Concerning collectionwise
Hausdorff spaces

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In this paper the notion of a space which has property (ω)
pointwise is studied. The primary application of this concept is
a reformulation of Tall's existence theorem for normal non-
metrizable metacompact Moore spaces in terms of families which
are finite on convergent sequences. Since the first countability
in Tall's theorem yields an abundance of convergent sequences,
this reformulation places the existence theorem in a natural
setting.

1. Preliminaries

A topological space is *collectionwise Hausdorff* if for each discrete
collection of points $\{p_\alpha : \alpha \in A\}$ there is a disjoint collection of open
sets $\{U_\alpha : \alpha \in A\}$ such that $p_\alpha \in U_\alpha$, for each $\alpha \in A$. Tall [7] has
reduced the question of the existence of a normal Moore space which is not
metrizable to the following theorem.

THEOREM 1.1 [Tall]. *If there is a normal first countable space
which is not collectionwise Hausdorff, then there exists a normal non-
metrizable metacompact Moore space.*

The set theoretic consequences of the answer to the Moore metrization
problem are well documented in the various works of Tall. Tall's Theorem
places this question in a highly simplified setting and reduces the
consideration of the metrizability of normal metacompact Moore spaces, from
discrete collections of closed sets to discrete sets of points.

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The primary purpose of this paper is to present a reformulation of
Tall's Theorem (Corollary 3.2) in terms of families which are finite on
convergent sequences. For this purpose the notion of a space with property
(ω) pointwise is introduced.

2, Definitions

A space will be called *strongly collectionwise Hausdorff* if for each
discrete set of points $\{p_\alpha : \alpha \in A\}$ there is a discrete collection of
open sets $\{U_\alpha : \alpha \in A\}$ such that $p_\alpha \in U_\alpha$, for each $\alpha \in A$. Every
normal collectionwise Hausdorff space is strongly collectionwise Hausdorff.
A collection of subsets G of a space X is said to be *cs-finite* at p ,
$p \in X$, if for each sequence $\{p_i\}$ in X , such that $p_i \to p$,
$G \cap \{p_i : i \in N\} \neq \emptyset$ for at most a finite number of $G \in G$. The
collection G is *cs-finite* if it is cs-finite at every point in X .
A space X is said to have *property* (ω) [2] (*pointwise collectionwise*
normal [3]) if for each discrete collection of closed sets
$F = \{F_\alpha : \alpha \in A\}$, there exists a cs-finite (point-finite) collection of
open sets $G = \{G_\alpha : \alpha \in A\}$ such that $F_\alpha \subset G_\alpha$, for each $\alpha \in A$ and
$F_\alpha \cap G_\beta = \emptyset$, if $\alpha \neq \beta$. Every collectionwise normal space has property
(ω) , and if a space has property (ω) then it is pointwise collectionwise
normal. Every metacompact space is pointwise collectionwise normal. A
Moore space is metrizable if and only if it has property (ω) [2]. Two of
the results reported in [3] are:

> a space is metacompact if and only if it is θ-refinable [8] and
> pointwise collectionwise normal, and

> a developable pointwise collectionwise normal space is hereditarily
> metacompact.

DEFINITION. A space is said to have *property* (ω) *pointwise* if for
each discrete set of points $\{p_\alpha : \alpha \in A\}$ there exists a *cs-finite*
collection of open sets $G = \{G_\alpha : \alpha \in A\}$ such that $p_\alpha \in G_\alpha$, for each
$\alpha \in A$ and $p_\alpha \notin G_\beta$, if $\alpha \neq \beta$.

Every space with property (ω) has property (ω) pointwise and

every strongly collectionwise Hausdorff space has property (ω) pointwise.

3.   Theorems

THEOREM 3.1. *Every regular sequential space which has property (ω) pointwise is strongly collectionwise Hausdorff.*

Proof. Let $\{p_\alpha : \alpha \in A\}$ be any discrete set of points in a regular space X . Let $\{U_\alpha : \alpha \in A\}$ be a cs-finite collection of open sets such that $p_\alpha \in U_\alpha$ for each $\alpha \in A$ and $p_\beta \notin U_\alpha$, if $\beta \neq \alpha$. For each $\alpha \in A$, let V_α be an open set such that $p_\alpha \in V_\alpha \subset \mathrm{cl}(V_\alpha) \subset U_\alpha$. Then $\{\mathrm{cl}(V_\alpha) : \alpha \in A\}$ is a cs-finite collection of closed sets in a sequential space. Accordingly, $\{\mathrm{cl}(V_\alpha) : \alpha \in A\}$ is locally finite by $[1$, Lemma 2.1]. If O_α is an open subset of X such that

$$p_\alpha \in O_\alpha \subset \mathrm{cl}(O_\alpha) \subset V_\alpha - \mathrm{U}\{\mathrm{cl}(V_\beta) : \beta \neq \alpha\}$$

for each $\alpha \in A$, then $(O_\alpha : \alpha \in A\}$ is a discrete collection of open sets such that $p_\alpha \in O_\alpha$ for each $\alpha \in A$. This completes the proof.

COROLLARY 3.2. *If there exists a normal first countable space which fails to have property (ω) pointwise, then there is a normal non-metrizable metacompact Moore space.*

Additional corollaries to Theorem 3.1 are apparent, for those theorems involving normal collectionwise Hausdorff spaces, in the papers of Fitzpatrick [5], Christian [4], and Tall [6].

THEOREM 3.3. *Let $f : X \to Y$ be an open and closed continuous surjection defined on a pointwise collectionwise normal space X . If X is sequential and has property (ω) pointwise, then Y has property (ω) pointwise.*

Proof. Since pointwise collectionwise normality is invariant under closed mappings [3], Y is pointwise collectionwise normal. Let $\{p_\alpha : \alpha \in A\}$ be any discrete set in Y . There exists a point-finite collection of open sets, $V = \{V_\alpha : \alpha \in A\}$, such that $p_\alpha \in V_\alpha$, for each

$\alpha \in A$ and $p_\beta \notin V_\alpha$, if $\beta \neq \alpha$. If $q_\alpha \in f^{-1}(p_\alpha)$ for each $\alpha \in A$,
$\{q_\alpha : \alpha \in A\}$ is a discrete subset of X . There exists a cs-finite
collection of open sets, $\{G_\alpha : \alpha \in A\}$, such that $q_\alpha \in G_\alpha$ for each
$\alpha \in A$ and $q_\alpha \notin G_\beta$, if $\alpha \neq \beta$. For each $\alpha \in A$, let
$U_\alpha = (f(G_\alpha) - \{p_\gamma : \gamma \neq \alpha, \gamma \in A\}) \cap V_\alpha$. Then $p_\alpha \in U_\alpha$, for each $\alpha \in A$
and $p_\alpha \notin U_\beta$, if $\alpha \neq \beta$. Let $U = \{U_\alpha : \alpha \in A\}$. Since V is point-
finite, U is point-finite. Assume U is not cs-finite. There exists
some convergent sequence $\{r_i\}$ in Y , say $r_i \to r$, such that
$A' = \{\alpha \in A : U_\alpha \cap \{r_i : i \in N\} \neq \emptyset\}$ is infinite. The point-finiteness of
U allows us to suppose $r_i \neq r$, for each $i \in N$. Let i_1 be the first
index such that $r_{i_1} \in U_{\alpha_1}$, for some $\alpha_1 \in A'$. Since U is point-
finite, there exists a second index i_2 such that $i_1 < i_2$, and
$r_{i_2} \in U_{\alpha_2}$ for some $\alpha_2 \in A' - \{\alpha_1\}$. Suppose this process has been
continued $n - 1$ times. That is, for each $j \in N$, such that
$1 < j \leq n-1$, i_j is an index such that $i_1 < i_2 < \ldots < i_j$ and
$r_{i_j} \in U_{\alpha_j}$ for some $\alpha_j \in A' - \{\alpha_1, \alpha_2, \ldots, \alpha_{j-1}\}$. Since U is point-
finite, there is an index i_n , such that $i_1 < i_2 < \ldots < i_{n-1} < i_n$ and
$r_{i_n} \in U_{\alpha_n}$ for some $\alpha_n \in A' - \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$. Then $\{r_{i_n}\}$ is a
subsequence of $\{r_i\}$, $r_{i_n} \in U_{\alpha_n}$ for each $n \in N$, and $U_{\alpha_n} \neq U_{\alpha_k}$, if
$k \neq n$. Since $r_{i_n} \in U_{\alpha_n} \subset f(G_{\alpha_n})$, for each $n \in N$, $f^{-1}(r_{i_n}) \cap G_{\alpha_n} \neq \emptyset$.
Let $t_{i_n} \in f^{-1}(r_{i_n}) \cap G_{\alpha_n}$, for each $n \in N$. Then $\{t_{i_n} : n \in N\}$ is not
closed, because $\{r_{i_n} : n \in N\}$ is not closed. Since X is sequential,
$\{t_{i_n} : n \in N\}$ is not sequentially closed. Hence, there exists a sequence
in $\{t_{i_n} : n \in N\}$ which converges to some point in $f^{-1}(r)$. This

sequence must intersect infinitely many of the sets G_{α_n} , contradicting
the cs-finiteness of $\{G_\alpha : \alpha \in A\}$. Accordingly, U is cs-finite.
This completes the proof. Note that if Y is regular, then Y is
strongly collectionwise Hausdorff, because Y is sequential.

THEOREM 3.4. *The closed and open continuous image of a normal
collectionwise Hausdorff space is a normal collectionwise Hausdorff space.*

Proof. Let $f : X \to Y$ be a closed and open continuous surjection
defined on a normal collectionwise Hausdorff space X . Let $\{p_\alpha : \alpha \in A\}$
be a discrete set in Y . If $q_\alpha \in f^{-1}(p_\alpha)$ for each $\alpha \in A$, then
$\{q_\alpha : \alpha \in A\}$ is a discrete set in X . Let $U = \{U_\alpha : \alpha \in A\}$ be a
discrete collection of open subsets of X such that $p_\alpha \in U_\alpha$ for each
$\alpha \in A$. Since f is closed, Y is normal and for each $\alpha \in A$ we let G_α
be an open subset of Y such that

$$p_\alpha \in G_\alpha \subset \mathrm{cl}(G_\alpha) \subset f(U_\alpha) - \{p_\beta : \beta \neq \alpha,\ \beta \in A\} \ .$$

Since U was discrete, $\{f(U_\alpha) : \alpha \in A\}$ is hereditarily closure
preserving. Accordingly, if $V_\alpha = G_\alpha - \cup\{\mathrm{cl}(G_\beta) : \beta \neq \alpha,\ \beta \in A\}$ for each
$\alpha \in A$, then $\{V_\alpha : \alpha \in A\}$ is a disjoint collection of open subsets of Y
such that $p_\alpha \in V_\alpha$ for each $\alpha \in A$. This completes the proof.

THEOREM 3.5. *In a first countable space, the set of points, S , at
which a point-finite collection of sets $G = \{G_\alpha : \alpha \in A\}$ fails to be
cs-finite is precisely the set of points at which G fails to be locally
finite. Thus S is closed.*

Proof. Let p be any point in S , and let $\{p_i\}$ be such that
$p_i \to p$ and $G_\alpha \cap \{p_i : i \in N\} \neq \emptyset$ for infinitely many $\alpha \in A$. Since G
is point-finite, there exists a subsequence $\left\{p_{i_n}\right\}$ of $\{p_i\}$ such that
$p_{i_n} \in G_{\alpha_n}$, for each $n \in N$ and $G_{\alpha_n} \neq G_{\alpha_k}$, if $n \neq k$. Since $p_{i_n} \to p$,
every neighborhood of p intersects infinitely many different $G_{\alpha_n} \in G$.

Hence G is not locally finite at p . To prove the converse, let p be
any point at which G is not locally finite, and let $\{U_i : i \in N\}$ be any
countable neighborhood base at p , in which $U_{i+1} \subset U_i$, for each $i \in N$.
For $i = 1$, let $\alpha_1 \in A$, such that $U_1 \cap G_{\alpha_1} \neq \emptyset$. Let $p_1 \in \dot{U}_1 \cap G_{\alpha_1}$.
Suppose that for each i , $1 \leq i \leq n-1$, $p_i \in U_i \cap G_{\alpha_i}$ and $G_{\alpha_i} \neq G_{\alpha_k}$,
for each k , $1 \leq k < i$. Since G is not locally finite at p ,
$U_n \cap G_{\alpha_n} \neq \emptyset$, for some $G_{\alpha_n} \neq G_{\alpha_k}$, for each k , $1 \leq k \leq n-1$. Let
$p_n \in U_n \cap G_{\alpha_n}$. The sequence $\{p_n\}$ converges to p and
$\{p_n : n \in N\} \cap G_\alpha \neq \emptyset$, for infinitely many $G_\alpha \in G$. Thus $p \in S$. It is
clear that the set of points at which a collection fails to be locally
finite is closed. This completes the proof.

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