6
Gauge fields

What is a gauge theory? This question may have more answers than there are physicists. In this discursive chapter we digress into discussing a few general definitions of a gauge theory and, in particular, a non-Abelian theory.

At the simplest level, a non-Abelian gauge theory is merely an embellishment of electromagnetism with an internal symmetry. Electromagnetic fields form the components of an antisymmetric tensor which is a four-dimensional curl of a vector potential

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{6.1}$$

Yang and Mills (1954) proposed adding an isospin index to A_μ and $F_{\mu\nu}$

$$A_\mu \to A_\mu^\alpha, \tag{6.2}$$

$$F_{\mu\nu} \to F_{\mu\nu}^\alpha. \tag{6.3}$$

This trivial modification becomes not so trivial with the addition of a further antisymmetric piece to $F_{\mu\nu}$

$$F_{\mu\nu}^\alpha = \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + g_0 f^{\alpha\beta\gamma} A_\mu^\beta A_\nu^\gamma. \tag{6.4}$$

Here g_0 is the bare gauge coupling constant, and $f^{\alpha\beta\gamma}$ are the structure constants for some continuous group G.

We consider here only unitary groups. An element g of G is a matrix in the fundamental or defining representation. We parametrize the elements of G using a set of generators

$$g = e^{i\omega^\alpha \lambda^\alpha}. \tag{6.5}$$

Here the ω^α are parameters and the λ^α are a set of Hermitian matrices which generate the group. The structure constants are defined from the commutation relations

$$[\lambda^\alpha, \lambda^\beta] = i f^{\alpha\beta\gamma} \lambda^\gamma. \tag{6.6}$$

The generators are conventionally orthonormalized such that

$$\mathrm{Tr}\,(\lambda^\alpha \lambda^\beta) = \tfrac{1}{2}\delta^{\alpha\beta}. \tag{6.7}$$

The simplest non-Abelian theory uses the group $SU(2)$ which is generated by the Pauli matrices (eqs 5.10–12)

$$\lambda^\alpha = \tfrac{1}{2}\sigma^\alpha, \tag{6.8}$$

$$f^{\alpha\beta\gamma} = \epsilon^{\alpha\beta\gamma}. \tag{6.9}$$

Maxwell's equations for electrodynamics follow from the Lagrangian density

$$\mathscr{L} = \tfrac{1}{4}F_{\mu\nu}F_{\mu\nu} + j_\mu A_\mu. \tag{6.10}$$

Here j_μ represents an external source for the electromagnetic field. The non-Abelian theory begins with the same Lagrangian, except for an assumed sum over a suppressed isospin index and $F_{\mu\nu}$ includes the extra term in eq. (6.4). The classical equation of motion for electromagnetism

$$\partial_\mu F_{\mu\nu} = j_\nu, \tag{6.11}$$

picks up an extra piece in the non-Abelian theory and becomes

$$(D_\mu F_{\mu\nu})^\alpha = j_\nu^\alpha. \tag{6.12}$$

Here the 'covariant derivative' is defined

$$(D_\mu F_{\mu\nu})^\alpha = \partial_\mu F_{\mu\nu}^\alpha + g_0 f^{\alpha\beta\gamma} A_\mu^\beta F_{\mu\nu}^\gamma. \tag{6.13}$$

The motivation for this definition will become clear when we discuss gauge transformations. The antisymmetry of $F_{\mu\nu}$ requires that the source satisfy

$$(D_\mu j_\mu)^\alpha = 0. \tag{6.14}$$

This is the non-Abelian analog of current conservation.

A convenient notation follows from using the group generators to define a matrix potential

$$A_\mu = A_\mu^\alpha \lambda^\alpha. \tag{6.15}$$

Using eq. (6.7) we can invert this relation

$$A_\mu^\alpha = 2\,\mathrm{Tr}\,(\lambda^\alpha A_\mu). \tag{6.16}$$

Similarly we define matrices for $F_{\mu\nu}$ and j_μ. The expression for $F_{\mu\nu}$ in terms of A_μ takes the simple form

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i} g_0 [A_\mu, A_\nu]. \tag{6.17}$$

In this notation the Lagrangian density becomes

$$\mathscr{L} = \tfrac{1}{2}\mathrm{Tr}\,(F_{\mu\nu}F_{\mu\nu}) + 2\,\mathrm{Tr}\,(j_\mu A_\mu). \tag{6.18}$$

We now turn to a second and probably the most popular definition of a gauge theory as a system possessing a local symmetry. Modification of the fields in a local region of space-time can leave the action unchanged. For electromagnetism this is the usual gauge symmetry under

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda, \tag{6.19}$$

where the gauge function Λ is an arbitrary function of the space-time coordinates. In the non-Abelian case, a gauge transformation is specified by a mapping of space into the gauge group. We associate a group element $g(x)$ with each space-time point. In matrix notation, A_μ transforms as

$$A_\mu \rightarrow g^{-1} A_\mu g + (\mathrm{i}/g_0)\, g^{-1} \partial_\mu g. \tag{6.20}$$

To recover the electrodynamic transformation of eq. (6.19), consider $g(x)$

to be a simple phase
$$g(x) = \mathrm{e}^{-\mathrm{i}g_0\Lambda(x)}. \tag{6.21}$$
Thus we can regard electromagnetism as a $U(1)$ gauge theory. Under eq. (6.20), $F_{\mu\nu}$ transforms particularly simply
$$F_{\mu\nu} \rightarrow g^{-1} F_{\mu\nu}\, g. \tag{6.22}$$

The covariant derivative of eq. (6.13) can be generalized to act on any field transforming under the gauge transformation as some representation of the gauge group. Suppose the field ϕ_i transforms as
$$\phi_i \rightarrow R_{ij}(g)\,\phi_j. \tag{6.23}$$
Here the matrices R_{ij} satisfy the representation property
$$R_{ij}(g)\,R_{jk}(g') = R_{ik}(gg'). \tag{6.24}$$
For example, the field $F^{\alpha}_{\mu\nu}$ transforms as the adjoint representation
$$F^{\alpha}_{\mu\nu} \rightarrow R^{\alpha\beta}(g)\,F^{\beta}_{\mu\nu}, \tag{6.25}$$
where
$$g^{-1}\lambda^{\alpha} g = R^{\alpha\beta}(g)\,\lambda^{\beta}. \tag{6.26}$$
Denote the generating matrices for the representation R by v^{α}_{ij} such that
$$R_{ij}(\mathrm{e}^{\mathrm{i}\omega^{\alpha}\lambda^{\alpha}}) = (\mathrm{e}^{\mathrm{i}\omega^{\alpha}v^{\alpha}})_{ij}. \tag{6.27}$$
These generators satisfy an analog of eq. (6.6)
$$[v^{\alpha}, v^{\beta}] = \mathrm{i} f^{\alpha\beta\gamma} v^{\gamma}. \tag{6.28}$$
We now define the covariant derivative of ϕ_i
$$(D_{\mu}\phi)_i = \partial_{\mu}\phi_i + \mathrm{i}g_0 A^{\alpha}_{\mu} v^{\alpha}_{ij}\phi_j. \tag{6.29}$$
The motivation for this definition is the simple gauge transformation property
$$(D_{\mu}\phi)_i \rightarrow R_{ij}(g)\,(D_{\mu}\phi)_j. \tag{6.30}$$
Note that for the equation of motion eq. (6.12) to remain simple under a gauge change, we must require that our source transform with the adjoint representation
$$j^{\alpha}_{\mu} \rightarrow R^{\alpha\beta}(g)\,j^{\beta}_{\mu} \tag{6.31}$$
or, in matrix notation
$$j_{\mu} \rightarrow g^{-1} j_{\mu}\, g. \tag{6.32}$$

We now turn to a third definition of a gauge theory as a theory of phases. Mandelstam (1962) and Yang (1975) have emphasized that the interaction of a particle with a gauge field involves a phase factor associated with any possible world line that the particle might traverse. In a non-Abelian theory, these path-dependent phase factors become matrices in the gauge group. Whenever a material particle traverses some contour in space-time, its wave function acquires a factor from electromagnetic interactions
$$\psi \rightarrow \psi \exp\left(\mathrm{i}g_0 \int_P A_{\mu}\,\mathrm{d}x_{\mu}\right) = U(P)\,\psi, \tag{6.33}$$

where the integral is along the path in question. This factor is particularly simple for a particle at rest

$$U(P) = \exp(\mathrm{i}g_0 A_0 t), \tag{6.34}$$

where t is the total time length of the path. The particle picks up an extra time oscillation at a rate proportional to its charge and the scalar potential. Thus its energy is increased by the scalar potential. Equation (6.33) generalizes this concept to any Lorentz frame.

For a non-Abelian theory we associate an element of the gauge group with any path. Consider a path

$$x_\mu(s), \quad s \in [0, 1], \tag{6.35}$$

where s represents some parametrization of the points along the path in question. We define a group element for the portion of the path from $x_\mu(0)$ to $x_\mu(s)$ via the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}s} U(s) = \frac{\mathrm{d}x_\mu}{\mathrm{d}s} \mathrm{i}g_0 A_\mu U(s). \tag{6.36}$$

For an initial condition we take

$$U(0) = 1. \tag{6.37}$$

In eq. (6.36) A_μ is a matrix in the sense of eq. (6.15). We can formally solve this system of equations

$$U(s) = \mathrm{P.O.}\left(\exp\left(\mathrm{i}g_0 \int_0^s \mathrm{d}s \frac{\mathrm{d}x_\mu}{\mathrm{d}s} A_\mu\right)\right), \tag{6.38}$$

where P.O. represents a 'path-ordering' instruction for the non-commuting matrices A_μ. In a power series expansion of the exponential, the matrices are to be ordered as encountered along the path, the largest values of the parameter s being to the left. That the matrix $U(s)$ remains in the gauge group follows because it is a product of group elements associated with infinitesimal pieces of the contour.

Under the local gauge transformation of eq. (6.20), this path-ordered exponential is only sensitive to the gauge function at the endpoints of the path

$$U(s) \to g^{-1}(x_\mu(s))\, U(s)\, g(x_\mu(0)). \tag{6.39}$$

Consider the case where the path is a closed contour C. The trace of the group element corresponding to such a contour

$$W(C) = \mathrm{Tr}\,(U(C)) \tag{6.40}$$

is independent of the starting point on the contour and is invariant under gauge changes. This is the Wilson loop operator and plays a key role in later chapters. The trace in this definition can be replaced by the character in any representation of the gauge group; however, unless otherwise specified, we use the fundamental defining representation.

https://doi.org/10.1017/9781009290395.007 Published online by Cambridge University Press

We end this chapter with a brief mention of yet another definition of a gauge theory. In a canonical Hamiltonian formalism one would like to write particle interactions in terms of operators involving local fields. Furthermore, discussions of Lorentz invariance are facilitated if these fields transform homogeneously under change of Lorentz frame. A gauge theory is one for which this is impossible (Weinberg, 1965). The interaction Hamiltonian necessarily involves the vector potential A_μ. A Lorentz transformation will in general change the gauge, in which case A_μ transforms inhomogeneously. Covariant gauges such as $\partial_\mu A_\mu = 0$ circumvent this problem but only at the expense of an indefinite metric quantum mechanical space.

That a description with local interactions requires the introduction of potentials is made clear in the Aharonov–Bohm (1959) experiment. A further consequence is the peculiar counting of degrees of freedom with a gauge particle. The potential A_μ in electrodynamics has four components, yet the photon has only two physical polarizations. The longitudinal component is unphysical in that its value depends on gauge choice. The second extra degree of freedom disappears because the time component A_0 is not dynamical. None of the equations of motion involve the time derivative of A_0 and thus its value is a function of the other variables. Elimination of A_0, however, generally introduces non-local objects. Indeed, Mandelstam (1962) has presented a non-local formulation of gauge theory without using potentials, but using the path-ordered integrals discussed above.

A lattice formulation rather severely mutilates Lorentz invariance at the outset. Thus this final definition of a gauge theory is not particularly useful here. The existence of unphysical degrees of freedom does persist on the lattice. We will return to this counting when we discuss the Hamiltonian formulation of lattice gauge theory.

Problems

1. Show that the structure constants $f^{\alpha\beta\gamma}$ defined in equation (6.6) are totally antisymmetric.

2. Verify the gauge transformation property of equation (6.39).

3. What are the generators v^α for the adjoint representation defined in eq. (6.26)?

4. Calculate a rectangular Wilson loop for the field theory of free photons. Using any convenient regulator, show how the leading divergence scales with the loop perimeter. Show that the ratio of two such loops with the same perimeter and number of corners is finite as the cutoff is removed.