overline{y}}\times]-\varepsilon_{\overline{y}},\varepsilon_{\overline{y}}[\,\setminus\{0\}\to H,\\
&(\overline{w},t)\mapsto t^{-1}(d^k f(x+tz,w_0;(0,w_1),\ldots,(0,w_k))\\
&\qquad\qquad - d^k f(x,w_0;(0,w_1),\ldots,(0,w_k)),
\end{aligned}$$

where $\overline{w} = (w_0,\ldots,w_k)$ the function takes values in U and extends continuously to some function $N_{\overline{y}}\times]-\varepsilon_{\overline{y}},\varepsilon_{\overline{y}}[\to U$. Exploiting compactness of K we cover it by finitely many of these neighbourhoods $N_{\overline{y}_1},\ldots,N_{\overline{y}_\ell}$. Hence if $|t|<\min_{1\le i\le\ell}\varepsilon_{\overline{y}_i}$, we see that

$$t^{-1}(d^k f(x+tz,v;(0,v_1),\ldots,(0,v_k)) - d^k f(x,v;(0,v_1),\ldots,(0,v_k))\in U.$$

In other words, $d^n\Delta(t,x,z)(\overline{v})\in U$ for all t small enough and $\overline{v}\in K$, hence (2.3) holds. Since $(d_1 f)^\vee$ is C^0 by Step 1, we see that $f^\vee$ is C^1.

Step 3: f is C^k for each $k \geq 2$.
Note that $h\colon (U \times E) \times V \to H$, $((x,z),y) \mapsto d_1 f(x,y;z)$ is smooth. Now we argue by induction, where Step 2 is the induction start. Also Step 2 shows that $df^\vee = h^\vee$. The induction hypothesis shows that $h^\vee$ is C^{k-1}, so $f^\vee$ must be C^k and since k was arbitrary, $f^\vee$ is smooth. □

2.11 Proposition *Let E be a finite-dimensional space, F be locally convex and $U \Subset E$. Then the evaluation map $\mathrm{ev}\colon C^\infty(U,F) \times U \to F$ is smooth.*

Proof We already know from Remark 2.2 that ev is continuous. Moreover, ev is linear in the first component and thus $d_1\, \mathrm{ev}(f,x;g) = \mathrm{ev}(g,x)$ (this implies, in particular, that all partial derivatives with respect to the first component exist). Let us now compute $d_2\, \mathrm{ev}$. For $f \in C^\infty(U,F)$ and $x \in U$, $w \in E$ and small t, we have

$$t^{-1}\,\mathrm{ev}(f,x+tw) - \mathrm{ev}(f,x) = t^{-1}(f(x+tw) - f(x)) \to df(x;w) \text{ as } t \to 0.$$

Hence $d_2\, \mathrm{ev}(f,x;w)$ exists and is given by

$$d_2\, \mathrm{ev}(f,x;w) = df(x;w) = \mathrm{ev}_1(df,(x,w)),$$

where $\mathrm{ev}_1\colon C^\infty(U \times E,F) \times (U \times E) \to F$, $(\gamma,z) \mapsto \gamma(z)$ is continuous. We conclude that

$$d\,\mathrm{ev}(f,x;g,v) = d_1\, \mathrm{ev}(f,x;g) + d_2\, \mathrm{ev}(f,x;v) = \mathrm{ev}(g,x) + \mathrm{ev}_1(df,(x,v)) \tag{2.5}$$

exists and is continuous (the mapping $C^\infty(U,F) \to C^\infty(U \times E,F)$, $f \mapsto df$ is clearly continuous and linear hence smooth). Thus ev (and also ev_1) is C^1. We see that inductively ev is C^k as its derivative is already C^{k-1}. □

We will now formulate and prove the exponential law, Theorem 2.12. To justify the name, denote the set of all functions from X to Y by Y^X. In this notation, the exponential law for arbitrary maps becomes $(Z^Y)^X \cong Z^{X\times Y}$, hence its name.

2.12 Theorem (Exponential law) *Let M be a compact manifold and $O \Subset E, U \Subset F$ be open subsets of locally convex spaces. Then*

(a) *If $f\colon U \times M \to O$ is smooth, so is $f^\vee\colon U \to C^\infty(M,O)$, $f^\vee(x)(y) := f(x,y)$.*
(b) *The evaluation map $\mathrm{ev}\colon C^\infty(M,O) \times M \to O$, $(\gamma,x) \mapsto \gamma(x)$ is smooth and the map $C^\infty(U \times M,O) \to C^\infty(U,C^\infty(M,O)$, $f \mapsto f^\vee$ is a bijection.*

In particular, the mapping f is smooth if and only if $f^\vee$ is smooth.

https://doi.org/10.1017/9781009091251.003 Published online by Cambridge University Press

Proof Since $C^\infty(M,O)$ is an open submanifold of $C^\infty(M,E)$, Lemma 1.39 shows that for the purpose of this proof we may assume without loss of generality that $O = E$.

(a) Pick a finite atlas $\{(\varphi_i, U_i)\}_{1\leq i\leq n}$ of the compact manifold M and obtain from Lemma 2.5 a topological embedding with closed image

$$\Psi\colon C^\infty(M,E) \to \prod_{i=1}^n C^\infty(U_i,E), \quad f \mapsto (f|_{U_i})_{1\leq i\leq n}.$$

Now since the image of Ψ is closed we apply Lemma 1.25: For any smooth $f\colon U \times M \to O$, the map $f^\vee\colon U \to C^\infty(M,O) \Subset C^\infty(M,E)$ will be smooth if and only if $\Psi \circ f^\vee$ is smooth. In other words, $f^\vee$ is smooth if and only if for every $1 \leq i \leq n$ the map $U \ni x \to f^\vee(x)|_{U_i} \in C^\infty(U_i,E)$ is smooth. Applying Exercise 2.1.2, this is equivalent to the smoothness of the mappings $U \ni x \to f^\vee(x)|_{U_i} \circ \varphi_i^{-1} \in C^\infty(\varphi_i(U_i),E)$ and these mappings are smooth by Lemma 2.10.

(b) The map $\mathrm{ev}\colon C^\infty(M,O) \times M \to O$ is smooth if locally around each point it is smooth. Hence we choose $(f,x) \in C^\infty(M,O) \times M$ and pick $\varphi\colon U \to V \Subset \mathbb{R}^d$ a chart of M around x. We obtain another evaluation map $\mathrm{ev}_\varphi\colon C^\infty(V,O) \times V \to O$ such that

$$\mathrm{ev}(\eta,z) = \mathrm{ev}_\varphi((\varphi^{-1})^*(\eta),\varphi(z)), \quad (\eta,z) \in C^\infty(M,O) \times U.$$

Note that $(\varphi^{-1})^*\colon C^\infty(M,O) \to C^\infty(V,O)$ is the restriction of the continuous linear (hence smooth) map $(\varphi^{-1})^*\colon C^\infty(M,E) \to C^\infty(V,E)$ to the open set $C^\infty(M,O)$ (see Exercise 2.1.2). Hence $(\varphi^{-1})^*$ is smooth and ev will be smooth if ev_φ is smooth for each chart φ of M. However, the smoothness of ev_φ was established in Proposition 2.11.

We finally have to check that $C^\infty(U \times M,O) \to C^\infty(U,C^\infty(M,O)$, $f \mapsto f^\vee$ is bijective. Obviously it is injective; hence we need to check surjectivity. If $f\colon U \to C^\infty(M,O)$ is smooth, then the following map is smooth:

$$U \times M \to C^\infty(M,O) \times M, \quad (u,m) \mapsto (f(u),m).$$

Composing with ev, the map $f^\wedge\colon U \times M \to O$, $(u,m) \mapsto f(u)(m)$ is smooth and satisfies $(f^\wedge)^\vee = f$. This establishes surjectivity. □

2.13 Remark One can even show that the map from Theorem 2.12(b) is a homeomorphism. Moreover, one can obtain similar results for finite orders of differentiability. We skip the details here and refer instead to Alzaareer and Schmeding (2015) for more information.

Note that compactness of the manifold M is a crucial ingredient and the statement of the exponential law becomes false for general non-compact M; see Michor (1980).

Exercises

2.2.1 Let M be compact and E be a finite-dimensional vector space. Show that the evaluation map $\mathrm{ev}\colon C^\infty(M,E) \times M \to E$ is a submersion. (If you fancy a real challenge, prove this for a locally convex space E.)

2.2.2 Let $f\colon U \times M \to O$ and $p\colon O \to N$ be smooth maps. As always, we denote by $p_*\colon C^\infty(M,O) \to C^\infty(M,N)$, $h \mapsto p \circ h$, the push-forward and assume that the exponential law, Theorem 2.12, holds for the spaces $C^\infty(U \times M, O)$ and $C^\infty(U \times M, N)$. Prove that

$$p_* \circ (f^\vee) = (p \circ f)^\vee.$$

2.3 Manifolds of Mappings

In this section, we discuss spaces of smooth mappings between manifolds as infinite-dimensional manifolds. We shall not directly construct the manifold structure for general spaces (see Appendix C for a sketch).

General Assumption In this section K will be a compact smooth manifold, M, N will be smooth (possibly infinite-dimensional) manifolds.

2.14 Definition A smooth manifold structure on $C^\infty(K,M)$ is *canonical* if

- the underlying topology is the compact open C^∞-topology, and
- for each (possibly infinite-dimensional) C^∞-manifold N and for a map $f\colon N \to C^\infty(K,M)$, said map is C^∞ if and only if

$$f^\wedge\colon N \times K \to M, \quad (x,y) \mapsto f(x)(y) \quad \text{is } C^\infty.$$

2.15 Remark A canonical manifold structure enforces a suitable version of the exponential law, Theorem 2.12. This enables differentiability properties of mappings to be verified on the underlying manifolds.

A similar notion of canonical manifold exists also for spaces of finitely often differentiable mappings (Amiri et al., 2020; Glöckner and Schmeding, 2022). We hasten to remark that the usual constructions of manifolds of mappings yield canonical manifold structures (see Appendix C).

2.16 Lemma *If $C^\infty(K,M)$ is endowed with a canonical manifold structure, then*

(a) *The evaluation map* $\mathrm{ev}\colon C^\infty(K,M)\times K\to M$, $\mathrm{ev}(\gamma,x):=\gamma(x)$ *is a C^∞-map.*

(b) *Canonical manifold structures are unique: writing $C^\infty(K,M)'$ for $C^\infty(K,M)$ with another canonical manifold structure, then*

$$\mathrm{id}\colon C^\infty(K,M)\to C^\infty(K,M)', \qquad \gamma\mapsto\gamma$$

is a C^∞-diffeomorphism.

(c) *Let $N\subseteq M$ be a submanifold such that the set $C^\infty(K,N)$ is a submanifold of $C^\infty(K,M)$. Then the submanifold structure on $C^\infty(K,N)$ is canonical.*

(d) *If M_1 and M_2 are smooth manifolds such that $C^\infty(K,M_1)$ and $C^\infty(K,M_2)$ have canonical manifold structures, then the manifold structure on the product manifold $C^\infty(K,M_1)\times C^\infty(K,M_2)\cong C^\infty(K,M_1\times M_2)$ is canonical.*

Proof (a) Since $\mathrm{id}\colon C^\infty(K,M)\to C^\infty(K,M)$ is C^∞ and $C^\infty(K,M)$ is endowed with a canonical manifold structure, it follows that $\mathrm{id}^\vee\colon C^\infty(K,M)\times K\to M$, $(\gamma,x)\mapsto \mathrm{id}(\gamma)(x)=\gamma(x)=\mathrm{ev}(\gamma,x)$ is C^∞.

(b) The map $f:=\mathrm{id}\colon C^\infty(K,M)\to C^\infty(K,M)'$ satisfies $f^\wedge=\mathrm{ev}$ and is thus C^∞, by (a). Since $C^\infty(K,M)'$ is endowed with a canonical manifold structure, it follows that f is C^∞. By the same reasoning, $f^{-1}=\mathrm{id}\colon C^\infty(K,M)'\to C^\infty(K,M)$ is C^∞.

(c) As $C^\infty(K,N)$ is a submanifold, the inclusion $\iota\colon C^\infty(K,N)\to C^\infty(K,M)$, $\gamma\mapsto\gamma$ is C^∞ (see Lemma 1.39). Likewise, the inclusion map $j\colon N\to M$ is C^∞. Let L be a manifold and $f\colon L\to C^\infty(K,N)$ be a map. If f is smooth, then $\iota\circ f$ is smooth, entailing that $(\iota\circ f)^\wedge\colon L\times K\to M$, $(x,y)\mapsto f(x)(y)$ is C^∞. As the image of this map is contained in N, which is a submanifold of M, we deduce that $f^\wedge=(\iota\circ f)^\wedge|^N$ is C^∞. If, conversely, $f^\wedge\colon L\times K\to N$ is C^∞, then also $(\iota\circ f)^\wedge=j\circ(f^\wedge)\colon L\times K\to M$ is C^∞. Hence $\iota\circ f\colon L\to C^\infty(K,M)$ is C^∞ (the manifold structure on the range being canonical). As $\iota\circ f$ is a C^∞-map with image in $C^\infty(K,N)$ which is a submanifold of $C^\infty(K,M)$, we deduce that f is C^∞.

(d) If L is a manifold and $f=(f_1,f_2)\colon L\to C^\infty(K,M_1)\times C^\infty(K,M_2)$ a map, then f is C^∞ if and only if f_1 and f_2 are C^∞. As the manifold structures are canonical, this holds if and only if $f_1^\wedge\colon L\times K\to M_1$ and $f_2^\wedge\colon L\times K\to M_2$ are C^∞, which holds if and only if $f^\wedge=(f_1^\wedge,f_2^\wedge)$ is C^∞. □

2.17 Proposition *Assume that $C^\infty(K,M)$ and $C^\infty(K,N)$ admit canonical manifold structures. If $\Omega\subseteq K\times M$ is an open subset and $f\colon\Omega\to N$ is a*

https://doi.org/10.1017/9781009091251.003 Published online by Cambridge University Press

C^∞-map, then $\Omega' := \{\gamma \in C^\infty(K,M) \mid \{(k,\gamma(k)), k \in K\} \subseteq \Omega\}$ is an open subset of $C^\infty(K,M)$ and

$$f_\star\colon \Omega' \to C^\infty(K,N), \qquad \gamma \mapsto f \circ (\mathrm{id}_K, \gamma)$$

is a C^∞-map.

Proof In Exercise B.2.1 it was proved that Ω' is open in $C(K,M)_{\text{c.o.}}$, and so it is open in the finer compact open C^∞-topology. By Lemma 2.16(a), the evaluation map $\mathrm{ev}\colon C^\infty(K,M) \times K \to M$ is C^∞, whence $C^\infty(K,M) \times K \to K \times M$, $(\gamma,x) \mapsto (x,\gamma(x))$ is C^∞. Since f is C^∞, the chain rule shows that

$$(f_\star)^\wedge\colon \Omega' \times K \to N, \qquad (\gamma,x) \mapsto f_\star(\gamma)(x) = f(x,\gamma(x)) = f(x,\mathrm{ev}(\gamma,x))$$

is C^∞. So $f_\star$ is C^∞, as the manifold structure on $C^\infty(K,N)$ is canonical. □

2.18 Corollary *Assume that $C^\infty(K,M)$ and $C^\infty(K,N)$ admit canonical manifold structures. If $f\colon K \times M \to N$ is a C^∞-map, then we obtain a smooth map*

$$f_\star\colon C^\infty(K,M) \to C^\infty(K,N), \qquad \gamma \mapsto f \circ (\mathrm{id}_K, \gamma).$$

Applying Corollary 2.18 with $f(x,y) := g(y)$, we get the following.

2.19 Corollary *Assume that $C^\infty(K,M)$ and $C^\infty(K,N)$ admit canonical manifold structures. If $g\colon M \to N$ is a C^∞-map, then the* pushforward *is smooth*

$$g_*\colon C^\infty(K,M) \to C^\infty(K,N), \qquad \gamma \mapsto g \circ \gamma.$$

To construct manifold structures on $C^\infty(K,M)$ one needs an additional structure on M. This so-called *local addition* replaces the vector space addition not present on M.

2.20 Definition Let M be a smooth manifold. A *local addition* is a smooth map

$$\Sigma\colon U \to M,$$

defined on an open neighbourhood $U \Subset TM$ of the *zero-section* of the tangent bundle $\mathbf{0}_M := \{0_p \in T_pM \mid p \in M\}$ such that $\Sigma(0_p) = p$ for all $p \in M$,

$$U' := \{(\pi_M(v), \Sigma(v)) \mid v \in U\}$$

is open in $M \times M$ and $\theta := (\pi_{TM}, \Sigma)\colon U \to U'$ is a diffeomorphism.

If $C^\infty(M,N)$ is canonical and we interpret a tangent vector as an equivalence class of smooth curves $[t \to c(t)]$, with $c\colon\,]-\varepsilon,\varepsilon[\to C^\infty(M,N)$, the derivative of c can be identified with the partial derivative of the adjoint map $c^\wedge\colon\,]-\varepsilon,\varepsilon[\times M \to N$. This shows that as a set we should have $TC^\infty(M,N) \cong C^\infty(M,TN)$. In the presence of a local addition, the set $C^\infty(M,TN)$ turns also

https://doi.org/10.1017/9781009091251.003 Published online by Cambridge University Press

into a canonical manifold and the bijection becomes an isomorphism of vector bundles over the identity. Summing up, this identification yields the following result.

2.21 Proposition *If M admits a local addition*[1]*, then $C^\infty(K,M)$ admits a canonical manifold structure and the tangent bundle can be identified with $C^\infty(K,TM)$.*

We refer to Appendix C for more information about the proof.

2.22 Assume that M,N admit local additions and $f\colon M \to N$ is a C^∞-map. Then the identification $TC^\infty(K,M) \cong C^\infty(K,TM)$ induces a commuting diagram (Exercise 2.3.6):

$$\begin{array}{ccc} TC^\infty(K,M) & \xrightarrow{\cong} & C^\infty(K,TM) \\ \big\downarrow{\scriptstyle T(f_*)} & & \big\downarrow{\scriptstyle (Tf)_*} \\ TC^\infty(K,N) & \xrightarrow{\cong} & C^\infty(K,TN). \end{array} \tag{2.6}$$

We have seen in Corollary 2.19 and Exercise 2.3.1 that the pushforward and the pullback of smooth functions are smooth with respect to canonical manifolds of mappings. Viewing these mappings as partial mappings of the full composition map

$$\mathrm{Comp}\colon C^\infty(K,M)\times C^\infty(L,K) \to C^\infty(L,M), \quad (f,g)\mapsto f\circ g,$$

we see that the full composition is separately smooth in its variables. This immediately prompts the question as to whether the full composition map is smooth. In the general case (of a possibly non-compact source manifold) when one has no exponential law available this is complicated, but in our situation it reduces to an easy observation.

2.23 Proposition *Let K,L be compact manifolds and assume that the manifolds $C^\infty(K,M),C^\infty(L,M)$ are canonical ($C^\infty(L,K)$ is automatically a canonical manifold as L admits a local addition). Then the composition map*

$$\mathrm{Comp}\colon C^\infty(K,M)\times C^\infty(L,K) \to C^\infty(L,M), \qquad (f,g)\mapsto f\circ g$$

is smooth.

[1] One can show that every paracompact strong Riemannian manifold (see Chapter 4), and thus every finite-dimensional paracompact manifold, admits a local addition. Moreover, Lie groups (see Chapter 3) admit local additions, C.2.

https://doi.org/10.1017/9781009091251.003 Published online by Cambridge University Press

Proof By the exponential law for canonical manifolds, Comp is smooth if and only if the adjoint map

$$\mathrm{Comp}^\wedge\colon C^\infty(K,M)\times C^\infty(L,K)\times L\to M,\quad (f,g,l)\mapsto f(g(l))$$

is smooth. However, this shows that $\mathrm{Comp}^\wedge(f,g,l) = \mathrm{ev}(f,\mathrm{ev}(g,l))$ and since the evaluation mappings are smooth for canonical manifolds, also the adjoint map and thus the composition are smooth. □

We have already seen that certain properties 'lift to the manifold of mappings'. For example, if $f\colon M \to N$ is smooth, the pushforward $f_*\colon C^\infty(K,M)\to C^\infty(K,N)$, $g\mapsto f\circ g$ is smooth. Another example of this is the following result whose proof is remarkably involved and technical (we omit the proof here and pose it as Exercise 2.3.7).

2.24 Lemma (Stacey–Roberts Lemma (Amiri and Schmeding, 2019, Lemma 2.4)) *Let $p\colon M\to N$ be a submersion between finite-dimensional manifolds. Endowing the function spaces with their canonical manifold structure, the pushforward $p_*\colon C^\infty(K,M)\to C^\infty(K,N)$ becomes a submersion.*

In the following chapters, we will study other structures from differential geometry which can be lifted from finite dimensions to spaces of differentiable functions. For Lie groups this leads to the so-called current groups (whose most prominent examples are the loop groups). In the context of Riemannian geometry, the lifting procedure gives rise to the L^2-metric and more generally to the Sobolev type Riemannian metrics on function spaces. Some examples of Sobolev type metrics will be discussed in §5.1 and Chapter 7.

Exercises

2.3.1 Let $h\colon L\to K$ be a smooth map. Assume that $C^\infty(K,M)$ and $C^\infty(L,M)$ are canonical manifolds.

(a) Show that the pullback $h^*\colon C^\infty(K,M)\to C^\infty(L,M)$, $f\mapsto f\circ h$ is smooth.

(b) Assume that K,L are compact and M admits a local addition. Then we identify $TC^\infty(K,M)\cong C^\infty(K,TM)$ (see C.12). Show that this identifies $T(h^*)$ with $h^*\colon C^\infty(K,TM)\to C^\infty(L,TM)$.

2.3.2 Let K be a compact manifold and $O \Subset E$ in a locally convex space. Prove that $C^\infty(K,O) \Subset C^\infty(K,E)$ (Proposition 2.4) is a canonical manifold.

https://doi.org/10.1017/9781009091251.003 Published online by Cambridge University Press

2.3.3 Let M be a finite-dimensional manifold and K be a compact manifold. Endow $C^\infty(K,M)$ with the canonical manifold structure from Appendix C.

(a) Show that for $x \in K$ the point evaluation $\mathrm{ev}_x : C^\infty(K,M) \to M$, $\gamma \mapsto \gamma(x)$ is a submersion.[2]

(b) Deduce that the set $S(x,y) := \{f \in C^\infty(K,M) \mid f(x) = y\}$ for some fixed $x \in K$, $y \in M$ is a split submanifold of $C^\infty(K,M)$.

(c) Is the set $\bigcap_{1 \leq i \leq n} S(x_i, y_i)$ also a submanifold of $C^\infty(K,M)$ if we pick points $x_i \in K$, $y_i \in M$ for $1 \leq i \leq n$ and $n \in \mathbb{N}$?

2.3.4 Consider a compact manifold K and M a manifold with a local addition. We endow $C^\infty(K,M)$ with the canonical manifold structure induced by the local addition; see Appendix C.3. Compute the tangent map of the evaluation map

$$\mathrm{ev}\colon C^\infty(K,M) \times K \to M, \quad (\varphi, m) \mapsto \varphi(m).$$

Hint: Apply the rule on partial differentials, Exercise 1.6.3. To compute the derivative for the variable in $C^\infty(K,M)$, exploit the fact that $TC^\infty(K,M) \cong C^\infty(K,TM)$; see C.12. After choosing a smooth curve $c\colon]-\varepsilon,\varepsilon[\to C^\infty(K,M)$, apply the exponential law to carry out the computation.

2.3.5 Assume that K, L are compact and M admits a local addition. Compute a formula for the tangent map of the smooth map (see Proposition 2.23)

$$\mathrm{Comp}\colon C^\infty(L,M) \times C^\infty(K,L) \to C^\infty(K,M), \quad (g,f) \mapsto g \circ f.$$

2.3.6 Use the identification $TC^\infty(M,N) \ni [t \mapsto c] \mapsto (x \mapsto \frac{\partial}{\partial t} c^\wedge(t,x)) \in C^\infty(M,TN)$ to establish the commutativity of the diagram (2.6).

2.3.7 Let K be a compact manifold and $p\colon M \to N$ a submersion between finite-dimensional paracompact manifolds. Establish the Stacey–Roberts Lemma by showing that the pushforward $p_*\colon C^\infty(K,M) \to C^\infty(K,N)$ becomes a submersion.

Hint: This is an involved exercise in *finite-dimensional* geometry which should only be attempted if one is familiar with Riemannian exponential maps, parallel transport and horizontal distributions. The

[2] It is also possible to prove that the evaluation map $\mathrm{ev}\colon C^\infty(K,M) \times K \to M$, $(\gamma, x) \mapsto \gamma(x)$ is a submersion; see Schmeding and Wockel (2016, Corollary 2.9).

idea is to construct a horizontal distribution $\mathcal{H}$ together with local additions η_M, η_N (constructed from suitable Riemannian exponential maps) such that the following diagram commutes:

$$\begin{array}{ccccc} TM = \mathcal{V} \oplus \mathcal{H} & \xleftarrow[\supseteq]{} & \Omega_M & \xrightarrow{\eta_M} & X \\ \Big\downarrow{\scriptstyle 0\oplus Tp|_{\mathcal{H}}} & & & & \Big\downarrow{\scriptstyle p} \\ TN & \xleftarrow[\supseteq]{} & \Omega_N & \xrightarrow{\eta_N} & M. \end{array}$$

Using these local additions, the canonical charts of the manifold of mappings become submersion charts.