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A PRÜFER APPROACH TO HALF-LINEAR STURM LIOUVILLE PROBLEMS

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We consider the half linear Sturm-Liouville problem

$$-(py') + qy = \lambda ry + \alpha y^+ + \beta y^-$$

on the interval [0, 1] subject to separated boundary conditions (which may be eigenparameter dependent at $x = 1$) and use Prüfer techniques to produce an oscillation theory for this problem. Both right definite ($r > 0$) and left definite (r of both signs) cases are discussed.

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1. Introduction

The eigenvalue problem to be considered in this paper takes the form

$$-(py')' + qy = \lambda ry + \alpha y^+ + \beta y^- \tag{1}$$

on [0, 1], where p, q, r, α, β are real valued continuous functions on [0, 1] and additionally, $p, r > 0$. The continuity requirement can be relaxed to integrability on [0, 1] but it is not our purpose here to pursue that line of refinement. Rather, our interest will focus on the terms involving y^+ and y^-, the positive and negative parts of the function y defined, as usual, by $y^+ = \max(y, 0)$, $y^- = (-y)^+$. We shall subject (1) to boundary conditions:

$$b_0 y(0) = d_0(py')(0) \tag{2}$$

$$b_1 y(1) = d_1(py')(1) \tag{3}$$

where b_0, d_0, b_1, d_1 are constants. Problems of this type have been considered previously by Berestycki [2] and, more recently, Rynne [6]. While they are nonlinear because of the terms involving $y^\pm$, they are positively homogeneous and linear in the cones $y > 0$ and $y < 0$, and have been termed "half linear" by Berestycki. "Half-eigenvalues", then, are points λ for which (1, 2, 3) has a non trivial solution y, the corresponding "half-

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eigenfunction". In this case the functions ty where $t>0$ are also solutions. Berestycki has produced, via non linear bifurcation techniques, a Sturm oscillation theorem for these problems to the effect that there are two sequences of half-eigenvalues, $\lambda_0^+ < \lambda_1^+ < \lambda_2^+ < \ldots$ and $\lambda_0^- < \lambda_1^- < \lambda_2^- < \ldots$ where the corresponding half-eigenfunction $y_k^\pm$ has k zeros in $(0, 1)$ and satisfies $\pm y_k^\pm > 0$ in a neighbourhood of 0 of the form $(0, \delta)$ – see [**2**, Theorem 2].

Our aim here is to establish this result via Prüfer angle techniques and subsequently, allow the boundary conditions at $x=1$ to depend on the eigenparameter λ. We shall also discuss the so called "left definite" case for both the standard boundary conditions (3) and for λ dependent ones. The results in this case are new even for the situation of standard separated boundary conditions. The arguments follow the ideas used in [**3**, **4**, **5**] where linear problems have been discussed from this point of view.

2. The Prüfer angle

We use the usual substitutions: $y=\rho\sin\theta$, $py'=\rho\cos\theta$ to obtain the first order equation

$$\theta' = p^{-1}\cos^2\theta + (\lambda r - q)\sin^2\theta + \sin\theta[\alpha(\sin\theta)^+ + \beta(\sin\theta)^{-1}] \tag{4}$$

subject to the initial condition

$$\theta(0) = \tan^{-1}(d_0/b_0). \tag{5}$$

Of course this initial condition is not sufficient to determine a solution of (4) since (5) does not specify the sign of $\sin\theta(0)$. For now we shall take

$$\theta(0) \in [0, \pi[$$

thereby forcing $\sin\theta(0) \geq 0$, and, in terms of the original equation (1), $y>0$ in some deleted neighbourhood of 0. Other cases will be discussed later. Equation (4) can also be written as

$$\theta' = p^{-1}\cos^2\theta + [\lambda r - q + \alpha(\sin\theta)^+/\sin\theta + \beta(\sin\theta)^-/\sin\theta]\sin^2\theta$$

from which we conclude the two useful comparisons:

$$\theta' \leq p^{-1}\cos^2\theta + [\lambda r - (q - |\alpha| - |\beta|)]\sin^2\theta$$
$$\theta' \geq p^{-1}\cos^2\theta + [\lambda r - (q + |\alpha| + |\beta|)]\sin^2\theta.$$

The right hand sides of these two inequalities are those that would arise from linear Sturm-Liouville problems with potentials $q_1 = q - |\alpha| - |\beta|$, $q_2 = q + |\alpha| + |\beta|$ respectively and with the same initial condition. If θ_1, θ_2 denote the solutions to those Prüfer equations we obtain

$$\theta_2(x, \lambda) \le \theta(x, \lambda) \le \theta_1(x, \lambda) \tag{6}$$

for all $x \in [0, 1]$ and for all λ. With this background we have

Theorem 1.

(i) *For any fixed* λ, $\theta(x, \lambda)$ *increases through integer multiples of* π.

(ii) *For any* x, $\theta(x, \lambda)$ *is an increasing function of* λ.

(iii) *For any* x, $\theta(x, \lambda) \to \infty$ *as* $\lambda \to \infty$, *and* $\theta(x, \lambda) \to 0$ *as* $\lambda \to -\infty$.

Proof. The first claim is proved using the fact that when θ is a multiple of π, $\theta' > 0$. (For the case when the coefficient functions are integrable and not necessarily continuous, an argument similar to that of Atkinson [**1**, pp. 209–211] can be used.) The second result is a consequence of standard theory. For the third result we note that these limits hold by standard Sturm theory for θ_1, θ_2 and so the comparisons (6) yield the desired conclusion.

3. Existence of half eigenvalues and oscillation theory

The *right hand Dirichlet problem (RDP)* associated with (1, 2) consists of taking the boundary condition (3) to be $y(1) = 0$. Half-eigenvalues for the RDP occur precisely at those values of λ for which $\theta(1, \lambda)$ is an integer multiple of π. Theorem 1 shows that there is an increasing sequence λ_k^{D+}, $k = 0, 1, 2, \ldots, \lambda_k^{D+} \to \infty$, satisfying $\theta(1, \lambda_k^{D+}) = (k+1)\pi$, $k = 0, 1, 2, \ldots$. The corresponding half-eigenfunctions y_k^{D+} have k zeros in (0, 1) and are positive in a deleted neighbourhood of 0.

We now consider the function

$$f(\lambda) = \cot \theta(1, \lambda)$$

and list its immediate properties.

Theorem 2. *The graph* $\mu = f(\lambda)$ *consists of countably many branches* B_k, $k = 0, 1, 2, \ldots$. *Interpreting* λ_{-1}^{D+} *as* $-\infty$, *we have for* $k \ge 0$,:

(1) $\theta(1, \lambda) \in]k\pi, (k+1)\pi]$ *for* $\lambda \in]\lambda_{k-1}^{D+}, \lambda_k^{D+}[$ *and* $\theta(1, \lambda_k^{D+}) = (k+1)\pi$.

(2) B_k *is defined for* $\lambda \in]\lambda_{k-1}^{D+}, \lambda_k^{D+}[$ *and* f *decreases over this interval with* $f(\lambda) \to -\infty$ *as* $\lambda \uparrow \lambda_k^{D+}$ *and* $f(\lambda) \to \infty$ *as* $\lambda \downarrow \lambda_k^{D+}$.

(3) $\lambda = \lambda_k^{D+}$ *are the vertical asymptotes of the graph of* f.

The following lemma sets the stage for the oscillation theory associated with our problem: it parallels [**5**, Lemma 2.2].

Lemma 3. *If* $k \geq 0$, *and* $\lambda \in]\lambda_{k-1}^{D+}, \lambda_k^{D+}[$ *then all solutions of* (1, 2) *which are positive in a deleted neighbourhood of 0 possess exactly k zeros in* (0, 1).

Proof. While the solutions of (1, 2) which are positive in a deleted neighbourhood do not form a vector space they are all of the form ty for some solution y, i.e. they constitute a positive ray of functions. Since $\theta(1, \lambda) \in]k\pi, (k+1)\pi[$ for the λ value in question and since $\theta(x, \lambda)$ increases through multiples of π, there is no $x \in (0, 1)$ for which $\theta(x, \lambda) = (k+1)\pi$. Further, we have $\theta(0, \lambda) \in [0, \pi[$, so that the result follows directly for $k = 0$, and for $k > 0$, we have $\theta(x, \lambda) = k\pi$ for some $x \in]0, \pi[$.

Theorem 4. *If the solutions of* (1, 2) *are also required to satisfy* (3), *then for each* $k \geq 0$, *there is a unique half-eigenvalue* $\lambda = \lambda_k^+$ *whose corresponding half-eigenfunction has exactly k zeros in*]0, 1[. *The* λ_k^+ *interlace the RDP eigenvalues in the sense that*

$$\lambda_k^+ \leq \lambda_k^{D+} \leq \lambda_{k+1}^+, k = 0, 1, \ldots$$

Proof. If we take $\beta = \tan^{-1}(d_1/b_1) \in]0, \pi]$, we see that the graph of $\mu = \cot\beta$ cuts B_k at (λ_k, β) say, when $\beta \neq \pi$, while if $\beta = \pi$ we set $\lambda_k^+ = \lambda_k^{D+}$. The lemma provides the remainder of the argument.

We have now produced the results of [**2**, Theorem 2] for the case in which the half-eigenfunctions are to be positive in a deleted neighbourhood of 0. To cover the alternate case in which the half-eigenfunctions are to be negative in a deleted neighbourhood of 0, we can either adjust the initial condition for (4) by requiring that $\theta(0) \in [-\pi, 0[$ and modify the subsequent results accordingly, or we can consider the original problem with α, β replaced by $-\beta, -\alpha$ respectively and note that if y is a solution to that problem which is positive in a deleted neighbourhood of 0, then $-y$ solves the original problem and is negative in a deleted neighbourhood of 0. Thus the full Berestycki result on the existence of half-eigenvalues and their oscillation properties is obtained.

At this stage one further simple result involving $\lambda_0^\pm$ is immediate. We give the case for λ_0^+ only.

Corollary 5. *The half-eigenvalue* λ_0^+ *coincides with the zero-th eigenvalue* λ_0 *of the linear problem*

$$-(py')' + (q - \alpha)y = \lambda ry$$

with boundary conditions (2, 3).

Proof. We note that for $\lambda \leq \lambda_0^{D+}$, $\theta(x, \lambda) \in [0, \pi]$ so that the equation (4) coincides with the corresponding Prüfer equation from the linear problem above.

We can use the comparisons (6) to produce comparisons for the half-eigenvalues with those from the linear problems with potentials $q \pm (|\alpha| + |\beta|)$ giving a slight improvement on the result of Berestycki [**2**, Theorem 1].

Theorem 6. *Let $\lambda_n^1, n \geq 0$, denote the eigenvalues from the linear Sturm-Liouville problem with potential $q - (|\alpha| + |\beta|)$ and $\lambda_n^2, n \geq 0$, those from the linear problem with potential $q + (|\alpha| + |\beta|)$. Then we have*

$$\lambda_n^1 \leq \lambda_n^+ \leq \lambda_n^2, n \geq 0$$

Proof. The result is an easy consequence of consideration of the equations $\theta_i(1, \lambda) = \tan^{-1}(-d_1/b_1) + n\pi$ and the corresponding equation for $\theta(1, \lambda)$.

4. Eigenparameter dependent boundary conditions

We now turn to the situation in which the boundary conditions at $x = 1$ are eigenvalue dependent and take the form:

$$(a_1\lambda + b_1)y(1) = (c_1\lambda + d_1)(py')(1) \tag{7}$$

Sturm-Liouville problems with this type of boundary condition have been the subject of recent investigation by the author and co-workers: [**3**, **4**, **5**]. There is a significant literature devoted to them and the introduction to [**4**] contains details of references. To date no non-linear problems with such boundary conditions have been studied.

We assume initially that $\delta = a_1d_1 - b_1c_1 > 0$ and $c_1 \neq 0$ a typical right definiteness condition for linear problems of this kind. The half-eigenvalue problem becomes that of finding the λ values at which the graphs of $f(\lambda)$ and

$$g(\lambda) = \frac{a_1\lambda + b_1}{c_1\lambda + d_1}$$

intersect. The graph of $\mu = g$ is a hyperbola with vertical asymptote at $\lambda = -d_1/c_1$. It is increasing on each of its branches and has a horizontal asymptote at $\mu = a_1/c_1$. It is thus easy to locate the intersections of these two graphs. We suppose that the vertical asymptote for g intersects the *K-th* branch B_K of f or is its right hand asymptote: i.e. we select K so that

$$\lambda_{K-1}^{D+} < -d_1/c_1 \leq \lambda_K^{D+}$$

We also denote by λ_k^{A+} the half eigenvalues for the so called "asymptotic problem" in which the boundary conditions at $x = 1$ take the form

$$a_1y(1) = c_1(py')(1).$$

Theorem 7. (i) *For the problem* $(1,2,7)$ *subject to* $\delta > 0, c_1 \neq 0$, *the half-eigenvalues whose corresponding half-eigenfunctions are positive in a deleted neighbourhood of 0 consist of a sequence* $\lambda_0^+ < \lambda_1^+ < \lambda_2^+ < \dots$ *where for each* $k = 0, 1, \dots$ *the corresponding half-eigenfunction has* k *zeros in* $]0,1[$, *together with an additional half-eigenvalue* $\lambda' \in]\lambda_{K-1}^{D+}, \lambda_K^{D+}]$ *whose corresponding half-eigenfunction has* K *zeros in* $]0,1[$.

(ii) $\lambda_k^{A+} < \lambda_k^+ < \lambda_{k+1}^{A+}$ *for all* $k \geq K$.

(iii) $\lambda_{k-1}^{D+} < \lambda_k^{A+} < \lambda_k^+ < \lambda_k^{D+}$ *for all* k.

Proof. The results come from considering the superposition of the graphs of f and g much along the lines of the corresponding results for the linear problem given in [**5**, Theorems 3.1, 3.3].

We have adopted here the numbering convention for the half-eigenvalues used by Binding and Browne in [**3**, **4**].

We also note in passing that it is possible to give comparison results for the case in which the coefficient functions depend on a parameter: cf. [**5**, Theorem 3.2]. Of course similar results are available for the half-eigenvalues λ_k^-. We leave the reader to formulate them.

As in [**5**], we can give an asymptotic result comparing the half-eigenvalues λ_k^+ with the asymptotic half-eigenvalues λ_k^{A+} for large k. We have

Theorem 8. *If* $pr \in AC[0,1]$, *then* $\lambda_k^+ - \lambda_k^{A+} = O(k^{-2})$ *as* $k \to \infty$.

Proof. We introduce a modified Prüfer transformation via the substitutions $(pr\lambda)^{1/2} y = \omega \sin\phi$, $py' = \omega\cos\phi$, whence $\tan\phi = (pr\lambda)^{1/2}\tan\theta$ and, after some calculation,

$$\phi' = \left(\frac{\lambda r}{p}\right)^{1/2} + \frac{(pr)'}{4pr}\sin 2\phi - (\lambda pr)^{-1/2} q \sin^2 + (\lambda pr)^{-1/2}[\alpha \sin\phi(\sin\phi)^+ + \beta\sin\phi(\sin\phi)^-]$$

This corresponds to [**5**, eqn. (3.3), Proof of Theorem 3.5]. The argument now follows the lines of that given in [**5**, Theorem 3.5]: we note that it is easy to show that, for example, $\sin\phi(\sin\phi)^+$ is smooth in ϕ. We leave details of the proof to the reader.

When $c_1 = 0$, $g(\lambda)$ takes the form

$$g(\lambda) = a_1\lambda/d_1 + b_1/d_1$$

which, since $\delta > 0$, is a line of positive slope. It intersects each branch B_k of f exactly once and so we see that the usual Sturm oscillation theorem holds. We summarize this as

Theorem 9. *If the problem* $(1,2,7)$ *satisfies* $\delta > 0$ *and* $c_1 = 0$, *then the half-eigenvalues of the problem can be ordered as* $\lambda_0^+ < \lambda_1^+ < \lambda_2^+ < \dots$ *where the half-*

eigenfunction corresponding to λ_k^+ *has k zeros in* $[0,1[$ *and is positive in a deleted neighbourhood of* 0.

We turn to the cases in which $\delta=0$ but $(a_1,c_1)\neq(0,0)$. Firstly if $c_1\neq 0$, then boundary condition (7) becomes

$$c_1^{-1}a_1(c_1\lambda+d_1)y(1)=(c_1\lambda+d_1)(py'(1))$$

so that half-eigenvalues arise from the right hand asymptotic problem and additionally, $\lambda=-d_1/c_1$ is a further half-eigenvalue. Thus if we define K by the condition

$$\lambda_{K-1}^{A+}<-d_1/c_1\leq\lambda_K^{A+}$$

we have the same situation as in the statement of Theorem 7 unless it should happen that $\lambda_K^{A+}=-d_1/c_1$. If $c_1=d_1=0\neq a_1$, then the boundary condition (7) becomes $(a_1\lambda+b_1)y(1)=0$ and half-eigenvalues arise from the RDP and additionally from $\lambda=-b_1/a_1$. Now we define K by $\lambda_{K-1}^{D+}<-b_1/a_1<\lambda_K^{A+}$ and obtain the same result as above. Finally if $c_1\neq 0$ and $-d_1/c_1=\lambda_K^{A+}$ or if $c_1=d_1=0\neq a_1$ and $-b_1/a_1=\lambda_K^{D+}$, the corresponding half-eigenfunctions satisfy (1, 2) and are unique up to positive scalar multiplication so that no "extra" half-eigenvalue or half-eigenfunction exists in these cases. The analysis of these special cases follows that of [5] for the linear situation.

It is possible to consider the case in which the condition on δ is replaced by $\delta<0$. The development will parallel that for the linear case given in [3] but we choose not to pursue this avenue nor problems in which both boundary conditions are λ-dependent, preferring to turn to the consideration of left definite problems.

5. Left definite problems

In this section we shall abandon the requirement $r>0$ and replace it by a demand that neither of r^+, r^- be identically zero and also that

$$q\geq 0,\ \alpha<0,\ \beta>0$$
$$b_0d_0\geq 0.$$

For the case in which the boundary conditions at $x=1$ are of the form (3) we require $b_1d_1\leq 0$ and for the case in which they are of the form (7) we require the matrix $-\delta M$ where

$$M=\begin{pmatrix} a_1b_1 & -b_1c_1\\ -b_1c_1 & c_1d_1\end{pmatrix}$$

to be positive definite. This requirement has a number of consequences; see [3, Section 2]: in particular we note that $a_1c_1<0$.

We note in passing that the assumptions above prevent $\lambda=0$ being a half-eigenvalue for our problem for, if it were and if y were a corresponding half-eigenfunction we would have

$$-(py')+qy=\alpha y^{+}+\beta y^{-}$$

$$\int_0^1 -(py')'y+qy^2=\int_0^1 \alpha y^{+}y+\beta y^{-}y$$

and now routine calculations show the left hand side to be positive while the right hand side is negative. Essentially the same argument shows that $\lambda=0$ is not an eigenvalue for the linear case (i.e. $\alpha=\beta=0$). The Prüfer transformation to be used here is given by

$$\lambda y=\rho\sin\phi,\ py'=\rho\cos\phi$$

and the corresponding Prüfer equation is

$$\phi'=\frac{\lambda\cos^2\phi}{p}+\left(r-\frac{q}{\lambda}\right)\sin^2\phi+\alpha\sin\phi\left(\frac{\sin\phi}{\lambda}\right)^{+}+\beta\sin\phi\left(\frac{\sin\phi}{\lambda}\right)^{-},\ \lambda\neq 0. \tag{8}$$

The initial condition for this equation is

$$\phi(0,\lambda)=\tan^{-1}\left(\lambda\frac{d_0}{b_0}\right)\in]-\pi/2,\pi/2[,\ \text{if } b_0\neq 0,$$

$$\phi(0,\lambda)=sign(\lambda)\pi/2,\ \text{if } b_0=0.$$

Note that this choice of $\phi(0,\lambda)$ forces y to be positive in a deleted neighbourhood of 0. In developing properties of ϕ we shall concentrate on the case $\lambda>0$ since corresponding properties for $\lambda>0$ can be obtained via the transformation

$$\psi(\lambda)=-\phi(-\lambda)$$

where we realize that ψ satisfies the same differential equations as ϕ but with r replaced by $-r$. As in Section 2, we can form two useful comparisons:

$$\phi'\le\frac{\lambda\cos^2\phi}{p}+\left(r-\frac{q-|\alpha|-|\beta|}{\lambda}\right)\sin^2\phi$$

$$\phi'\ge\frac{\lambda\cos^2\phi}{\lambda}+\left(r-\frac{q+|\alpha|+|\beta|}{\lambda}\right)\sin^2\phi$$

and compare the right hand sides with Prüfer equations from linear problems with potentials $q\pm(|\alpha|+|\beta|)$. Such Prüfer angle functions have been analyzed in [4, Section 3] and, much as in the argument of Theorem 1 we can claim

Theorem 10.

(i) $\phi(x,\lambda)$ *increases in* λ *for any fixed* x.

(ii) $\phi(1,\lambda)\downarrow 0$ *as* $\lambda\downarrow 0$ *and* $\phi(1,\lambda)\uparrow 0$ *as* $\lambda\uparrow 0$

(iii) $\phi(1,\lambda)\to\pm\infty$ *as* $\lambda\to\pm\infty$

(iv) *For* $\lambda>0$, $\phi(x,\lambda)$ *increases through multiples of* π.

We are now interested in the graph of

$$f(\lambda):=\tan\phi(1,\lambda),\ \lambda\neq 0,$$
$$:=0,\ \lambda=0$$

whose important properties we summarise. When the boundary condition at $x=1$ is the Dirichlet condition $y(1)=0$, the resulting "right hand Dirichlet problem" has half-eigenvalues denoted by $\lambda_{k\pm}^{D+}$, with $\lambda_{k\pm}^{N+}$ denoting the half-eigenvalues for the "right hand Neumann" problem where the boundary condition at $x=1$ is $y'(1)=0$.

Theorem 11. *The graph of* $\mu=f(\lambda)$

(i) *has vertical asymptotes at* $\lambda=\lambda_{k\pm}^{N+}$ *and increases on each of its branches*

(ii) *is continuous at* $\lambda=0$

(iii) *crosses the* λ*-axis at* $\lambda=\lambda_{k\pm}^{D+}$.

Proof. The claims are easy consequences of the foregoing discussion.

For the general problem in which the boundary conditions at $x=1$ take the form (3) we seek intersections of the graph of f with that of

$$h(\lambda)=d_1\lambda/b_1$$

a straight line of negative slope. There is of course one intersection with each branch of f. The intersection on the branch passing through the origin occurs at the origin and must be discarded since, as we noted earlier, $\lambda=0$ is not an eigenvalue. It is important to note also that for $\lambda>0, \phi(1,\lambda)$ lies between $k\pi$ and $(k+1)\pi$ when λ lies between $\lambda_{(k-1)+}^{D+}$ and λ_{k+}^{D+} (here we interpret $\lambda_{(-1)+}^{D+}$ as 0). Hence half-eigenfunctions corresponding to half-eigenvalues in such a range would have k internal zeros. Similar statements hold for $\lambda<0$. Interlacing of half-eigenvalues is also immediate from the graphs of these two functions so that we can collect all of this information as the following.

Theorem 12. *The left definite problem* (1, 2, 3) *has two double infinite sequences of half-eigenvalues,* $\lambda_{k\pm}^{\pm}$, $k\geq 0$ *such that*

(i) *the half-eigenfunction corresponding to* $\lambda_{k\pm}^{+}$ *(respectively,* $\lambda_{k\pm}^{-}$*) has k internal zeros in*]0, 1] *and is positive (respectively, negative) in a deleted neighbourhood of* 0,

(ii) $\lambda_{k\pm}^{\pm} \to \pm\infty$ *as* $k \to \infty$,

(iii) *the following interlacing holds;*

$$\ldots \lambda_{1-}^{D+} < \lambda_{1-}^{+} < \lambda_{1-}^{N+} < \lambda_{0-}^{D+} < \lambda_{0-}^{+} < \lambda_{0-}^{N+} < 0 < \lambda_{0+}^{N+} < \lambda_{0+}^{+} < \lambda_{0+}^{D+} < \lambda_{1+}^{N+} < \lambda_{1+}^{+} < \lambda_{1+}^{D+} \ldots$$

with a similar interlacing for the half-eigenvalues $\lambda_{k\pm}^{-}$.

For the case when the boundary condition at $x = 1$ takes the form (7), we seek intersections of the graph of f above that of

$$g(\lambda) = \frac{\lambda(c_1\lambda + d_1)}{a_1\lambda + b_1}$$

which, as noted in [4, Section 4], is a hyperbola with vertical asymptote at $\lambda = -b_1/a_1$ and oblique asymptote

$$\mu = \frac{c_1}{a_1}\lambda + \frac{\delta}{a_1^2}.$$

The hyperbola decreases on both of its branches and crosses the λ-axis at $\lambda = 0$. Intersections of these two graphs will be simple, isolated and will accumulate only at $\pm\infty$. The situation is very much as in the linear case which is displayed graphically in [4, Section 4]. When $\delta > 0$, we see that each interval $]\lambda_{(k-1)+}^{D+}, \lambda_{k+}^{D+}[$ has precisely one intersection point with the exception of one in which there are two. This is the interval $]\lambda_{(K-1)+}^{D+}, \lambda_{K+}^{D+}[$ where K is defined by

$$\lambda_{(K-1)+}^{D+} < \frac{-d_1}{c_1} < \lambda_{K+}^{D+}. \tag{9}$$

The intersections in the region $\lambda < 0$ are regular in the sense that each interval $]\lambda_{(k+1)-}^{D+}, \lambda_{k-}^{D+}[$ will contain one half-eigenvalue whose corresponding half-eigenfunction will be positive in a deleted neighbourhood of 0 and will have k internal zeros in]0, 1[. Similar results hold for the cases in which $\delta < 0$ and for which we require the half-eigenfunctions to be negative in a deleted neighbourhood of 0. We thus have the following statement.

Theorem 13. *The left definite problem* (1, 2, 7) *has two double infinite sequences of half-eigenvalues,* $\lambda_{k\pm}^{\pm}$, $k \geq 0$ *such that*

(i) *the half-eigenfunction corresponding to* $\lambda_{k\pm}^{+}$ *(respectively,* $\lambda_{k\pm}^{-}$*) has k internal zeros in*]0, 1[*and is positive (respectively, negative) in a deleted neighbourhood of* 0,

(ii) $\lambda_{k\pm}^{\pm} \to \pm\infty$ *as* $k \to \infty$,

(iii) *the following interlacing holds;*

$$\ldots \lambda_{1-}^{D+} < \lambda_{1-}^{+} < \lambda_{1-}^{N+} < \lambda_{0-}^{D+} < \lambda_{0-}^{+} < \lambda_{0-}^{N+} < 0 < \lambda_{0+}^{N+} < \lambda_{0+}^{+} < \lambda_{0+}^{D+} < \lambda_{1+}^{N+} < \lambda_{1+}^{+} < \lambda_{1+}^{D+} \ldots$$

with a similar interlacing for the half-eigenvalues $\lambda_{k\pm}^{-}$.

If $\delta > 0$ (respectively < 0), there is an additional half-eigenvalue $\lambda^{+'} \in]\lambda_{(K-1)+}^{D+}, \lambda_{K+}^{N+}[$ (respectively, $]\lambda_{K-}^{N+}, \lambda_{(K-1)-}^{D+}[$) whose corresponding half-eigenfunction is positive in a deleted neighbourhood of 0 and has K internal zeros in $]0, 1[$, where K is defined by (9) (respectively,

$$\lambda^{D+} \le \frac{-c_1}{d_1} < \lambda_{(K-1)-}^{D+}$$

A similar statement holds with regard to an additional half-eigenvalue $\lambda^{-'}$ whose half-eigenfunction is negative in a deleted neighbourhood of 0.

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