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A NONLINEAR ERGODIC THEOREM FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Let X be a real uniformly convex Banach space satisfying the Opial's condition, C a bounded closed convex subset of X, and $T: C \to C$ an asymptotically nonexpansive mapping. Then we show that for each x in C, the sequence $\{T^n x\}$ almost converges weakly to a fixed point y of T, that is,

$$\text{weak-}\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} T^{i+k}x = y \quad \text{uniformly in } k \geqslant 0.$$

This implies that $\{T^n x\}$ converges weakly to y if and only if T is weakly asymptotically regular at x, that is, weak-$\lim_{n\to\infty}\left(T^{n+1}x - T^n x\right) = 0$. We also present a weak convergence theorem for asymptotically nonexpansive semigroups.

1. INTRODUCTION

Let C be a closed convex subset of a Banach space X and T be a mapping from C into itself. Then T is said to be a Lipschitzian mapping if there exists, for each integer $n \geqslant 1$, a corresponding real number $\lambda_n > 0$ such that

$$\|T^n x - T^n y\| \leqslant \lambda_n \|x - y\|$$

for all $x, y \in C$. A Lipschitzian mapping T is called nonexpansive if $\lambda_n = 1$ for all $n \geqslant 1$ and asymptotically nonexpansive if $\lim_{n\to\infty} \lambda_n = 1$, respectively. We denote by $F(T)$ the set of fixed points of T. The first nonlinear ergodic theorem for nonexpansive mappings was proved in 1975 by Baillon [1]: Let C be a bounded closed convex subset of a Hilbert space H and T be a nonexpansive mapping from C into itself. Then for each $x \in C$, the Cesaro means

$$S_n(x) = \frac{1}{n}\sum_{i=0}^{n-1} T^i x$$

converge weakly to some $y \in F(T)$. In 1979, Reich [13] and Bruck [2] independently generalised Baillon's theorem to a setting of a uniformly convex Banach space with a

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Frechet differentiable norm. (See Hirano [7] for another proof.) In 1982, Hirano [8] proved that the conclusion of Baillon's theorem is valid in a uniformly convex Banach space satisfying the Opial's condition. On the other hand, Hirano and Takahashi [9] proved that in a Hilbert space setting, Baillon's theorem holds true for asymptotically nonexpansive mappings. (This is in fact true [16] even for a wider class of mappings of asymptotically nonexpansive type [10].) However, whether Baillon's theorem is valid for asymptotically nonexpansive mappings in a Banach space setting remained open for a few years. Recently, the authors [14] have provided an affirmative answer to this question in a uniformly convex Banach space which has a Frechet differentiable norm. The purpose of this paper is to prove a counterpart to the result in [14]. That is, we show that if X is a uniformly convex Banach space satisfying the Opial's condition, C a bounded closed convex subset of X, and $T\colon C \to C$ an asymptotically nonexpansive mapping, then for each $x \in C$, the sequence $\{T^n x\}$ almost converges weakly to a fixed point of T, that is, there is a $y \in F(T)$ such that

$$\text{weak-}\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} T^{i+k}x = y \quad \text{uniformly in } k \geqslant 0.$$

This not only gives the above question another positive answer, but also implies that $\{T^n x\}$ converges weakly to y if and only if T is weakly asymptotically regular at x, that is, weak-$\lim\limits_{n\to\infty}\left(T^{n+1}x - T^n x\right) = 0$. We also present a weak convergence theroem for asymptotically nonexpansive semigroups. Our results generalise those of Hirano [8] and our proofs employ ideas of Hirano [8], Tan and Xu [14], and a technique of Bruck [2, 3].

2. Preliminaries and Lemmas

Recall that a Banach space X is said to satisfy the Opial's condition ([2]) if for any sequence $\{x_n\}$ in X, the condition $x_n \to x_0 \in X$ weakly implies that $\liminf\limits_{n\to\infty}\|x_n - x_0\| < \liminf\limits_{n\to\infty}\|x_n - x\|$, or equivalently $\limsup\limits_{n\to\infty}\|x_n - x_0\| < \limsup\limits_{n\to\infty}\|x_n - x\|$ for all $x \neq x_0$. It is known [12] that all Hilbert spaces and $\ell^p (1 < p < \infty)$ satisfy the Opial's condition. However, the $L^p (1 < p < \infty)$ spaces do not unless $p = 2$. A deeper result, shown by van Dulst [4], is that every separable Banach space can be equivalently renormed so that it possesses the Opial's condition.

Let F be a closed convex subset of a Banach space X and $\{x_n\}$ be a bounded sequence in X. Then we let

$$r(\{x_n\}, y) = \limsup_{n\to\infty}\|x_n - y\|,$$

and

$$r(\{x_n\}, F) = \min\{r(\{x_n\}, y)\colon y \in F\}.$$

We note here that, in Edelstein's terminology [5], the number $r(\{x_n\}, F)$ is called the asymptotic radius of the sequence $\{x_n\}$ with respect to the set F. We now establish some lemmas for later use. The following two lemmas are easy to prove (see Hirano [8]).

Lemma 2.1. *Let F be a closed convex subset of a reflexive Banach space X and $\{x_n\}$ be a bounded sequence in X such that for each $y \in F$, $\lim_n \|x_n - y\|$ exists. Then there is a $y_0 \in F$ such that*

$$\lim_n \|x_n - y_0\| = \min\{\lim_n \|x_n - y\| : y \in F\}.$$

Lemma 2.2. *Let F be a closed convex subset of a uniformly convex Banach space X and Λ be a set of bounded sequences in X which satisfies the following conditions:*

(i) *if $\{x_n\} \in \Lambda$, then for each $y \in F$, $\lim_n \|x_n - y\|$ exists;*

(ii) *if $\{x_n\}, \{y_n\} \in \Lambda$, then there exists $\{z_n\} \in \Lambda$ such that $r(\{z_n\}, y) \leqslant r(\{x_n\}, y)$ and $r(\{z_n\}, y) \leqslant r(\{y_n\}, y)$ for every $y \in F$.*

Let $r = \inf\{r(\{x_n\}, F) : \{x_n\} \in \Lambda\}$ and $\{\{x_n^{(i)}\} : i \geqslant 1\}$ be a sequence in Λ such that $\lim_i r\left(\{x_n^{(i)}\}, F\right) = r$. Then there exists a sequence $\{z_i\} \subset F$ such that $r\left(\{x_n^{(i)}\}, F\right) = r\left(\{x_n^{(i)}\}, z_i\right)$ for all $i \geqslant 1$ and $\{z_i\}$ converges strongly to a point in F.

Lemma 2.3. *Let X be a uniformly convex Banach space satisfying the Opial's condition, C a bounded closed convex subset of X, and $\{x_n\}$ a sequence in C such that $\limsup_{m\to\infty} \left(\limsup_{n\to\infty} \|T^m x_n - x_n\|\right) = 0$ and $\lim_{n\to\infty} \|x_n - y\|$ exists for each $y \in F(T)$. Then $\{x_n\}$ converges weakly to a point z in $F(T)$ such that $r(\{x_n\}, z) = r(\{x_n\}, F(T))$.*

Proof: We first note that by Geobel and Kirk [6], $F(T)$ is closed convex and nonempty. By Lemma 2.3 of [14], every weak limit point of the sequence $\{x_n\}$ is a fixed point of T. Suppose now $x_{n_i} \to u$ and $x_{m_j} \to v$ weakly; then $u, v \in F(T)$. If $u \neq v$, then the Opial's condition of X implies that

$$\begin{aligned} \lim_n \|x_n - u\| &= \lim_i \left\|x_{n_i} - u\right\| \\ &< \lim_i \left\|x_{n_i} - v\right\| = \lim_j \left\|x_{m_j} - v\right\| \\ &< \lim_j \left\|x_{m_j} - u\right\| = \lim_n \|x_n - u\|. \end{aligned}$$

This is a contradition, proving that $\{x_n\}$ converges weakly to some $z \in F(T)$. The equality $r(\{x_n\}, z) = r(\{x_n\}, F(T))$ now follows directly from the Opial's condition of X. The proof is complete. ∎

Lemma 2.4. ([14]). *Let C be a bounded closed convex subset of a uniformly convex Banach space X, and $T\colon C \to C$ an asymptotically nonexpansive mapping. Then for each $x \in C$, each integer $n \geqslant 1$, and arbitrary $\varepsilon > 0$, there exist integers i_n and k_ε depending only on n and ε, respectively, such that*

$$\left\|T^k S_n T^i x - S_n T^k T^i x\right\| \leqslant (1+\varepsilon) g^{-1}\left(\frac{1}{n} + \varepsilon M\right) \tag{2.1}$$

for all $k \geqslant k_\varepsilon$ and $i \geqslant i_n$, where $S_n = (1/n)\left(I + T + \ldots + T^{n-1}\right)$ with I the identity operator of X, $M = \operatorname{diam}(T^n x)$, and $g\colon [0,\infty) \to [0,\infty)$ is a strictly increasing convex continuous function such $g(0) = 0$.

Corollary 2.1. *Let C, T, and i_n be as in Lemma 2.4. Then for all sequences $\{j_n\}$, $\{k_n\}$ of integers such that $j_n \geqslant i_n$ for all $n \geqslant 1$ and $\lim_n k_n = \infty$, we have*

$$\lim_n \left\|T^{k_n} S_n T^{j_n} x - S_n T^{k_n + j_n} x\right\| = 0. \tag{2.2}$$

In the sequel, we always assume that the integers $\{i_n\}$ in Lemma 2.4 are chosen so that $i_1 < i_2 < \ldots < i_n < \ldots \to \infty$.

Lemma 2.5. *Let C be a bounded closed convex subset of a uniformly convex Banach space X with the Opial's condition, $T\colon C \to C$ an asymptotically nonexpansive mapping, and x an element of C. Suppose $\{k_n\}$ is a sequence of integers such that $k_n > i_{2^n}$ and $k_{n+1} > k_n + i_{2^n}$ for all $n \geqslant 1$ (where i_{2^n} is as selected in Lemma 2.4.) Then for each $y \in F(T)$, $\lim_n \left\|S_{2^n} T^{k_n} x - y\right\|$ exists and $\{S_{2^n} T^{k_n} x\}$ converges weakly to some $z \in F(T)$.*

Proof: For a fixed $y \in F(T)$, let $r := \liminf_{n\to\infty} \left\|S_{2^n} T^{k_n} x - y\right\|$. It follows from (2.1) that

$$\left\|T^k S_{2^n} T^{k_n} x - S_{2^n} T^k T^{k_n} x\right\| \leqslant (1+\varepsilon) g^{-1}\left(\frac{1}{2^n} + \varepsilon M\right)$$

for each $n \geqslant 1$ and all $k \geqslant k_\varepsilon$. Since T is asymptotically nonexpansive, we have an integer $\overline{k} > k_\varepsilon$ such that

$$\lambda_k < 1 + \varepsilon \quad \text{for} \quad k \geqslant \overline{k}. \tag{2.3}$$

We then have an integer n large enough so that

$$\left\|S_{2^n} T^{k_n} x - y\right\| < r + \varepsilon, \quad k_{n+1} - k_n > \overline{k}, \quad \text{and} \quad 2^{-n} < \varepsilon. \tag{2.4}$$

It follows from (2.1), (2.3) and (2.4) that

$$\begin{aligned}
&\left\|S_{2^{n+1}}T^{k_{n+1}}x - y\right\| \\
&\quad = \left\|\left(T^{k_{n+1}}x + T^{k_{n+1}+1}x + \ldots + T^{k_{n+1}+2^{n+1}-1}x\right)/2^{n+1} - y\right\| \\
&\quad = \left\|\frac{1}{2}\left(S_{2^n}T^{k_{n+1}}x + S_{2^n}T^{k_{n+1}+2^n}x\right) - y\right\| \\
&\quad \leqslant \left(\left\|S_{2^n}T^{k_{n+1}}x - T^{k_{n+1}-k_n}S_{2^n}T^{k_n}x\right\| + \left\|T^{k_{n+1}-k_n}S_{2^n}T^{k_n}x - y\right\|\right)/2 \\
&\qquad + \left(\left\|S_{2^n}T^{k_{n+1}+2^n}x - T^{k_{n+1}-k_n+2^n}S_{2^n}T^{k_n}x\right\|\right. \\
&\qquad + \left.\left\|T^{k_{n+1}-k_n+2^k}S_{2^n}T^{k_n}x - y\right\|\right)/2 \\
&\quad \leqslant (1+\varepsilon)g^{-1}\left(2^{-n} + \varepsilon M\right) + (1+\varepsilon)(r+\varepsilon) \\
&\quad \leqslant (1+\varepsilon)g^{-1}(\varepsilon(1+M)) + (1+\varepsilon)(r+\varepsilon).
\end{aligned}$$

In the same way, we can prove

$$\left\|S_{2^{n+i}}T^{k_{n+i}}x - y\right\| \leqslant (1+\varepsilon)g^{-1}(\varepsilon(1+M)) + (1+\varepsilon)(r+\varepsilon)$$

for all $i \geqslant 1$, from which it follows that

$$\begin{aligned}
\limsup_{i\to\infty}\left\|S_{2^i}T^{k_i}x - y\right\| &= \limsup_{i\to\infty}\left\|S_{2^{n+i}}T^{k_{n+i}}x - y\right\| \\
&\leqslant (1+\varepsilon)g^{-1}(\varepsilon(1+M)) + (1+\varepsilon)(r+\varepsilon).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\limsup_{n\to\infty}\left\|S_{2^n}T^{k_n}x - y\right\| \leqslant \liminf_{n\to\infty}\left\|S_{2^n}T^{k_n}x - y\right\|,$$

showing $\lim_n \left\|S_{2^n}T^{k_n}x - y\right\|$ exists. Noticing for each $u \in C$ and each fixed integer $m \geqslant 1$,

$$\|S_n(T^m u) - S_n(u)\| \leqslant \frac{m}{n}\operatorname{diam}(C) \to 0 \quad \text{as} \quad n \to \infty,$$

we get by Lemma 2.4 that

$$\begin{aligned}
&\limsup_{m\to\infty}\left(\limsup_{n\to\infty}\left\|T^m S_{2^n}T^{k_n}x - S_{2^n}T^{k_n}x\right\|\right) \\
&\quad \leqslant \limsup_{m\to\infty}\left(\limsup_{n\to\infty}\left\{\left\|T^m S_{2^n}T^{k_n}x - S_{2^n}T^m T^{k_n}\right\|\right.\right. \\
&\qquad + \left.\left.\left\|S_{2^n}T^{m+k_n}x - S_{2^n}T^{k_n}x\right\|\right\}\right) \\
&\quad \leqslant (1+\varepsilon)g^{-1}(\varepsilon M) \to 0 \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}$$

Now applying Lemma 2.3, we complete the proof of the lemma. ∎

3. The nonlinear ergodic theorem

In this section, we prove the main result of the paper. We begin by recalling the notion of almost convergence due to Lorentz [11].

Definition: Let X be a Banach space. A sequence $\{x_n\}_{n=0}^{\infty}$ in X is said to be weakly almost convergent to an element y of X if

$$\text{weak-}\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}x_{i+k}=y \quad \text{uniformly in } k\geqslant 0.$$

Theorem 3.1. *Let X be a uniformly convex Banach space satisfying the Opial's condition, C a bounded closed convex subset of X, and $T\colon C\to C$ an asymptotically nonexpansive mapping. Then for each $x\in C$, the sequence $\{T^n x\}$ is weakly almost convergent to a fixed point of T. That is, there is a $z\in F(T)$ such that*

$$\text{weak-}\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}T^{i+k}x=z \quad \text{uniformly in } k\geqslant 0.$$

Proof: We first observe that T has a fixed point by Goebel and Kirk [6]. For a fixed $x\in C$, let

$$\Lambda=\{\{S_{2^n}T^{h_n}x\}: h_n>i_{2^n} \text{ and } h_{n+1}>h_n+i_{2^n} \text{ for all } n\geqslant 1\},$$

where i_{2^n} is chosen as in Lemma 2.4. Then each $\{S_{2^n}T^{h_n}x\}$ in Λ is bounded since C is bounded and by Lemma 2.5, $\lim_n\|S_{2^n}T^{h_n}x-y\|$ exists for every $y\in F(T)$ and $\{S_{2^n}T^{h_n}x\}$ converges weakly to a fixed point of T. Now let $\{S_{2^n}T^{h_n}x\}$ and $\{S_{2^n}T^{r_n}x\}$ be in Λ and let $p_n=\max(h_n,r_n)+n$. Then it is readily seen that $p_n>i_{2^n}$ and $p_{n+1}>p_n+i_{2^n}$ for all n and hence $\{S_{2^n}T^{p_n}x\}\in\Lambda$. Moreover, in view of Lemma 2.5 and Corollary 2.1, we derive for each $y\in F(T)$ that

$$\begin{aligned}
&\lim_n\|S_{2^n}T^{p_n}x-y\|\\
&\leqslant\lim_n\left(\|S_{2^n}T^{p_n}x-T^{p_n-h_n}S_{2^n}T^{h_n}x\|+\|T^{p_n-h_n}S_{2^n}T^{h_n}x-y\|\right)\\
&\leqslant\lim_n\left(\|S_{2^n}T^{p_n-h_n}T^{h_n}x-T^{p_n-h_n}S_{2^n}T^{h_n}x\|+\lambda_{p_n-h_n}\|S_{2^n}T^{h_n}x-y\|\right)\\
&=\lim_n\|S_{2^n}T^{h_n}x-y\|,
\end{aligned}$$

that is,

$$\lim_n\|S_{2^n}T^{p_n}x-y\|\leqslant\lim_n\|S_{2^n}T^{h_n}x-y\|. \tag{3.1}$$

Similarly, we have

$$\lim_n\|S_{2^n}T^{p_n}x-y\|\leqslant\lim_n\|S_{2^n}T^{r_n}x-y\|.$$

It then follows that Λ satisfies the hypotheses of Lemma 2.2 with $F = F(T)$. Set

$$r = \inf\{r(\{S_{2^n}T^{h_n}x\}, F(T)) : \{S_{2^n}T^{h_n}\} \in \Lambda\}$$

and choose a sequence $\{\{S_{2^n}T^{h_n^{(j)}}x\}\}_{j\geqslant 1}$ in Λ such that $\lim_j r\left(\{S_{2^n}T^{h_n^{(j)}}x\}, F(T)\right) = r$. Then by Lemma 2.2, there exists a sequence $\{y_j\}$ in $F(T)$ which satisfies the equality $r\left(\{S_{2^n}T^{h_n^{(j)}}x\}, F(T)\right) = r\left(\{S_{2^n}T^{h_n^{(j)}}x\}, y_j\right)$ for all $j \geqslant 1$, and converges strongly to some $y \in F(T)$. Define $h_n = \max\left(h_n^{(j)} : 1 \leqslant j \leqslant n\right) + n$ for all $n \geqslant 1$. Then it is easily seen that $\{S_{2^n}T^{h_n}x\} \in \Lambda$. Similarly to (3.1), we can prove that

$$\begin{aligned} r(\{S_{2^n}T^{h_n}x\}, y) &= \lim_j r(\{S_{2^n}T^{h_n}x\}, y_j) \\ &\leqslant \lim_j r\left(\{S_{2^n}T^{h_n^{(j)}}x\}, y_j\right) \\ &= r. \end{aligned}$$

It thus follows that

$$r(\{S_{2^n}T^{h_n}x\}, F(T)) = r(\{S_{2^n}T^{h_n}x\}, y) = r \tag{3.2}$$

and $\{S_{2^n}T^{h_n}x\}$ converges weakly to y by the Opial's condition and Lemma 2.5. We now prove the following

CLAIM: Each $\{S_{2^n}T^{t_n}x\} \in \Lambda$, with $t_n \geqslant h_n + n$ for all n, must converge weakly to y.

In fact, by Lemma 2.5, $\{S_{2^n}T^{t_n}x\}$ converges weakly to a point, say z, in $F(T)$. If $z \neq y$, then it follows from (3.1) and the Opial's condition of X that

$$\begin{aligned} r &\leqslant \lim_n \left\|S_{2^n}T^{t_n}x - z\right\| < \lim_n \left\|S_{2^n}T^{t_n}x - y\right\| \\ &\leqslant \lim_n \left\|S_{2^n}T^{h_n}x - y\right\| = r. \end{aligned}$$

This contradiction proves the claim. Since $r\left(\{S_{2^n}T^{h_n+k_n2^n+j_n}(x)\}, y\right) = r$ for any sequences $\{k_n\}$ and $\{j_n\}$, by the same way as above, we can prove that $\{S_{2^n}T^{h_n+k2^n+j}(x)\}$ converges weakly to y as $n \to \infty$ uniformly in $k, j \geqslant 0$. We are now in a position to complete the proof of the theorem. For any integers n and m with $m > h_n$, we have

$$\begin{aligned} S_mT^ix &= \frac{1}{m}\sum_{k=0}^{m-1} T^{k+i}x \\ &= \frac{1}{m}\left\{\sum_{k=0}^{h_n-1} T^{k+i}x + 2^n\left(\sum_{k=0}^{j-1} S_{2^n}T^{h_n+k2^n+i}x\right) + \sum_{k=h_n+j2^n}^{m-1} T^{k+i}x\right\}, \end{aligned}$$

where $m = j2^n + h_n + r$, $0 \leqslant r < 2^n$. Since $\{S_{2^n} T^{h_n + k2^n + i} x\}$ converges weakly to y as $n \to \infty$ uniformly in $k, i \geqslant 0$, we conclude that $\{S_m T^i x\}$ converges weakly to y as $m \to \infty$ uniformly in $i \geqslant 0$. This completes the proof. ∎

Recall that T is said to be weakly asymptotically regular at $x \in C$ if $\text{weak-}\lim_n (T^{n+1}x - T^n x) = 0$.

Theorem 3.2. *Let C, X and T be as in Theorem 3.1. Then for each $x \in C$, the sequence $\{T^n x\}$ converges weakly to a fixed point of T if and only if T is weakly asymptotically regular at x.*

Proof: Necessity is trivial. Sufficiency follows from Theorem 3.1 and the fact that the weak asymptotic regularity of T at x is a Tauberian condition for weak almost convergence of $\{T^n x\}$ (see Lorentz [11]). ∎

4. Nonlinear semigroups

Let C be a closed convex subset of a Banach space X. A one parameter family $\mathcal{F} = \{T(t) : t \geqslant 0\}$ of mappings from C into itself is said to be a Lipschitzian semigroup on C if the following conditions are satisfied:

(1) $T(0)x = x$ for $x \in C$;
(2) $T(t+s)x = T(t)T(s)x$ for $x \in C$ and $t, s \geqslant 0$;
(3) for each $x \in C$, the mapping $T(t)x$ is continuous for $t \in [0, \infty)$;
(4) for each $t > 0$, there exists a real number $\lambda_t > 0$ such that

$$\|T(t)x - T(t)y\| \leqslant \lambda_t \|x - y\| \quad \text{for} \quad x, y \in C.$$

A Lipschitzian semigroup $\mathcal{F}$ is said to be nonexpansive if $\lambda_t = 1$ for all $t > 0$ and asymptotically nonexpansive if $\lim_{t\to\infty} \lambda_t = 1$, respectively. We denote by $F(\mathcal{F})$ the set of common fixed points of the semigroup $\mathcal{F}$, that is, $F(\mathcal{F}) = \bigcap_{t>0} F(T(t))$. If C is assumed to be a bounded closed convex subset of a uniformly convex Banach space and if $\mathcal{F} = \{T(t) : t \geqslant 0\}$ is an asymptotically nonexpansive semigroup on C, then it has been shown (see [15]) that $F(\mathcal{F})$ is closed, convex and nonempty. In this case, the metric projection P from X onto $F(\mathcal{F})$ is well-defined. If we assume, in addition, that $\mathcal{F} = \{T(t) : t \geqslant 0\}$ is nonexpansive and the space X either has a Frechet differentiable norm or satisfies the Opial's condition, then it has also been shown (see [2], [3], [8]) that for each $x \in C$, $\{T(t)x\}$ converges weakly to a common fixed point of $\mathcal{F}$ if and only if $\mathcal{F}$ is weakly asymptotically regular at x, that is, $\text{weak-}\lim_{t\to\infty} (T(t+h)x - T(t)x) = 0$ for all $h > 0$. The same conclusion was recently shown true by the authors [15] for an asymptotically nonexpansive semigroup $\mathcal{F}$ on C in the case when X has a Frechet differentiable norm. The object of this section is to show a counterpart in the case, when X satisfies the Opial's condition.

THEOREM 4.1. *Let X be a uniformly convex Banach space satisfying the Opial's condition, C a bounded closed convex subset of X, and $\mathcal{F} = \{T(t) : t \geqslant 0\}$ an asymptotically nonexpansive semigroup on C. Then for each $x \in C$, $\{T(t)x\}$ converges weakly to a member of $F(\mathcal{F})$ if and only if $\mathcal{F}$ is weakly asymptotically regular at x, that is, weak-$\lim_{t\to\infty}(T(t+h)x - T(t)x) = 0$ for all $h > 0$.*

PROOF: It suffices to show the sufficiency part. We first show that if $u =$ weak-$\lim_k T(t_k)x$ for some sequence $\{t_k\}$ of real numbers such that $\lim_k t_k = \infty$, then $u \in F(\mathcal{F})$. Under this assumption, since $\mathcal{F}$ is weakly asymptotically regular at x, we see that weak-$\lim_k T(t_k + s)x = u$ for all $s \geqslant 0$. Let

$$r_s = \limsup_{k\to\infty} \|T(t_k + s)x - u\|.$$

Using the Opial's condition, we get for all $s, t \geqslant 0$,

$$\begin{aligned} r_{s+t} &= \limsup_{k\to\infty} \|T(t_k + s + t)x - u\| \\ &\leqslant \limsup_{k\to\infty} \|T(t)T(t_k + s)x - T(t)u\| \\ &\leqslant \lambda_t \limsup_{k\to\infty} \|T(t_k + s)x - u\| \\ &= \lambda_t r_s, \quad \text{namely,} \end{aligned}$$

$$r_{s+t} \leqslant \lambda_t r_s \quad \text{for all} \quad s, t \geqslant 0. \tag{4.1}$$

From this, it follows that $\lim_{t\to\infty} r_t =: r$ exists and $r \leqslant r_s$ for all $s \geqslant 0$. If $r = 0$, then it is immediate that $u \in F(\mathcal{F})$. So, we assume $r > 0$. In this case, we show that $T(t)u \to u$ strongly as $t \to \infty$. Suppose not; then there is a sequence $\{\bar{t}_n\}$ for which $\lim_n \bar{t}_n = \infty$ such that

$$\|T(\bar{t}_n)u - u\| \geqslant \varepsilon_0, \qquad n = 1, 2, \ldots \tag{4.2}$$

for some $\varepsilon_0 > 0$. Choose $0 < \eta < \varepsilon_0$ so small that

$$(r + \eta)(1 - \delta(\varepsilon_0/(r + \eta))) < r, \tag{4.3}$$

where δ is the modulus of convexity of X. Choose N and s_0 so that

$$\lambda_{\bar{t}_N} r_{s_0} < r + \eta,$$

where $\lambda_{\bar{t}_N}$ is the Lipschitz constant of $T(\bar{t}_N)$. Using the Opial's condition of X and combining (4.1), (4.2) and (4.4), it yields

$$\begin{aligned} r \leqslant r_{s_0+\bar{t}_N} &= \limsup_{k\to\infty} \left\|T(t_k+s_0+\bar{t}_N)s-u\right\| \\ &\leqslant \limsup_{k\to\infty} \left\|T(t_k+s_0+\bar{t}_N)x-\frac{1}{2}(T(\bar{t}_N)u+u)\right\| \\ &\leqslant \lambda_{\bar{t}_N} r_{s_0}\left(1-\delta\left(\varepsilon_0/\left(\lambda_{\bar{t}_N} r_{s_0}\right)\right)\right) \\ &\leqslant (r+\eta)(1-\delta(\varepsilon_0/(r+\eta))), \end{aligned}$$

which contradicts (4.3) and therefore, $T(t)u \to u$ strongly as $t \to \infty$. This implies that $u \in F(\mathcal{F})$ by continuity of $\mathcal{F}$. Now we set

$$d(t) = \|T(t)x - PT(t)x\|, \quad t \geqslant 0,$$

where P is the metric projection of X onto $F(\mathcal{F})$. Since

$$\begin{aligned} d(t+s) &= \|T(t+s)x - PT(t+s)x\| \\ &\leqslant \|T(t+s)x - PT(t)x\| \\ &= \|T(s)T(t)x - T(s)PT(t)x\| \\ &\leqslant \lambda_s \|T(t)x - PT(t)x\| \\ &= \lambda_s d(t) \end{aligned}$$

for all $t,\ s \geqslant 0$, it follows that $d := \lim_{t\to\infty} d(t)$ exists and $d \leqslant d(t)$ for all $t \geqslant 0$. We now claim that $\{PT(t)x\}$ is norm Cauchy. This is trivially valid if $d = 0$. Suppose now $d > 0$. For any $\varepsilon > 0$, choose first $\eta > 0$ such that

$$(d+\eta)(1-\delta(\varepsilon/(d+\eta))) < d \tag{4.5}$$

and then t_0 such that

$$d(t) < d + \frac{1}{2}\eta \quad \text{and} \quad \lambda_t\left(d+\frac{1}{2}\eta\right) < d+\eta \tag{4.6}$$

for all $t \geqslant t_0$. Now let $t_1,\ t_2 \geqslant t_0$ be arbitrary but fixed. If $\|PT(t_1)x - PT(t_2)x\| \geqslant \varepsilon$, then, since

$$\begin{aligned} \|T(t_0+t_1+t_2)x - PT(t_1)x\| &= \|T(t_0+t_2)T(t_1)x - T(t_0+t_2)PT(t_1)x\| \\ &\leqslant \lambda_{t_0+t_2} \|T(t_1)x - PT(t_1)x\| \\ &= \lambda_{t_0+t_2} d(t_1) < \lambda_{t_0+t_2}\left(d+\frac{1}{2}\eta\right) < d+\eta, \end{aligned}$$

we get

$$\begin{aligned} d \leqslant d(t_0+t_1+t_2) &= \|T(t_0+t_1+t_2)x - PT(t_0+t_1+t_2)x\| \\ &\leqslant \left\| T(t_0+t_1+t_2)x - \frac{1}{2}(PT(t_1)x + PT(t_2)x) \right\| \\ &\leqslant (d+\eta)(1-\delta(\varepsilon/(d+\eta))), \end{aligned}$$

a contradiction to (4.5). This shows $\|PT(t_1)x - PT(t_2)x\| < \varepsilon$ and hence $\{PT(t)x\}$ is norm Cauchy. Let $y = \lim_{t\to\infty} PT(t)x$ and $u = \text{weak-}\lim_k T(t_k)x$ be an arbitrary weak limit point of $\{T(t)x\}$. If $u \neq y$, using the Opial's condition of X, we then obtain

$$\begin{aligned} \lim_k \|T(t_k)x - y\| &= \lim_k \|T(t_k)x - PT(t_k)x\| \\ &\leqslant \lim_k \|T(t_k)x - u\| \\ &< \lim_k \|T(t_k)x - y\| . \end{aligned}$$

This is a contradiction. We have therefore $u = y$ and $\{T(t)x\}$ converges weakly to y. The proof is complete. ∎

References

[1] J.B. Baillon, 'Un theoreme de type ergodique pour les contractions non lineaires dans un espace de Hilbert', *C.R. Acad. Sci. Paris* **280** (1975), 1511–1514.

[2] R.E. Bruck, 'A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces', *Israel J. Math.* **32** (1979), 107–116.

[3] R.E. Bruck On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces, *Israel J. Math.* **34** (1981), 304–314.

[4] D. van Dulst, 'Equivalent norms and the fixed point property for nonexpansive mappings', *J. London Math. Soc.* **25** (1982), 139–144.

[5] M. Edelstein, 'The construction of an asymptotic center with a fixed-point property', *Bull. Amer. Math. Soc.* **78** (1972), 206–208.

[6] K. Goebel and W.A. Kirk, 'A fixed point theorem for asymptotically nonexpansive mappings', *Proc. Amer. Math. Soc.* **35** (1972), 171–174.

[7] N. Hirano, 'A proof of the mean ergodic theorem for nonexpansive mappings in Banach spaces', *Proc. Amer. Math. Soc.* **78** (1980), 361–365.

[8] N. Hirano, 'Nonlinear ergodic theorems and weak convergence theorems', *J. Math. Soc. Japan* **34** (1982), 35–46.

[9] N. Hirano and W. Takahashi, 'Nonlinear ergodic theorems for nonexpansive mappings in Hilbert spaces', *Kodai Math. J.* **2** (1979), 11–25.

[10] W.A. Kirk, 'Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type', *Israel J. Math.* **17** (1974), 339–346.

[11] G.G. Lorentz, 'A contribution to the theory of divergent series', *Acta Math.* **80** (1948), 167–190.

[12] Z. Opial, 'Weak convergence of the sequence of successive approximations for nonexpansive mappings', *Bull. Amer. Math. Soc.* **73** (1967), 591–597.

[13] S. Reich, 'Weak convergence theorems for nonexpansive mappings in Banach spaces', *J. Math. Anal. Appl.* **57** (1979), 274–276.

[14] K.K. Tan and H.K. Xu, 'The nonlinear ergodic theorem for asymptotically nonexpansive mappings in Banach spaces', *Proc. Amer. Math. Soc.* (to appear).

[15] K.K. Tan and H.K. Xu, 'An ergodic theorem for nonlinear semigroups of Lipschitzian mappings in Banach spaces', *Nonlinear Anal.* (to appear).

[16] Z.Y. You and H.K. Xu, 'An ergodic convergence theorem for mappings of asymptotically nonexpansive type', *Chinese Ann. of Math.* **11A** (1990), 519–523. (in Chinese) .

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