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ON INDECOMPOSABLE DECOMPOSITIONS OF CS-MODULES

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Abstract

It is shown that, over any ring R, the direct sum $M = \oplus_{i\in I} M_i$ of uniform right R-modules M_i with local endomorphism rings is a CS-module if and only if every uniform submodule of M is essential in a direct summand of M and there does not exist an infinite sequence of non-isomorphic monomorphisms $M_{i_1} \xrightarrow{f_1} M_{i_2} \xrightarrow{f_2} \cdots \rightarrow M_{i_n} \xrightarrow{f_n} \cdots$, with distinct $i_n \in I$. As a consequence, any CS-module which is a direct sum of submodules with local endomorphism rings has the exchange property.

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1. Introduction

A module M is called a CS-*module* (or extending module) if every submodule of M is essential in a direct summand of M. CS-modules have been studied extensively in recent years (see for example [2, 3, 5, 7, 8, 12, 14, 15, 17, 18]), and it appears that several classical theorems on injective modules have natural generalizations for CS-modules. However, in some sense, the CS property is quite far from injectivity and several questions on CS-modules still remain unsolved. An interesting open question is to find necessary and sufficient conditions for a CS-module to have an indecomposable decomposition (see [14, Open problem 8, p. 106]). A very closely related question is to determine when a direct sum of indecomposable modules is a CS-module. The purpose of this paper is to settle this latter question in the case when each indecomposable summand has a local endomorphism ring.

More precisely, we prove that, over any ring R, the direct sum $M = \oplus_{i\in I} M_i$ of uniform right R-modules M_i with local endomorphism rings is a CS-module if and only if every uniform submodule of M is essential in a direct summand of M and there does not exist an infinite sequence $M_{i_1} \xrightarrow{f_1} M_{i_2} \xrightarrow{f_2} \cdots \rightarrow M_{i_n} \xrightarrow{f_n} \cdots$ of non-

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isomorphic monomorphisms f_n, with distinct $i_n \in I$. As a consequence, we deduce that if M is a CS-module which has an indecomposable decomposition $M = \oplus_{i\in I} M_i$, where each M_i has a local endomorphism ring, then this decomposition complements direct summands and M has the exchange property. A special case of this corollary was essentially observed earlier in Kamal and Müller [12, Theorem 12], where the ring was assumed to be right Noetherian.

There is another motivation for the paper. A well-known open problem asks whether a (left and right) perfect right self-injective ring R must be quasi-Frobenius, or briefly QF (if R is also left self-injective, the answer is "yes" by Kato [13] and Osofsky [16]). To prove that such R is QF is equivalent to showing that each projective right R-module is injective, that is any infinite direct sums of indecomposable injective projective right R-modules are also injective. This was the approach in Clark and Huynh [4] where it was proved that a semiperfect right self-injective ring R is QF if and only if every uniform submodule of any projective right R-module is contained in a finitely generated submodule. We will show that this characterization of QF-rings can be deduced as an immediate consequence of our results on CS-modules, but our method of proof is quite different.

2. Definitions and preliminaries

Throughout this paper, all rings are associative with identity and all modules are unitary right modules.

A submodule N of a module M is said to be *essential* in M if $N \cap K \neq 0$ for every nonzero submodule K of M. A nonzero module M is called *uniform* if every nonzero submodule of M is essential in M. A submodule C of M is called a *complement submodule* in M provided C has no proper essential extensions in M.

Following [3], a module M is called a CS-*module* if every complement submodule of M is a direct summand of M, or equivalently, every submodule of M is essential in a direct summand of M. It is obvious that an indecomposable module is CS if and only if it is uniform.

A non-empty family $\{A_i \mid i \in I\}$ of submodules of a module M is called a *local direct summand* of M if $\sum_{i\in I} A_i$ is direct and $\sum_{i\in F} A_i$ is a direct summand of M for any finite subset $F \subseteq I$. If, furthermore, $\sum_{i\in I} A_i$ is a direct summand of M, then we say that the local direct summand $\{A_i \mid i \in I\}$ is a *direct summand* of M. A family of modules $\{M_i \mid i \in I\}$ is called *locally semi-T-nilpotent* if, for any countable set of non-isomorphisms $\{f_n : M_{i_n} \to M_{i_{n+1}}\}$ with all i_n distinct in I, and for any $x \in M_{i_1}$, there exists k (depending on x) such that $f_k \cdots f_1(x) = 0$.

A module M is said to have the *exchange property* (Crawley and Jónsson [6]) if for any index set I, whenever $M \oplus N = \oplus_{i\in I} A_i$ for modules N and A_i, then

$M \oplus N = M \oplus (\oplus_{i\in I} B_i)$ for submodules $B_i \subseteq A_i$. A decomposition $M = \oplus_{i\in I} M_i$ is said to *complement direct summands* (Anderson and Fuller [1]) if, for any direct summand A of M, there exists a subset $J \subseteq I$ such that $M = A \oplus (\oplus_{i\in J} M_i)$.

For the reader's convenience, we list now a number of known results which will be used in the next section.

LEMMA 2.1. *An indecomposable module M has the exchange property if and only if* End M *is local.*

PROOF. See Warfield [19, Proposition 1].

LEMMA 2.2. *Let $M = \oplus_{i\in I} M_i$ be a direct sum of modules with local endomorphism rings. Then the following are equivalent:*

(a) $S =$ End M *is a semi-regular ring, that is $S/J(S)$ is von Neumann regular and the idempotents in $S/J(S)$ can be lifted over $J(S)$, where $J(S)$ is the Jacobson radical of S;*
(b) *Every local direct summand of M is a direct summand;*
(c) *The decomposition $M = \oplus_{i\in I} M_i$ complements direct summands;*
(d) *The family $\{M_i \mid i \in I\}$ is locally semi-T-nilpotent;*
(e) *M has the exchange property;*

PROOF. The equivalence of (a), (b), (c) and (d) is due to Harada [11]. The equivalence of (d) and (e) is due to Zimmermann-Huisgen and Zimmermann [22].

Let R be a ring and M a right R-module. For each subset X of M, we denote the *annihilator* of X in R by

$$r_R(X) = \{r \in R \mid xr = 0 \quad \text{for all} \quad x \in X\}.$$

When there is no ambiguity, we write $r(X)$ instead of $r_R(X)$.

The chain condition in the next lemma appeared as the condition (A_2) in Mohamed and Müller [14, p.4].

LEMMA 2.3. *Let $\{M_i \mid i \in I\}$ be right R-modules. Then $\oplus_{i\in I} M_i$ is quasi-injective if and only if M_i is M_j-injective for all i, $j \in I$ and for every choice of $x_n \in M_{i_n}$, with distinct $i_n \in I$, such that $\bigcap_{n=1}^{\infty} r_R(x_n) \supseteq r_R(y)$ for some $y \in M_j$ $(j \in I)$, the ascending sequence $\bigcap_{k=n}^{\infty} r_R(x_k) (n \in \mathbb{N})$ becomes stationary.*

PROOF. See Mohamed and Müller [14, Proposition 1.18]

3. CS-modules which are direct sums of modules with local endomorphism rings

Our primary aim in this section is to establish necessary and sufficient conditions for a direct sum of modules with local endomorphism rings to be a CS-module. Since each indecomposable summand of a CS-module is uniform, it is enough to consider the direct sums of uniform modules with local endomorphism rings. The proof of our main theorem will be based on a number of preparatory lemmas.

LEMMA 3.1. *Let $M = \oplus_{i\in I} M_i$ be a direct sum of uniform modules with local endomorphism rings such that every uniform submodule of M is essential in a direct summand of M. Let j be an index in I, A a nonzero submodule of M_j and $f : A \to \oplus_{i\in I\setminus j} M_i$ a non-monomorphic homomorphism. Then f can be extended to a homomorphism $f^* : M_j \to \oplus_{i\in I\setminus j} M_i$.*

PROOF. Define $A^* = \{a - f(a) \mid a \in A\}$. Then A^* is a submodule of M and clearly $A^* \cap (\oplus_{i\in I\setminus j} M_i) = 0$. Also, since $A^* \approx A$, A^* is uniform and so by hypothesis A^* is essential in a direct summand D of M. Then D is also uniform and, by [1, Theorem 12.6], D is isomorphic to some M_k ($k \in I$). Hence D has a local endomorphism ring, and it follows by Lemma 2.1 that D has the exchange property. There are submodules N_i of M_i such that $D \oplus (\oplus_{i\in I} N_i) = M$. Then, for each i, N_i is a direct summand of M_i, and since M_i is uniform, we get $N_i = M_i$ or $N_i = 0$. Hence it follows easily that there exists an index $t \in I$ such that $D \oplus (\oplus_{i\in I\setminus t} M_i) = \oplus_{i\in I} M_i$. Suppose that $t \neq j$. Then we have $M = D \oplus M_j \oplus (\oplus_{I\setminus t\cup j} M_i)$. So, in particular, we have $D \cap M_j = 0$. This implies that $A^* \cap M_j = 0$, and since M_j is uniform, $A^* \cap A = 0$. This means that $f(a) \neq 0$ for each nonzero $a \in A$, that is f is a monomorphism, a contradiction. Thus we have $t = j$, hence $M = D \oplus (\oplus_{i\in I\setminus j} M_i)$. Let p be the projection $D \oplus (\oplus_{i\in I\setminus j} M_i) \to \oplus_{i\in I\setminus j} M_i$, and we denote by f^* the restriction of p to M_j. Then clearly f^* extends $f : A \to \oplus_{i\in I\setminus j} M_i$ which proves the lemma.

LEMMA 3.2. *Let R be any ring and $M = \oplus_{i\in I} M_i$ a direct sum of uniform right R-modules with local endomorphism rings. Assume further that every uniform submodule of M is essential in a direct summand of M. Then for any choice of elements $x_n \in M_{i_n}$, with distinct $i_n \in I$, such that $\cap_{n=1}^{\infty} r_R(x_n) \supseteq r_R(y)$ for some $y \in M_j$, $j \in I$, the ascending sequence $\cap_{n=m}^{\infty} r_R(x_n)$, $m = 1, 2, 3, \ldots$ becomes stationary.*

PROOF. Without loss of generality, we may assume for simplicity that $I = \{0, 1, 2, \ldots\}$ and $M = \oplus_{i=0}^{\infty} M_i$. We shall write $r(x)$ instead of $r_R(x)$, for $x \in M$. Choose any $x_i \in M_i$, $i = 0, 1, 2, \ldots$, such that $r(x_0) \subseteq r(x_i)$ for all $i = 1, 2, \ldots$. We have to show that the sequence $K_n = \cap_{i=n}^{\infty} r(x_i)$, $n = 1, 2, \ldots$, becomes stationary.

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Let m be the first natural number such that $r(x_0) \neq K_m = \cap_{i=m}^{\infty} r(x_i)$. Then we have

$$r(x_0) = \cap_{i=1}^{\infty} r(x_i) = \cdots = \cap_{i=m-1}^{\infty} r(x_i) \subsetneqq \cap_{i=m}^{\infty} r(x_i) \subseteq \cdots .$$

Then, instead of $x_0, x_1, x_2, \ldots$, we shall consider the subsequence $x_0, x_m, x_{m+1}, \ldots$ Thus, without loss of generality, we may assume that $m = 1$, that is $r(x_0) \neq \cap_{i=1}^{\infty} r(x_i)$.

We define a homomorphism

$$\varphi : x_0 R \longrightarrow \prod_{i=1}^{\infty} M_i$$

by the following rule: for $t \in R$, $\varphi(x_0 t) = (x_1 t, x_2 t, \ldots)$. If $x_0 t = x_0 t'$, then $x_i t = x_i t'$ for all $i \geq 1$, hence φ is a well-defined homomorphism.

Let $I = \cup_{n=1}^{\infty} K_n = \cup_{n=1}^{\infty}(\cap_{i=n}^{\infty} r(x_i))$; then I is a right ideal of R and $x_0 I$ is a submodule of $x_0 R$. Let $\bar{\varphi}$ be the restriction of φ to $x_0 I$. Take any $t \in I$; then $t \in K_n$ for some $n \geq 1$, hence $x_i t = 0$ for each $i \geq n$. Thus

$$\bar{\varphi}(x_0 t) = (x_1 t, \ldots, x_{n-1} t, 0, 0, 0, \ldots) \in \oplus_{i=1}^{\infty} M_i,$$

so $\bar{\varphi}$ is a homomorphism from $x_0 I$ to $\oplus_{i=1}^{\infty} M_i$. Since $r(x_0) \neq \cap_{i=1}^{\infty} r(x_i)$ there exists a nonzero element $t' \in R$ such that $t' \in \cap_{i=1}^{\infty} r(x_i)$ but $t' \notin r(x_0)$. It follows that $x_0 t' \neq 0$ but $\bar{\varphi}(t') = 0$, hence $\bar{\varphi}$ is not a monomorphism. By Lemma 3.1, $\bar{\varphi}$ can be extended to a homomorphism $\psi : M_0 \longrightarrow \oplus_{i=1}^{\infty} M_i$. Let $\bar{\psi}$ be the restriction of ψ to $x_0 R$, we have

$$\varphi(x_0) I = \bar{\varphi}(x_0 I) = \bar{\psi}(x_0 I) \subseteq \bar{\psi}(x_0 R) = \bar{\psi}(x_0) R \subseteq \oplus_{i=1}^{m} M_i$$

for some $m \geq 1$. But $\varphi(x_0) I = (x_1, \ldots, x_n, \ldots) I$, so it follows that $x_{m+1} I = x_{m+2} I = \cdots = 0$. This means that $I \subseteq r(x_{m+1}) \cap r(x_{m+2}) \cap \cdots = K_{m+1}$. Therefore $I = K_{m+1}$ which completes the proof.

LEMMA 3.3. *Let $M = \oplus_{i \in I} M_i$ be a direct sum of uniform modules with local endomorphism rings, and assume that every uniform submodule of M is essential in a direct summand of M. Then for any infinite sequence of non-monomorphisms f_n*

$$M_{i_1} \xrightarrow{f_1} M_{i_2} \xrightarrow{f_2} \cdots \to M_{i_n} \xrightarrow{f_n} \cdots$$

with distinct $i_n \in I$, and for any nonzero element $x_1 \in M_{i_1}$, there exists $m \geq 1$ such that $f_m f_{m-1} \cdots f_1(x_1) = 0$.

PROOF. Take any nonzero element $x_1 \in M_{i_1}$ and put $x_n = f_{n-1} \cdots f_1(x_1)$, for $n \geq 2$. Then $x_n \in M_{i_n}$ and clearly $r(x_1) \subseteq r(x_2) \subseteq \cdots \subseteq r(x_n) \subseteq \cdots$. Since

$r(x_n) = \cap_{i=n}^{\infty} r(x_i)$, it follows by Lemma 3.2 that this ascending sequence must become stationary. Thus there exists $m \geq 1$ such that $r(x_m) = r(x_{m+1})$, that is $r(x_m) = r(f_m(x_m))$. This means that $x_m t = 0$ for $t \in R$ if and only if $f_m(x_m t) = 0$. Let $\bar{f}_m$ be the restriction of f_m to $x_m R$; then $\bar{f}_m$ is a monomorphism. Suppose that $x_m R \neq 0$. Then because M_m is uniform, it follows that f_m is also a monomorphism, a contradiction to the hypothesis. Therefore we have $x_m R = 0$, that is $x_m = f_{m-1} \cdots f_1(x_1) = 0$.

We are now ready to give necessary and sufficient conditions for a direct sum of modules with local endomorphism rings to be a CS-module.

THEOREM 3.4. *Let R be any ring and $M = \oplus M_i$ a direct sum of uniform right R-modules M_i with local endomorphism rings. Then the following conditions are equivalent:*

(a) *Every uniform submodule of M is essential in a direct summand of M, and there does not exist an infinite sequence of non-isomorphic monomorphisms f_n:*

$$M_{i_1} \xrightarrow{f_1} M_{i_2} \xrightarrow{f_2} \cdots \to M_{i_n} \xrightarrow{f_n} \cdots$$

with all i_n distinct in I;

(b) *M is a* CS-*module*;

PROOF. (a) implies (b). Suppose that (a) holds. First we show that the family $\{M_i \mid i \in I\}$ is locally semi-T-nilpotent. Take any countable sequence $i_1, i_2, i_3, \ldots$ of pairwise different elements of I and a sequence of non-isomorphisms f_n:

$$M_{i_1} \xrightarrow{f_1} M_{i_2} \xrightarrow{f_2} \cdots \to M_{i_n} \xrightarrow{f_n} \cdots$$

By hypothesis, there exist infinitely many f_n's which are non-monomorphisms, say $f_{n(1)}, f_{n(2)}, \ldots, f_{n(k)}, \ldots$ with $n(1) < n(2) < \cdots < n(k) < \cdots$. Then we get an infinite sequence of non-monomorphisms $f_{n(k+1)-1} \cdots f_{n(k)} : M_{i_{n(k)}} \to M_{i_{n(k+1)}}$. Thus, by applying Lemma 3.3, we get easily that for any nonzero element $x \in M_{i_1}$, there exists $m \geq 1$ such that $f_m \cdots f_1(x_1) = 0$. Therefore, the family $\{M_i \mid i \in I\}$ is locally semi-T-nilpotent. By Lemma 2.2, every local direct summand of M is also a direct summand.

Now let A be any nonzero complement submodule of M. Any cyclic submodule of M has finite uniform dimension, so in particular A contains a nonzero uniform submodule U. Let U' be a maximal essential extension of U in A, then U' is uniform and a complement submodule of A. It follows by [3, Proposition 2.2] that U' is also a complement submodule of M. By Zorn's lemma, there exists a maximal local direct summand $\{V_\alpha \mid \alpha \in \Omega\}$ in A, where each V_α is an uniform summand of M. As we have shown above, $V = \oplus_{\alpha \in \Omega} V_\alpha$ is also a direct summand of M. Let $M = V \oplus M'$

for some submodule M', then $A = V \oplus (M' \cap A)$. If $M' \cap A \neq 0$, then since $M' \cap A$ is also a complement submodule of M by [3, Proposition 2.2], $M' \cap A$ contains a nonzero uniform summand of M which contradicts the maximality of the local direct summand $V = \oplus_{\alpha \in \Omega} V_\alpha$. Hence $M' \cap A = 0$, that is $A = V$ is a direct summand of M. This proves that M is a CS-module.

(b) implies (a). Suppose that M is a CS-module. That every uniform submodule of M is essential in a direct summand is obvious by the definition of CS-modules. The latter part of (a) was already proved in Baba-Harada [2, Proposition 3], by using the factor categories technique (cf. [10, 11]). We include here a direct module-theoretic proof using an idea due to R. Wisbauer. Suppose that there exists an infinite sequence of non-isomorphic monomorphisms f_n:

$$M_{i_1} \xrightarrow{f_1} M_{i_2} \xrightarrow{f_2} \cdots \to M_{i_n} \xrightarrow{f_n} \cdots$$

with distinct i_n. For simplicity we may assume that $I = \{1, 2, 3, \dots\}$ and write $i_n = n$. Define $M_n^* = \{x_n - f_n(x_n) \mid x_n \in M_n\}$. Then it is easy to check that $\sum_{i=1}^{\infty} M_i^*$ is direct and $(\oplus_{i=1}^n M_i^*) \oplus M_{n+1} = \oplus_{i=1}^{n+1} M_i$ for each n. Hence $M^* = \oplus_{i=1}^{\infty} M_i^*$ is a local direct summand of M. Since M is a CS-module, M^* is essential in a direct summand N of M and we have $M = N \oplus L$. Assume that $L \neq 0$. By [1, Theorem 12.6 (1)], L contains a nonzero indecomposable direct summand X. Let $L = Y \oplus X$, then $M = N \oplus Y \oplus X$. Since $N \oplus Y$ is a maximal direct summand in the sence of [1], by [1, Theorem 12.6 (2)] we have $M = N \oplus Y \oplus M_k$ for some M_k. This implies that $M_k \oplus M_k^*$ is a direct summand of M. However, it is obvious that $M_k \oplus M_k^*$ is essential in $M_k \oplus M_{k+1}$. Thus $M_k \oplus M_k^* = M_k \oplus M_{k+1}$ implying that f_k is epimorphic, hence an isomorphism, a contradiction. Therefore $L = 0$, that is M^* is essential in M. In particular, we have $M^* \cap M_1 \neq 0$. Take any nonzero element $x_1 \in M_1 \cap M^*$. Then

$$x_1 = y_1 - f_1(y_1) + y_2 - f_2(y_2) + \cdots + y_n - f_n(y_n),$$

where $y_i \in M_i$, $1 \leq i \leq n$. It follows that $y_1 = x_1$, $y_i = f_{i-1}(y_{i-1})$ for $2 \leq i \leq n$ and $f_n(y_n) = 0$. Hence $f_n f_{n-1} \cdots f_1(x_1) = 0$ which is a contradiction because all $f_1, \dots, f_n$ are monomorphisms.

COROLLARY 3.5. *Let R be any ring and $M = \oplus_{i \in I} M_i$ a direct sum of right R-modules M_i with local endomorphism rings. Assume that M is a* CS-*module. Then this decomposition complements direct summands and M has the exchange property. Moreover, in this case $S =$* End *M is a semi-regular ring.*

PROOF. By the proof of Theorem 3.4 ((a) implies (b)), we have that if the CS-module M has the decomposition $M = \oplus_{i \in I} M_i$, where each M_i has a local endomorphism ring, then the family $\{M_i \mid i \in I\}$ is locally semi-T-nilpotent. Now the result follows from Lemma 2.2.

The result below is an easy consequence of Lemma 3.2.

COROLLARY 3.6. *Let* $M = \oplus_{i\in I} M_i$ *be a direct sum of indecomposable quasi-injective modules* M_i. *Then the following conditions are equivalent:*

(a) M *is quasi-injective.*

(b) M_i *is* M_j*-injective for all* $i \neq j$ *in* I, *and every uniform submodule of* M *is essential in a direct summand of* M.

PROOF. (a) implies (b). This is immediate by the well-known facts that if A and B are modules with $A \oplus B$ quasi-injective, then A is B-injective and B is A-injective (see for example [14, Proposition 1.17]), and that each quasi-injective module is CS.

(b) implies (a). Suppose that (b) holds. Note that each M_i is uniform with a local endomorphism ring. By Lemma 3.2, for any choice of elements $x_n \in M_{i_n}$, $n = 1, 2, \ldots$, with distinct $i_n \in I$, such that $\bigcap_{n=1}^{\infty} r(x_n) \supseteq r(y)$ for some $y \in M_j$, $j \in I$, the ascending sequence $\bigcap_{n=m}^{\infty} r(x_n)$, $m = 1, 2, 3, \ldots$, becomes stationary. Now it follows by Lemma 2.3 that M is quasi-injective.

The next corollary generalizes the Clark-Huynh result in [4] which was mentioned in the Introduction.

COROLLARY 3.7. *Let* R *be a semiperfect right self-injective ring. Then* R *is* QF *if and only if every uniform submodule of the free right* R*-module* $R^{(\mathbb{N})}$ *is contained in a finitely generated submodule of* $R^{(\mathbb{N})}$.

PROOF. If R is QF then $R^{(\mathbb{N})}$ is injective, and so every uniform submodule U of $R^{(\mathbb{N})}$ is contained in an injective envelope A of U in $R^{(\mathbb{N})}$. Since A is indecomposable projective, A must be cyclic by [1, Proposition 27.10]. Conversely, suppose that R is semiperfect right self-injective, and consider the free right R-module $R^{(\mathbb{N})}$. Then $R^{(\mathbb{N})}$ can be decomposed as $R^{(\mathbb{N})} = \oplus_{i\in I} A_i$, where each A_i is an indecomposable injective right R-module. Let U be any uniform submodule of $R^{(\mathbb{N})}$, then by hypothesis, there is a finite subset $F \subset I$ such that $U \subseteq \oplus_{i\in F} A_i$. But $\oplus_{i\in F} A_i$ is injective, hence U is essential in a direct summand of $\oplus_{i\in F} A_i$, thus of $R^{(\mathbb{N})}$. By Corollary 3.6, we have that $R^{(\mathbb{N})}$ is quasi-injective, so $R^{(\mathbb{N})}$ is $(R-)$ injective. Therefore R is a QF-ring (see for example [1, Theorems 25.1 and 31.9]).

As a further application of Theorem 3.4, we next give a criterion of when a ring R is right Σ-CS, that is any direct sum of copies of R_R is a CS-module. Right nonsingular right Σ-CS rings are precisely the right nonsingular rings over which nonsingular right modules are projective (see Goodearl [9, Chapter 5]). Right Σ-CS-rings in the general case were studied extensively by Oshiro (for example [15]) under the name right co-H-rings (in honor of Harada). Fore more recent results on Σ-CS rings and modules we refer to Clark and Wisbauer [5], Dung and Smith [7].

A module M is called *continuous* if M is a CS-module and every submodule of M which is isomorphic to a direct summand of M is also a direct summand of M (see [14]).

COROLLARY 3.8. *Let R be a ring with a finite set of orthogonal primitive idempotents $e_1, \dots, e_n$. Then the following conditions are equivalent:*

(a) *Each e_iR $(1 \leq i \leq n)$ is a continuous right R-module, and every uniform submodule of $R_R^{(\mathbb{N})}$ is essential in a direct summand of $R_R^{(\mathbb{N})}$;*
(b) *R is right* Σ-CS;

PROOF. (a) implies (b). Assume (a). Note that each e_iR is uniform and has a local endomorphism ring (see [14, Proposition 3.5]). Consider the free module $R_R^{(\mathbb{N})}$ which can be decomposed as $R_R^{(\mathbb{N})} = \oplus_{j\in J} M_j$, where each M_j is isomorphic to some e_iR, $(1 \leq i \leq n)$. Suppose that there exists an infinite sequence of non-isomorphic monomorphisms f_n

$$M_{j_1} \xrightarrow{f_1} M_{j_2} \xrightarrow{f_2} \cdots \to M_{j_n} \xrightarrow{f_n} \cdots .$$

Since among $\{M_j \mid j \in J\}$ there is only a finite number of non-isomorphic members, there are k and ℓ with $k < \ell$ such that $M_{j_k} \approx M_{j_\ell}$. Since M_{j_k} is indecomposable continuous, $f_{\ell-1} \dots f_k$ is a monomorphism $M_{j_k} \to M_{j_\ell}$, it follows that $f_{\ell-1} \dots f_k$ is an isomorphism, and this clearly implies that all $f_{\ell-1}, \dots, f_k$ are isomorphisms, a contradiction. Thus there does not exist an infinite sequence of non-isomorphic monomorphisms

$$M_{j_1} \xrightarrow{f_1} M_{j_2} \xrightarrow{f_2} \cdots \to M_{j_n} \xrightarrow{f_n} \cdots .$$

By Theorem 3.4, we have that $R_R^{(\mathbb{N})}$ is a CS-module. Now the result follows from the fact that if the ring R is semiperfect and $R_R^{(\mathbb{N})}$ is CS, then R is right Σ-CS (see Clark and Wisbauer [5] or Vanaja [18]).

(b) implies (a). This follows from Oshiro [15, Lemma 3.7] (cf. Clark and Wisbauer [5, Theorem 2.6] for an alternative proof).

REMARK. The proof of Corollary 3.8 yields also Theorem 1 ((a) implies (c)) in Clark and Huynh [4] which states that a ring R is QF if (and only if) R is semiperfect right continuous such that $R_R^{(n)}$ is CS for each $n \geq 1$ and every uniform submodule of any projective right R-module is contained in a finitely generated submodule. In fact, under these hypotheses, the above proof shows that every free right R-module is CS, that is R is right Σ-CS. Since R is right continuous, we have $J(R) = Z(R_R)$, where $J(R)$ and $Z(R_R)$ are the Jacobson radical and the singular right ideal of R, respectively. By Oshiro [15, Theorem 4.3], we deduce that R is QF.

In view of Corollary 3.8, below we present an example of a semiperfect Noetherian ring R such that every uniform submodule of any projective right R-module P is essential in a direct summand of P, but R is not semiprimary, so in particular R is not right Σ-CS (see [15, Theorem 3.18]).

EXAMPLE 3.9. Let R be a (left and right) hereditary serial Noetherian semiprime ring with zero left and right socles. For the existence of such rings R we refer to Warfield [20, Theorems 5.11 and 5.14]. Clearly R is semiperfect, but not semiprimary. By [20, Theorem 4.6], every finitely generated nonsingular right R-module is projective. Since R is right nonsingular, this means that every finitely generated projective right R-module is a CS-module. Now let P be any projective right R-module and A a uniform submodule of P. By [1, Theorem 27.11], we have $P = \oplus_{i \in I} P_i$, where each P_i is isomorphic to an indecomposable direct summand of R_R. It is easy to see that A can be imbedded into a finite direct sum $P_{i_1} \oplus \cdots \oplus P_{i_n}$. Since each P_{i_k} is Noetherian, it follows that A is Noetherian, in particular A is finitely generated. Thus A is contained in a finite direct sum $P_{j_1} \oplus \cdots \oplus P_{j_m}$ which is a CS-module. Therefore A is essential in a direct summand of $P_{j_1} \oplus \cdots \oplus P_{j_m}$, and hence of P.

We conclude the paper with a result which shows that, under certain chain conditions on the ring R, there is a rather large class of CS right R-modules which have decompositions into indecomposable summands with local endomorphism rings. The proof is inspired by some ideas in Clark and Wisbauer [5]. Recall that a module M is said to be *Σ-quasi-injective* if any direct sum of copies of M is also quasi-injective, and it is well known that M is Σ-quasi-injective if and only if $M^{(\mathbb{N})}$ is quasi-injective.

THEOREM 3.10. *Let R be a ring and M a finitely generated right R-module. Suppose that $M^{(\mathbb{N})}$ is a* CS-*module and R satisfies* ACC *on $r_R(X)$ for finite subsets X of M. Then M is a finite direct sum of indecomposable Σ-quasi-injective submodules.*

PROOF. Since M can be decomposed as a finite direct sum of uniform submodules (see for example [14, Theorem 2.17 and Proposition 2.18]), we may assume, without loss of generality, that M is uniform, and we have to show that M is Σ-quasi-injective.

Since R satisfies ACC on $r_R(X)$ for finite subsets X of M, it follows that R has ACC on $r_R(a)$ for elements $a \in M^{(\mathbb{N})}$. Since $M^{(\mathbb{N})}$ is CS, by [14, Proposition 2.18] we have that every local direct summand of $M^{(\mathbb{N})}$ is also a direct summand. Next we show that $S = \operatorname{End} M$ is local. Let $a_1, a_2, \ldots, a_n, \ldots$ be any sequence of elements in S and consider the descending chain of principal left ideals of S

$$Sa_1 \supset Sa_2a_1 \supset \cdots \supset Sa_na_{n-1}\cdots a_1 \supset \cdots .$$

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Let $M^{(\mathbb{N})} \approx \oplus_{n=1}^{\infty} M_n$ with $M_n \approx M$ for each $n \geq 1$. We may consider a_n as a homomorphism from M_n to M_{n+1}. Define $M_n^* = \{x_n - a_n(x_n) \mid x_n \in M_n\}$. As in the proof of Theorem 3.4 ((b) implies (a)), we have that $\oplus_{n=1}^{\infty} M_n^*$ is a local direct summand of $M^{(\mathbb{N})}$, hence $\oplus_{n=1}^{\infty} M_n^*$ is a direct summand of $M^{(\mathbb{N})}$. Since M is finitely generated, by Wisbauer [21, 43.3], there exist a positive integer m and an element $h \in S$ such that $a_{m-1} \cdots a_1 = h a_m a_{m-1} \cdots a_1$. This implies that $S a_{m-1} \cdots a_1 = S a_m a_{m-1} \cdots a_1$. It follows that S satisfies DCC on principal left ideals, that is S is a right perfect ring. But M is indecomposable, hence S contains no nontrivial idempotents, so S is a local ring.

By hypothesis, M is a CS-module, so by Theorem 3.4 ((b) implies (a)), every monomorphism $g : M \to M$ is also an isomorphism. Since End M is local and $M \oplus M$ is CS, the above fact and [8, Lemma 3] show that M is M-injective, that is M is quasi-injective. By Corollary 3.6, we have that $M^{(\mathbb{N})}$ is quasi-injective, hence M is Σ-quasi-injective which completes the proof.

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