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# COHERENT POWER SERIES RING AND WEAK GORENSTEIN GLOBAL DIMENSION

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**Abstract.** In this paper we compute the weak Gorenstein global dimension of a coherent power series ring. It is shown that the weak Gorenstein global dimension of $R[[x]]$ is equal to the weak Gorenstein global dimension of $R$ plus one, provided $R[[x]]$ is coherent.

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**1. Introduction.** All rings are commutative with identity throughout this paper. Let $R$ be a ring, and let $M$ be an $R$-module. As usual, we use $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$ and $\mathrm{fd}_R(M)$ to denote, respectively, the classical projective dimension, injective dimension and flat dimension of $M$.

The global dimension (or global homological dimension) of ring $R$, denoted $\mathrm{gldim} R$, is a non-negative integer or infinity which is a homological invariant of the ring. It is defined to be the supremum of the set of projective dimensions of all $R$-modules. Similarly, the supremum of the set of flat dimensions of all $R$-modules is called weak global dimension and denoted $\mathrm{wdim} R$. Jøndrup and Small in [**11**] gave a connection between a weak dimension of a coherent power series ring over a commutative ring $R$ and the weak dimension of $R$. Let $R$ be a ring, and let $x$ be an indeterminate over $R$. If $R[[x]]$ is a coherent ring, then Jøndrup and Small showed that $\mathrm{wdim}(R[[x]]) = \mathrm{wdim}(R) + 1$.

In this paper, we give an extension of this result to the Gorenstein weak dimension. More precisely, we prove that the weak Gorenstein global dimension of $R[[x]]$ is equal to the weak Gorenstein global dimension of $R$ plus one, provided $R[[x]]$ is coherent.

In the following we recall the definitions of the Gorenstein global dimension and the weak Gorenstein global dimension.

The Gorenstein projective dimension is a refinement of the classical notion of projective dimension of a module in the sense that it is always less than or equal to the projective dimension and the equality holds when projective dimension is finite. It was introduced by Enochs and Jenda in [**7**] as a generalization of the G-dimension defined by Auslander–Bridger [**1**] some 20 years earlier. In [**6**] the Gorenstein injective dimension was introduced as a dual notion of Gorenstein projective dimension. To complete the analogy with the classical homological dimension, Enochs et al. [**8**] introduced the Gorenstein flat dimension.

Recall that an $R$-module $M$ is called *Gorenstein projective* if there exists an exact sequence of projective $R$-modules:

$$\mathbf{P}: \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

such that $M \cong \mathrm{Im}(P_0 \to P^0)$ and the functor $\mathrm{Hom}_R(-, Q)$ leaves $\mathbf{P}$ exact whenever $Q$ is a projective $R$-module. The complex $\mathbf{P}$ is called a complete projective resolution. The Gorenstein injective $R$-modules are defined dually.

An $R$-module $M$ is called *Gorenstein flat* if there exists an exact sequence of flat $R$-modules:

$$\mathbf{F}: \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

such that $M \cong \mathrm{Im}(F_0 \to F^0)$ and the functor $I \otimes_R -$ leaves $\mathbf{F}$ exact whenever $I$ is an injective $R$-module. The complex $\mathbf{F}$ is called a *complete flat resolution*.

The Gorenstein projective, injective and flat dimensions are defined in terms of resolutions and denoted by $\mathrm{Gpd}(-)$, $\mathrm{Gid}(-)$ and $\mathrm{Gfd}(-)$ respectively.

In [**4**], the authors proved the following equality:

$$\sup\{\mathrm{Gpd}_R(M) \mid \text{M is an } R\text{-module}\} = \sup\{\mathrm{Gid}_R(M) \mid \text{M is an } R\text{-module}\}.$$

They called the common value of the above quantities the Gorenstein global dimension of $R$ and denoted it by $\mathrm{Ggldim}(R)$. Similarly, they set

$$\mathrm{wGgldim}(R) = \sup\{\mathrm{Gfd}_R(M) \mid \text{M is an } R\text{-module}\},$$

which they called Gorenstein weak dimension of $R$.

## 2. Main results.

The main results of this paper are stated as follows.

THEOREM 2.1. *Let $R$ be a ring and let $x$ be an indeterminate over $R$. If $R[[x]]$ is a coherent ring, then*

$$\mathrm{wGgldim}(R[[x]]) = \mathrm{wGgldim}(R) + 1.$$

*Proof.* First we show that $\mathrm{wGgldim}(R[[x]]) \geq \mathrm{wGgldim}(R) + 1$. The short exact sequence

$$0 \longrightarrow R[[x]] \xrightarrow{x} R[[x]] \longrightarrow R \longrightarrow 0$$

implies that $R \cong R[[x]]/xR[[x]]$. Since $R$ is finitely presented as $R[[x]]$-module, we have that $R$ is coherent (see [**9**, Theorem 4.1.1(1)].

If $\mathrm{wGgldim}(R[[x]]) = \infty$, then there is nothing to prove. Therefore, we assume that $\mathrm{wGgldim}(R[[x]]) = n < \infty$. Using [**9**, Theorem 1.3.3] and [**10**, Proposition 2.27], we have $\mathrm{Gpd}_{R[[x]]}(R) = \mathrm{pd}_{R[[x]]}(R) = 1$ and so by [**5**, Theorem 7] we have that

$\text{wGgldim}(R[[x]]) = n \geq 1$. Now let $M$ be a finitely presented $R$-module. By [**9**, Theorem 2.1.8], $M$ is a finitely presented $R[[x]]$-module. Thus, by [**9**, Theorem 1.3.5] and [**5**, Theorem 7], $\text{Ext}^n_R(M, R) = \text{Ext}^{n+1}_{R[[x]]}(M, R[[x]]) = 0$. Therefore, by [**5**, Theorem 7], $\text{wGgldim}(R) \leq n - 1$ and so $\text{wGgldim}(R) \leq \text{wGgldim}(R[[x]]) - 1$.

To show that $\text{wGgldim}(R[[x]]) \leq \text{wGgldim}(R) + 1$, we may assume $\text{wGgldim}(R) = n < \infty$. Let $M$ be a finitely presented $R[[x]]$-module and consider a short exact sequence of $R[[x]]$-modules

$$(*)\ 0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0\,,$$

where $P$ is a finitely generated projective $R[[x]]$-module. By [**9**, Theorem 2.5.1], $K$ is also finitely presented since $R[[x]]$ is coherent. If $M$ is projective, then $\text{Gpd}_{R[[x]]}(M) = \text{pd}_{R[[x]]}(M) = 0$. Otherwise, $K \neq 0$. Thus, by the Nakayama's lemma, we deduce that $K \neq xK$. Moreover, by [**9**, Theorem 2.1.8], $K/xK$ is a finitely presented $R$-module. On the other hand, from the short exact sequence $(*)$ we have $\text{Tor}_1^{R[[x]]}(K, R) = \text{Tor}_2^{R[[x]]}(M, R) = 0$ since $\text{fd}_{R[[x]]}(R) \leq 1$. $K$ is a submodule of a projective $R[[x]]$-module $P$, and $P$ is a submodule of a free $R[[x]]$-module. The element x is regular on any free $R[[x]]$-module, therefore regular on K. Thus, from [**3**, Theorem 5.1], $\text{Gpd}_{R[[x]]}(K) = \text{Gpd}_R(K/xK) \leq n$. Accordingly, by [**3**, Lemma 2.4], $\text{Gpd}_{R[[x]]}(M) \leq \text{Gpd}_{R[[x]]}(K) + 1 \leq n + 1$. Consequently, by [**5**, Theorem 7], $\text{wGgldim}(R[[x]]) \leq n + 1$. Thus, we have the desired equality. □

Recall that a ring $R$ is called an arithmetical ring if every finitely generated ideal is locally principal. If $\text{wdim}(R) \leq 1$, then $R$ is an arithmetical ring (see, for instance, [**2**]). So we lead to ask the following question: If $\text{wGgldim}(R) \leq 1$, then is $R$ arithmetical ring? In the following result we give a partial answer to this question. Recall that a ring is called quasi-Frobenius if it is Noetherian and self-injective.

Proposition 2.2. Let $R$ be a quasi-Frobenius ring which is not a field. Then the following statements hold:

(1) $\text{wGgldim}(R[[x]]) = \text{Ggldim}(R[[x]]) = 1$.
(2) If $R$ is local with maximal ideal $\mathfrak{m}$, then $R[[x]]$ is not an arithmetical ring.

*Proof.* (1) Using [**4**, Proposition 2.8 and Theorem 2.9], we have that $\text{wGgldim}(R) = \text{Ggldim}(R) = 0$. Thus, by Theorem 2.1, $\text{wGgldim}(R[[x]]) = 1$. On the other hand, $R$ is Noetherian and so $R[[x]]$ is also Noetherian. Therefore, by [**4**, Theorem 2.9], $\text{Ggldim}(R[[x]]) = 1$.

(2) Assume that $R[[x]]$ is an arithmetical ring. Let $a$ be a non-zero, non-invertible element of $R$ and let $I := aR[[x]] + xR[[x]]$. Since $R[[x]]$ is a local arithmetical ring, we have $I = PR[[x]]$ for some $P := \sum_i a_i x^i \in R[[x]]$ (where $a_i \in R$).

We have $P \in I = aR[[x]] + xR[[x]]$, so $P = aQ_1 + xQ_2$ for some $Q_1 (:= \sum_i c_i x^i)$, $Q_2 \in R[[x]]$. Hence, $a_0 = ac_0$. On the other hand, we have $a = PQ$ for some $Q = \sum_i b_i x^i \in R[[x]]$. Hence, $a = a_0 b_0$. We claim that $b_0 \in \mathfrak{m}$. If this is not the case, then $b_0$ is invertible in $R$ and so $Q$ is invertible in $R[[x]]$; hence, we may assume that $P = a$ (since $aR[[x]] = PQR[[x]] = PR[[x]] = I$). But, $x \in I = aR[[x]]$ implies that $x = a \sum_i d_i x^i$ for some $d_i \in R$ and so $1 = ad_1$. This implies that $a$ is invertible in $R$, which is a contradiction. Therefore, $b_0 \in m$. Since $a_0 = ac_0$ and $a = a_0 b_0$, we have that $a = ab_0 c_0$ and so $a(1 - b_0 c_0) = 0$. But $1 - b_0 c_0$ is invertible in $R$ since $b_0 c_0 \in M$. Thus, $a = 0$, a contradiction. Therefore, $R[[x]]$ is not an arithmetical ring as desired. □

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