15

Open Problems and Related Topics

In this final chapter we summarize the open problems we have encountered in the text and we add additional ones including some discussion on their significance. We conclude with a brief account of various related topics.

15.1 Open Problems

We start with a basic local uniqueness problem.

Open problem 1 Let (M, g) be a surface with boundary and let $x \in \partial M$ be a point such that the boundary is strictly convex near x. Let $\mathcal{O}$ be a sufficiently small open set containing x. Given a smooth function on $\mathcal{O}$ that integrates to zero along every geodesic in $\mathcal{O}$ running between boundary points, is it true that f must be zero near x?

For the case of a ball in the plane with the standard flat metric a positive answer is given by Theorem 1.2.9 and in dimensions ≥ 3 this question is resolved in Uhlmann and Vasy (2016).

The next three questions are for simple surfaces.

Open problem 2 Let $\mathcal{G}$ denote the set of C^∞ simple metrics g on the surface M. Describe the range of the scattering relation $g \mapsto \alpha_g$.

In general, very little is known about the range of non-linear forward maps such as the scattering relation. The description of the range is of importance when implementing numerical schemes for solving inverse problems, particularly when initializing algorithms. For the non-Abelian X-ray transform, a fairly satisfactory solution to the range description problem is given in Bohr and Paternain (2021) in terms of a non-linear analogue of the map

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P that appears in Proposition 9.6.1. This map is constructed using Birkhoff factorizations of invertible Hermitian first integrals.

Given the close connection between the scattering relation and the Calderón problem, as explained in Theorem 11.5.1, it should be mentioned that a description of the range for the Dirichlet-to-Neumann map for simply connected surfaces is given in Sharafutdinov (2011, Theorem 1.3).

Open problem 3 Let (M, g) be a simple surface and let $W: L^2(M) \to C^\infty(M)$ be the smoothing operator introduced in Chapter 9. Let $f \in L^2(M)$ be such that $Wf \pm if = 0$. Is it true that $f = 0$?

Note that by the arguments in Section 9.3 the question has a positive answer if g is sufficiently close to a metric of constant curvature in the C^3-topology, so that W becomes a contraction in L^2. If a positive answer holds for any simple surface, then in the formula from Theorem 9.4.11 we may solve for f in the left-hand side by inverting $\mathrm{Id} + W^2$ thus providing a full inversion formula for I_0.

Open problem 4 Let (M, g) be a simple surface and let $a \in \oplus_{-N}^{N} \Omega_k$ be an attenuation with finite vertical Fourier expansion. Is it true that $I_{a,0}$ is injective?

As we mentioned at the end of Chapter 12 there is no characterization of those weights ρ for which I_ρ is injective. Restricting to attenuations with finite Fourier content in the simple case seems to be a reasonable next step. Even an answer to the question for the case of $a \in \Omega_k$ for $k \neq 0, \pm 1$ would be of great interest.

We can of course ask all these questions for non-trapping surfaces with strictly convex boundary, but we limit ourselves to the most basic one.

Open problem 5 Let (M, g) be a compact non-trapping surface with strictly convex boundary. Is it true that I_0 is injective?

A solution to the local uniqueness problem would give an answer to this question by a layer stripping argument, using the fact that any surface (M, g) as above admits a strictly convex function (Betelú et al., 2002; Paternain et al., 2019).

15.2 Related Topics

In this text we have focused mostly on geodesic X-ray transforms and related rigidity questions on simple or non-trapping manifolds with strictly convex boundary, with an emphasis on the two-dimensional case. There are several ways in which one can relax these requirements and each one takes to an active avenue of research. There are also other related geometric inverse problems that have not been discussed in this text. In this section we briefly discuss some of these topics without being exhaustive.

X-ray transforms and boundary ridigity in dimensions $n \geq 3$. When $n = \dim M \geq 3$ the methods in Chapters 10–14 that were largely based on holomorphic integrating factors are not available. However, the problem of inverting the geodesic X-ray transform is formally overdetermined when $n \geq 3$ (the measurement If lives on a $(2n-2)$-dimensional manifold whereas the unknown f depends on n variables), and there is a set of methods that only applies when $n \geq 3$. One of the main results is Uhlmann and Vasy (2016), which states that the local geodesic X-ray transform is injective near any point where the boundary is strictly convex. By a layer stripping argument this implies that the X-ray transform is injective on strictly convex non-trapping manifolds that satisfy a foliation condition (i.e. admit a foliation by strictly convex hypersurfaces). Such manifolds may have conjugate points, but when $n \geq 3$ it is not known if simple manifolds satisfy the foliation condition.

The method in Uhlmann and Vasy (2016) is microlocal, and it is based on studying a localized normal operator in the scattering calculus of Melrose. This method is used in Stefanov et al. (2016, 2021) to study the boundary rigidity problem and X-ray transforms on 1- and 2-tensors on manifolds satisfying the foliation condition. The case of matrix weights is considered in Paternain et al. (2019), which also contains a detailed analysis of the foliation condition. There are several related results and we refer to the surveys Ilmavirta and Monard (2019); Stefanov et al. (2019) for references.

Analytic microlocal methods. In the study of X-ray transforms and related problems one may be able to obtain improved results if the underlying structures (the manifold, metric and weight) are real-analytic. The main idea is that, in this context, the normal operator of the X-ray transform is an elliptic analytic pseudodifferential operator, and it can be inverted modulo an analytic smoothing operator. One can then combine analyticity with infinite order vanishing at the boundary to show that the normal operator is injective, instead of just invertible modulo smoothing.

This scheme was employed in Boman and Quinto (1987) to show that the weighted Euclidean X-ray transform is invertible for real-analytic weights. In Stefanov and Uhlmann (2005) it was proved that the X-ray transform on 2-tensors is solenoidal injective on generic simple manifolds including real-analytic ones, and this was used to show local uniqueness and stability near generic simple metrics in the boundary rigidity problem. These results were extended to some non-simple real-analytic manifolds in Stefanov and Uhlmann

(2009). Local injectivity results for the X-ray transform on analytic simple manifolds are given in Krishnan (2009); Krishnan and Stefanov (2009).

Closed manifolds. There are well-known similarities between the main setting treated in this book – that of simple manifolds – and the case of closed manifolds with Anosov geodesic flows. In particular, the Pestov identity applies equally well in both settings and in the Anosov case the link with the transport equation is established via Livsic theorems. The geodesic X-ray transform in the closed case corresponds to integration along periodic geodesics, and tensor tomography problems appear naturally. Related inverse problems involve transparent connections, marked length spectral rigidity, and spectral rigidity.

Spectral rigidity of negatively curved surfaces goes back to Guillemin and Kazhdan (1980a), and this was extended to any dimension in Guillemin and Kazhdan (1980b); Croke and Sharafutdinov (1998). Marked length spectral rigidity for negatively curved surfaces was established in Otal (1990); Croke (1990). Spectral rigidity of closed Anosov surfaces is due to Paternain et al. (2014a), and the X-ray transform on tensors is studied in Guillarmou (2017a). New results on marked length spectral rigidity are given in Guillarmou and Lefeuvre (2019). Transparent connections were first studied in Paternain (2009) and further results are in Guillarmou et al. (2016). We refer the reader to Lefeuvre (2021) for a recent survey and more references on these topics.

Non-convex boundaries. If we drop the assumption that the boundary is strictly convex but we keep the non-trapping property, the exit time function τ may no longer be continuous and one can have glancing geodesics. However, some good progress has been made in this direction. For instance Stefanov and Uhlmann (2009) shows that it is possible to determine the jet of a Riemannian metric at the boundary from its (possibly discontinuous) lens data, while Dairbekov (2006) proves tensor tomography results. The more recent work Guillarmou et al. (2021) essentially manages to remove the strict convexity assumption in two dimensions for many of the results in the present text.

Trapping. Allowing for some form of trapping in geometric inverse problems presents considerable challenges. There is a particularly successful scenario in which one allows a specific form of trapping by demanding the trapped set to be a *hyperbolic* set for the geodesic flow. The work by Guillarmou (2017b) provides a major breakthrough in this direction for the lens rigidity problem. The non-Abelian X-ray transform in the presence of a hyperbolic trapped set is studied in Guillarmou et al. (2016). For other developments, see Guillarmou and Monard (2017); Guillarmou and Mazzucchelli (2018); Lefeuvre (2020).

When the trapped set is not assumed to be hyperbolic, very little is known. Notable exceptions are given in Croke (2014); Croke and Herreros (2016). In Croke (2014) the flat cylinder in any dimensions ≥ 3 is shown to be scattering rigid, while Croke and Herreros (2016) discuss the two-dimensional situation for lens rigidity (it turns out that the flat Möbius band is not scattering rigid).

Obstacles. Another interesting variation is the introduction of *obstacles* so that the geodesics reflect at their boundaries and one studies the geodesic X-ray transform over broken rays. The known injectivity results in this case are for non-positive curvature and when there is just one obstacle with strictly concave boundary (as seen from the manifold), see Ilmavirta and Salo (2016); Ilmavirta and Paternain (2020). A similar broken X-ray transform arises in the Calderón problem with partial data (Kenig and Salo, 2013, 2014) and there are related open questions even in the unit disk. See the thesis Ilmavirta (2014) for references to known results.

Non-compact manifolds. Most of the theory in this monograph is in the setting of compact manifolds with boundary. However, it is also natural to study geodesic X-ray transforms and inverse problems on non-compact manifolds and for functions satisfying certain decay conditions at infinity. The most classical case is $\mathbb{R}^n$ (see Chapter 1), and there are analogous results on homogeneous and symmetric spaces based on Fourier methods (Helgason, 2011). Geodesic X-ray transforms and rigidity questions have also been studied on Cartan–Hadamard manifolds (Lehtonen et al., 2018), asymptotically hyperbolic manifolds (Graham et al., 2019) and asymptotically conic manifolds (Guillarmou et al., 2020).

Curves other than geodesics. It would be natural to extend all this theory to more general classes of curves. By this we mean replacing geodesics by other natural set of curves like magnetic geodesics or geodesics of affine connections with torsion (thermostats). Concerning magnetic geodesics, the tensor tomography problem in two dimensions is solved in Ainsworth (2013) using the ideas presented here and the results in Dairbekov et al. (2007). See also Assylbekov and Dairbekov (2018).

Calderón problem. We have only discussed the Calderón problem of determining a metric g up to gauge from the Dirichlet-to-Neumann map Λ_g in the two-dimensional case. This problem is open in dimensions ≥ 3 but there are positive results when the metric is real-analytic (Lee and Uhlmann, 1989; Lassas and Uhlmann, 2001; Lassas et al., 2003a, 2020), or Einstein (Guillarmou and Sá Barreto, 2009). In the absence of real-analyticity, it is known that one can determine g in a fixed conformal class if one restricts to certain conformally transversally anisotropic manifolds (Dos Santos Ferreira et al., 2009, 2016). These works also address the problem of determining a potential $q(x)$ in the Schrödinger equation $(-\Delta_g + q)u = 0$ in M. Incidentally, the previous works employ the attenuated geodesic X-ray transform when recovering the coefficients. In the two-dimensional case the problem of determining a potential q has been solved on any compact Riemann surface with boundary, even with partial data (Guillarmou and Tzou, 2011). However, if one measures the Dirichlet and Neumann data on disjoint sets there are counterexamples to uniqueness (Daudé et al., 2019). There is a very large literature on various aspects of this problem in the Euclidean case. We refer to the survey Uhlmann (2014) for references.