A NOTE ABOUT LOCALLY SPHERICAL SPHERES

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1. Introduction. A 2-sphere S in E^3 is said to be locally spherical if for each point p in S and each $\epsilon > 0$ there is a 2-sphere S' such that $p \in \operatorname{Int} S'$, $S' \cap S$ is a continuum, and $\operatorname{Diam} S' < \epsilon$. It is not known whether locally spherical spheres are tame; however, there are several partial results. Burgess (**2**) showed that S is tame if $S' \cap S$ is a simple closed curve and Loveland (**3**) proved that S is tame if S can be side approximated missing the continuum $S \cap S'$. In this paper we demonstrate that S is tame if the continuum $S \cap S'$ is irreducible with respect to separating S. This result is stated more precisely in Theorem 3. Theorem 2, which is used in the proof of Theorem 3, is a generalization of a theorem recently proved by Loveland (**4**).

2. Preliminary definitions. A 2-sphere S in E^3 is said to be strongly locally spherical at point p if for each $\epsilon > 0$ there is a 2-sphere S' such that $p \in \operatorname{Int} S'$, $S \cap S'$ is a continuum irreducible with respect to separating S, and $\operatorname{Diam} S' < \epsilon$. A 2-sphere S is said to be strongly locally spherical if S is strongly locally spherical at each of its points.

3. Strongly locally spherical spheres are tame.

LEMMA 1. *If D is a disk, f is a map from D into E^3, $\{D_i\}_{i=1}^n$ is a finite collection of disjoint disks such that $f(\operatorname{Bd} D) \cap (\bigcup_{i=1}^n D_i) = \emptyset$, and C is the component of $D - f^{-1}(\bigcup_{i=1}^n D_i)$ that contains* Bd *D, then there is a map g from D into E^3 such that*

(1) *$g(x) \in \bigcup_{i=1}^n D_i$ if $x \in D - C$,*

(2) *$g(x) = f(x)$ if $x \in C$.*

Proof. This follows from the Tietze Extension Theorem as indicated in (**2**, Lemma 1).

THEOREM 2. *If a 2-sphere S in E^3 is locally spherical and $E^3 - S$ is simply connected, then S is tame.*

Proof. We show that each complementary domain of S is 1-ULC (uniformly locally simply connected) and use Bing's (**1**) characterization of tame spheres. Suppose that $\epsilon > 0$; then we may cover $S \cup \operatorname{Int} S$ with interiors of 2-spheres as follows:

(1) if $p \in \operatorname{Int} S$, there is a 2-sphere S' such that $p \in \operatorname{Int} S'$, $S' \subset \operatorname{Int} S$, and $\operatorname{Diam} S' < \epsilon$,

Received March 11, 1968. This research was supported in part by grant NSF GP–5420.

(2) if $p \in S$, there is a 2-sphere S' such that $p \in \operatorname{Int} S', S \cap S'$ is a continuum, and $\operatorname{Diam} S' < \epsilon$.

Let δ be a Lebesgue number for this covering. If f is a map of the boundary of a disk D such that $\operatorname{Diam} f(\operatorname{Bd} D) < \delta$ and $f(\operatorname{Bd} D) \subset \operatorname{Int} S$, then $f(\operatorname{Bd} D)$ is contained in the interior of a sphere of type (1) or type (2). If $f(\operatorname{Bd} D)$ is contained in the interior of a 2-sphere S', where S' satisfies (1), then, since $S' \cup \operatorname{Int} S'$ is simply connected, f can be extended over D so that $f(D) \subset \operatorname{Cl}(\operatorname{Int} S')$. If $f(\operatorname{Bd} D)$ is contained in the interior of S', where S' satisfies (2), then, since $\operatorname{Int} S$ is simply connected, f may be extended over D so that $f(D) \subset \operatorname{Int} S$. Since $S' \cap S$ is a continuum, we may cover $f(D) \cap S'$ with a finite collection of disjoint disks $\{D_i\}_{i=1}^n$ such that $D_i \subset (S' \cap \operatorname{Int} S)$. It follows from Lemma 1 that there is a map g such that $g(x) = f(x)$ for $x \in \operatorname{Bd} D$ and

$$g(D) \subset (\operatorname{Int} S \cap \operatorname{Cl}(\operatorname{Int} S')).$$

In each case, $\operatorname{Diam} \operatorname{Cl}(\operatorname{Int} S') < \epsilon$, thus $\operatorname{Int} S$ is 1-ULC. Similarly, the exterior of S is 1-ULC.

THEOREM 3. *A strongly locally spherical 2-sphere S in E^3 is tame.*

Proof. We show that $\operatorname{Int} S$ and $\operatorname{Ext} S$ are simply connected and then use Theorem 2. Let D be a disk and let f be a map of $\operatorname{Bd} D$ into $\operatorname{Int} S$. Since $\operatorname{Cl}(\operatorname{Int} S)$ is simply connected, there is a map f_0 from D into $\operatorname{Cl}(\operatorname{Int} S)$ such that $f = f_0|\operatorname{Bd} D$. Since $S \cap f_0(D)$ is compact and S is strongly locally spherical, there is a finite collection of 2-spheres $\{S_i\}_{i=1}^n$ and a finite collection of sets $\{R_i\}_{i=1}^n$ such that

(1) R_i is a component of $S \cap \operatorname{Int} S_i$,

(2) $\{R_i\}_{i=1}^n$ covers $f_0(D) \cap S$,

(3) $S_i \cap S$ is a continuum irreducible with respect to separating S, and

(4) $f(\operatorname{Bd} D) \subset \operatorname{Ext} S_i$.

We complete the proof by constructing a finite sequence of maps $\{f_i\}_{i=1}^n$ from D into $\operatorname{Cl}(\operatorname{Int} S)$ such that $f_i|\operatorname{Bd} D = f$ and $f_i(D) \cap S \subset \bigcup_{j=i+1}^n R_j$. The map f_n will then be an extension of f over D such that $f_n(D) \subset \operatorname{Int} S$.

Loveland has shown, in the proof of (**4**, Lemma 1), that there is a component R_i' of $S_i - S$ such that $\operatorname{Cl} R_i'$ separates R_i from $f_0(\operatorname{Bd} D)$ in $\operatorname{Cl}(\operatorname{Int} S)$. We assume that $S \not\subset f_0(D)$ and that the subscripts on the S_i's have been chosen so that $S_1 \cap S$ is not a subset of $f_0(D)$. Consequently, since $S_1 \cap S$ is irreducible with respect to separating S, the components of $f_0(D) \cap S_1 \cap S$ do not separate S; furthermore, we have $\operatorname{Bd} R_1 = \operatorname{Bd} R_1' = S_1 \cap S$. Applying the techniques of the topology of E^2, we cover $f_0(D) \cap \operatorname{Bd} R_1$ with the interiors of a finite collection $\{D_i\}_{i=1}^m$ of disjoint disks in S such that $\bigcup_{i=1}^m D_i \subset \bigcup_{i=2}^n R_i$. Let e be a point of $S_1 - \operatorname{Cl} R_1'$. Since the components of $f_0(D) \cap \operatorname{Bd} R_1'$ do not separate S_1, we can cover $f_0(D) \cap \operatorname{Bd} R_1'$ with the interiors of a finite collection $\{E_i\}_{i=1}^k$ of disjoint disks in S_1 so that $(\bigcup_{i=1}^k E_i) \cap S \subset \bigcup_{i=1}^m D_i$ and $e \notin (\bigcup_{i=1}^k E_i)$.

Since $A = (f_0(D) \cap \mathrm{Cl}\, R_1') - \bigcup_{i=1}^k \mathrm{Int}\, E_i$ is a compact subset of R_1', there is a disk E in R_1' which contains A in its interior. We assume, without loss of generality, that $\mathrm{Bd}\, E$ and $\bigcup_{i=1}^k \mathrm{Bd}\, E_i$ are in general position so that $E \cup (\bigcup_{i=1}^k E_i) = F$ is a disk with a finite number of holes. Note that $F \cap S \subset \bigcup_{i=1}^m D_i$, $f_0(D) \cap \mathrm{Cl}\, R_1' \subset \mathrm{Int}\, F$, $F \subset S_1$, and $e \notin F$.

We now adjust F so that it becomes a disk. Let H be the component of $S_1 - F$ that contains e and suppose that there is another component K of $S_1 - F$. If $K \cap S = \emptyset$, then adjust F by adding K to it. If $K \cap S \neq \emptyset$, then, since $S \cap S_1$ is the boundary of each component of $S_1 - S$, K intersects the component, L, of $S_1 - S$ that contains e. Let q be a point of $K \cap L$ and qe an arc from q to e in L that is in general position with $\mathrm{Bd}\, F$. Let b be that last point of $qe \cap \mathrm{Bd}\, K$ and c the first point of $be \cap (\mathrm{Bd}\, F - \mathrm{Bd}\, K)$. F is now adjusted by thickening the arc bc slightly and subtracting this open neighbourhood of bc from F. In each case, the adjusted F has one less hole than F. This process is continued until all holes have been eliminated. If G is the resulting disk, then $G \cap S \subset \bigcup_{i=1}^m D_i$ and $f_0(D) \cap \mathrm{Cl}\, R_1' \subset G$.

We now adjust the map f_0 in two steps to obtain map f_1. Since $\mathrm{Cl}\, R_1'$ separates R_1 from $f_0(\mathrm{Bd}\, D)$ in $\mathrm{Cl}(\mathrm{Int}\, S)$, $f_0(D) \cap \mathrm{Cl}\, R_1' \subset G$, and

$$G \cap (R_1 - (\bigcup_{i=2}^n R_i)) = \emptyset,$$

it follows that G separates $f_0(D) \cap (R_1 - (\bigcup_{i=2}^n R_i))$ from $f_0(\mathrm{Bd}\, D)$ in $f_0(D)$. Let C be the component of $D - f_0^{-1}(G)$ that contains $\mathrm{Bd}\, D$. It follows from Lemma 1 that there is a map g from D into E^3 such that $g(x) = f_0(x)$ for $x \in C$ and $g(x) \in G$ for $x \in D - C$. The exterior of S may intersect $g(D)$; however, $\bigcup_{i=1}^m D_i$ separates $g(D) \cap \mathrm{Ext}\, S$ from $g(\mathrm{Bd}\, D)$ in $g(D)$. If J is the component of $D - g^{-1}(\bigcup_{i=1}^m D_i)$ that contains $\mathrm{Bd}\, D$, then by Lemma 1 there is a map f_1 from D into E^3 such that $f_1(x) = g(x)$ for $x \in J$ and $f_1(x) \in \bigcup_{i=1}^m D_i$ for $x \in D - J$. Now $f_1(x) = f_0(x)$ for $x \in \mathrm{Bd}\, D$, $f_1(D) \subset \mathrm{Cl}(\mathrm{Int}\, S)$, and $f_1(D) \cap S \subset \bigcup_{j=2}^n R_j$ since $\bigcup_{i=1}^m D_i \subset \bigcup_{i=2}^n R_i$.

Repeating the above procedure we obtain, inductively, maps $f_2, \ldots, f_n$ from D into $\mathrm{Cl}(\mathrm{Int}\, S)$ such that $f_i | \mathrm{Bd}\, D = f$ and $f_i(D) \cap S \subset \bigcup_{j=i+1}^n R_j$. The map f_n is an extension of f over D such that $f_n(D) \subset \mathrm{Int}\, S$.

References

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