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Kiguradze-type Oscillation Theorems for Second Order Superlinear Dynamic Equations on Time Scales

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Abstract. Consider the second order superlinear dynamic equation

$$x^{\Delta\Delta}(t) + p(t)f(x(\sigma(t))) = 0 \tag{$*$}$$

where $p \in C(\mathbb{T}, \mathbb{R})$, $\mathbb{T}$ is a time scale, $f\colon \mathbb{R} \to \mathbb{R}$ is continuously differentiable and satisfies $f'(x) > 0$, and $xf(x) > 0$ for $x \neq 0$. Furthermore, $f(x)$ also satisfies a superlinear condition, which includes the nonlinear function $f(x) = x^\alpha$ with $\alpha > 1$, commonly known as the Emden–Fowler case. Here the coefficient function $p(t)$ is allowed to be negative for arbitrarily large values of t. In addition to extending the result of Kiguradze for $(*)$ in the real case $\mathbb{T} = \mathbb{R}$, we obtain analogues in the difference equation and q-difference equation cases.

1 Introduction

Consider the second order superlinear dynamic equation

$$x^{\Delta\Delta}(t) + p(t)f(x^\sigma(t)) = 0, \tag{1.1}$$

where $p \in C(\mathbb{T}, \mathbb{R})$, $\mathbb{T}$ is a time scale, $f\colon \mathbb{R} \to \mathbb{R}$ is continuously differentiable and satisfies $f'(x) > 0$, and $xf(x) > 0$ for $x \neq 0$. The prototype of equation (1.1) is the so-called superlinear Emden–Fowler equation

$$x^{\Delta\Delta}(t) + p(t)x^\alpha(\sigma(t)) = 0,$$

where $\alpha > 1$ is the quotient of odd positive integers. Here we are interested in the oscillation of solutions of (1.1) when $f(x)$ satisfies, in addition, the superlinearity conditions

$$0 < \int_\epsilon^\infty \frac{dx}{f(x)}, \quad \int_{-\infty}^{-\epsilon} \frac{dx}{f(x)} < \infty, \quad \text{for all } \epsilon > 0. \tag{1.2}$$

Examples of $f(x)$ satisfying (1.2), which are not of Emden–Fowler type, are

$$f(x) = \sum_{i=1}^n a_i x^{\alpha_i},$$

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where the constants $a_i > 0$, $1 \le i \le n$, and the α_i are all quotients of odd positive integers, with $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$, and $\alpha_n > 1$.

When $\mathbb{T} = \mathbb{R}$, the dynamic equation (1.1) is the second order superlinear differential equation

$$x''(t) + p(t)f(x(t)) = 0. \tag{1.3}$$

When $p(t)$ is nonnegative, Atkinson [1] established a necessary and sufficient condition for the oscillation of (1.3), which is

$$\int^{\infty} t p(t)\, dt = \infty. \tag{1.4}$$

When $p(t)$ is allowed to take on negative values, Kiguradze [10] proved that (1.4) is sufficient for all solutions of the differential equation (1.3) to be oscillatory for the same case considered by Atkinson.

When $\mathbb{T} = \mathbb{Z}_0$, the dynamic equation (1.1) is the second order superlinear difference equation

$$\Delta^2 x(n) + p(n)x^{\alpha}(n+1) = 0. \tag{1.5}$$

When $\alpha > 1$ and $p(n)$ is nonnegative, J. W. Hooker and W. T. Patula [8, Theorem 4.1], and A. Mingarelli [11], respectively, proved that

$$\sum^{\infty} n p(n) = \infty \tag{1.6}$$

is a necessary and sufficient condition for the oscillation of all solutions of the difference equation (1.5) (also see [9, Theorem 6.23] for a related result).

In this paper, we obtain Kiguradze-type oscillation theorems (Theorems 2.2, 2.5, and 2.3) for (1.1). In particular, for the case when $\mathbb{T} = \mathbb{R}$, and for f satisfying (1.2), condition (1.4) implies that all solutions of (1.3) are oscillatory, which is a substantial improvement of Kiguradze's result. We also note that the proof is essentially different from that of Kiguradze. As a special case, we get that with no sign assumption on $p(n)$, the condition (1.6) is sufficient for the oscillation of the difference equation (1.5). To be precise, we prove that the superlinear difference equation

$$\Delta^2 x(n) + p(n)x^{\alpha}(n+1) = 0,$$

is oscillatory, if there exists a real number β, $0 < \beta \le 1$ such that $\sum_{n=1}^{\infty} n^{\beta} p(n) = \infty$. Moreover, it follows from our results that all solutions of the superlinear q-difference equation $x^{\Delta\Delta}(t) + p(t)x^{\alpha}(qt) = 0$, where $t \in q^{\mathbb{N}_0}$, $q > 1$, are oscillatory, if there exists a real number β, $0 < \beta \le 1$ such that

$$\int_1^{\infty} t^{\beta} p(t) \Delta t = \infty.$$

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In particular, under the assumption (1.2), we can show that the difference equation

$$\Delta^2 x(n) + \left[\frac{a}{n^b(n+1)} + \frac{c(-1)^n}{n^b}\right] f(x(n+1)) = 0,$$

for $a > 0$, $0 < b \leq 1$, is oscillatory. In [4], this result is shown to be true only for $0 < b < 1$ and $0 < bc < a < c(1-b)$, since the condition (A), that is, the condition

$$\limsup_{t\to\infty} \int_T^t p(s)\Delta s \geq 0$$

and not identically zero for all sufficiently large T, was necessary in the proof.

For completeness, (see [5,6] for elementary results for the time scale calculus), we recall some basic results for dynamic equations and the calculus on time scales. Let $\mathbb{T}$ be a time scale, *i.e.,* a closed nonempty subset of $\mathbb{R}$, with $\sup \mathbb{T} = \infty$. The forward jump operator is defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, and the backward jump operator is defined by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$, where $\sup \varnothing = \inf \mathbb{T}$, where $\varnothing$ denotes the empty set. If $\sigma(t) > t$, we say t is right-scattered, while if $\rho(t) < t$, we say t is left-scattered. If $\sigma(t) = t$, we say t is right-dense, while if $\rho(t) = t$ and $t \neq \inf \mathbb{T}$, we say t is left-dense. Given a time scale interval $[c,d]_{\mathbb{T}} := \{t \in \mathbb{T} : c \leq t \leq d\}$ in $\mathbb{T}$ the notation $[c,d]^{\kappa}{}_{\mathbb{T}}$ denotes the interval $[c,d]_{\mathbb{T}}$ in case $\rho(d) = d$ and denotes the interval $[c,d)_{\mathbb{T}}$ in case $\rho(d) < d$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$, and for any function $f\colon \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. We say that $x\colon \mathbb{T} \to \mathbb{R}$ is differentiable at $t \in \mathbb{T}$ provided

$$x^{\Delta}(t) := \lim_{s\to t} \frac{x(t) - x(s)}{t - s},$$

exists when $\sigma(t) = t$ (here by $s \to t$ it is understood that s approaches t in the time scale) and when x is continuous at t and $\sigma(t) > t$

$$x^{\Delta}(t) := \frac{x(\sigma(t)) - x(t)}{\mu(t)}.$$

Note that if $\mathbb{T} = \mathbb{R}$, then the delta derivative is just the standard derivative, and when $\mathbb{T} = \mathbb{Z}$ the delta derivative is just the forward difference operator. Hence our results contain the discrete and continuous cases as special cases and generalize these results to arbitrary time scales.

2 Main Theorem

In the case when $\mathbb{T}$ is such that $\mu(t)$ is not eventually identically zero, we define the set of all right-scattered points by $\hat{\mathbb{T}} := \{t \in \mathbb{T} : \mu(t) > 0\}$ and note that $\hat{\mathbb{T}}$ is necessarily countable. We let χ denote the characteristic function of $\hat{\mathbb{T}}$. The following condition, which will be needed later, imposes a lower bound on the graininess function $\mu(t)$, for $t \in \hat{\mathbb{T}}$. More precisely, we introduce the following (see [7]).

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Condition (C) We say that $\mathbb{T}$ satisfies condition (C) if there is an $M > 0$ such that

$$\chi(t) \leq M\mu(t), \quad t \in \mathbb{T}.$$

We note that if $\mathbb{T}$ satisfies condition (C), then the subset $\check{\mathbb{T}}$ of $\mathbb{T}$ defined by

$$\check{\mathbb{T}} = \{t \in \mathbb{T} \mid t > 0 \text{ is right-scattered or left-scattered}\}$$

is also necessarily countable and, of course, $\hat{\mathbb{T}} \subset \check{\mathbb{T}}$. So we can suppose that

$$\check{\mathbb{T}} = \{t_i \in \mathbb{T} \mid 0 < t_1 < t_2 < \cdots < t_n < \cdots\}.$$

We will need the following second mean value theorem (see [6, Theorem 5.45]).

Lemma 2.1 *Let h be a bounded function that is integrable on $[a, b]_{\mathbb{T}}$. Let m_H and M_H be the infimum and supremum, respectively, of the function $H(t) := \int_a^t h(s)\Delta s$ on $[a, b]_{\mathbb{T}}$. Suppose that g is nonincreasing with $g(t) \geq 0$ on $[a, b]_{\mathbb{T}}$. Then there is some number Λ with $m_H \leq \Lambda \leq M_H$ such that*

$$\int_a^b h(t)g(t)\Delta t = g(a)\Lambda.$$

To clarify the arguments below, we let $A := \{n \in \mathbb{N} : (t_{n-1}, t_n) \subset \mathbb{T}\}$ so that we can write $\mathbb{T} = \check{\mathbb{T}} \cup [\bigcup_{n \in A}(t_{n-1}, t_n)]$.

Theorem 2.2 *Assume that $\mathbb{T}$ satisfies condition (C) and that f satisfies* (1.2)*. Let*

$$\check{\mathbb{T}} = \{t_i \in \mathbb{T} \mid 0 < t_1 < t_2 < \cdots < t_n < \cdots\}.$$

If there exists a real number β, $0 < \beta \leq 1$ such that

$$\int_{t_1}^{\infty} (\sigma(t))^{\beta} p(t)\Delta t = \infty,$$

then (1.1) *is oscillatory.*

Proof Assume that (1.1) is nonoscillatory. Then without loss of generality there is a solution $x(t)$ of (1.1) and a $T \in \mathbb{T}$ with $x(t) > 0$, for all $t \in [T, \infty)_{\mathbb{T}}$. Multiplying (1.1) by $\frac{(\sigma(t))^{\beta}}{f(x(\sigma(t)))}$, integrating from T to t, and using integration by parts [5, Theorem 1.77(v)] on the first term we get

$$\frac{t^{\beta}x^{\Delta}(t)}{f(x(t))} - \frac{T^{\beta}x^{\Delta}(T)}{f(x(T))} - \int_T^t \Big(\frac{s^{\beta}}{f(x(s))}\Big)^{\Delta_s} x^{\Delta}(s)\Delta s + \int_T^t (\sigma(s))^{\beta} p(s)\Delta s = 0.$$

Then using the quotient rule [5, Theorem 1.20] and the Pötzsche chain rule [5, Theorem 1.90], we get

$$\begin{aligned} \text{(2.1)} \quad & \frac{t^{\beta}x^{\Delta}(t)}{f(x(t))} - \frac{T^{\beta}x^{\Delta}(T)}{f(x(T))} - \int_T^t \frac{(s^{\beta})^{\Delta_s}x^{\Delta}(s)}{f(x(\sigma(s)))}\Delta s, \\ & + \int_T^t \frac{s^{\beta}\int_0^1 f'(x_h(s))\,dh[x^{\Delta}(s)]^2}{f(x(s))f(x(\sigma(s)))}\Delta s + \int_T^t (\sigma(s))^{\beta} p(s)\Delta s = 0, \end{aligned}$$

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where $x_h(t) = x(t) + h\mu(t)x^\Delta(t) = (1-h)x(t) + hx(\sigma(t)) > 0$.

Since $0 < \beta \le 1$, one can use the Pötzsche chain rule to show that $(t^\beta)^\Delta$ is nonincreasing. Using the second mean value theorem (Lemma 2.1) we get that for each $t \in [T,\infty)_\mathbb{T}$

$$\int_T^t \frac{(s^\beta)^\Delta x^\Delta(s)}{f(x(\sigma(s)))}\Delta s = (s^\beta)^\Delta|_{s=T}\Lambda(t), \tag{2.2}$$

where $m_x \le \Lambda(t) \le M_x$, and where m_x and M_x denote the infimum and supremum, respectively, of the function $\int_T^s \frac{x^\Delta(\tau)}{f(x(\sigma(\tau)))}\Delta\tau$ for $s \in [T,t]_\mathbb{T}$.

Let $F(x) := \int_x^\infty \frac{dv}{f(v)}$. In the following, we will obtain an estimate for M_x, *i.e.*, an upper bound for the function $\int_T^t \frac{x^\Delta(s)}{f(x(\sigma(s)))}\Delta s$.

Assume that $t = t_{i-1} < t_i = \sigma(t)$. Then

$$\int_t^{\sigma(t)} \frac{x^\Delta(s)}{f(x(\sigma(s)))}\Delta s = \frac{x^\Delta(t)\mu(t)}{f(x(\sigma(t)))} = \frac{x(\sigma(t)) - x(t)}{f(x(\sigma(t)))}. \tag{2.3}$$

We consider the two possible cases:

(i) $x(t) \le x(\sigma(t))$,

(ii) $x(t) > x(\sigma(t))$.

First, if $x(t) \le x(\sigma(t))$, we have that

$$\frac{x(\sigma(t)) - x(t)}{f(x(\sigma(t)))} \le \int_{x(t)}^{x(\sigma(t))} \frac{1}{f(v)}\,dv = F(x(t)) - F(x(\sigma(t))), \tag{2.4}$$

since f is increasing. On the other hand, if $x(t) > x(\sigma(t))$, then

$$\frac{x(t) - x(\sigma(t))}{f\big(x(\sigma(t))\big)} \ge \int_{x(\sigma(t))}^{x(t)} \frac{1}{f(v)}\,ds = F(x(\sigma(t))) - F(x(t)),$$

which implies that

$$\frac{x(\sigma(t)) - x(t)}{f(x(\sigma(t)))} \le F(x(t)) - F(x(\sigma(t))). \tag{2.5}$$

Hence, whenever $t_{i-1} = t < \sigma(t) = t_i$, we have from (2.3) and (2.4) in the first case and (2.3) and (2.5) in the second case, that

$$\int_{t_{i-1}}^{t_i} \frac{x^\Delta(s)}{f(x(\sigma(s)))}\Delta s \le F(x(t_{i-1})) - F(x(t_i)). \tag{2.6}$$

If the real interval $[t_{i-1}, t_i] \subset \mathbb{T}$, then

$$\int_{t_{i-1}}^{t_i} \frac{x^\Delta(s)}{f(x(s))}\Delta s = \int_{x(t_{i-1})}^{x(t_i)} \frac{1}{f(v)}\,dv = F(x(t_{i-1})) - F(x(t_i)), \tag{2.7}$$

and so (2.6) also holds in this case.

Note that since $\mathbb{T}$ satisfies condition (C), we have from (2.6), (2.7), and the additivity of the integral that for $t \in [T, \infty)_{\mathbb{T}}$

$$\int_T^t \frac{x^\Delta(s)}{f(x(\sigma(s)))} \Delta s \le F(x(T)) - F(x(t)) \le F(x(T)).$$

So

$$\Lambda(t) \le M_x \le F(x(T)) \tag{2.8}$$

for $t \in [T, \infty)_{\mathbb{T}}$. From (2.1), (2.2), and (2.8), we have that

$$\frac{t^\beta x^\Delta(t)}{f(x(t))} - \frac{T^\beta x^\Delta(T)}{f(x(T))} - (s^\beta)^\Delta|_{s=T} F(x(T)) \\ + \int_T^t \frac{s^\beta \int_0^1 f'(x_h(s))\, dh [x^\Delta(s)]^2}{f(x(s)) f(x(\sigma(s)))} \Delta s + \int_T^t (\sigma(s))^\beta p(s) \Delta s \le 0.$$

Since $\int_T^\infty (\sigma(s))^\beta p(s) \Delta s = \infty$, there exists a sufficiently large $T_1 > T$ such that for $t \ge T_1$

$$\begin{aligned} \frac{t^\beta x^\Delta(t)}{f(x(t))} + \int_{T_1}^t \frac{s^\beta \int_0^1 f'(x_h(s)) dh [x^\Delta(s)]^2}{f(x(s)) f(x(\sigma(s)))} \Delta s & \\ \le \frac{t^\beta x^\Delta(t)}{f(x(t))} + \int_T^t \frac{s^\beta \int_0^1 f'(x_h(s))\, dh [x^\Delta(s)]^2}{f(x(s)) f(x(\sigma(s)))} \Delta s & \\ \le -\int_T^t (\sigma(s))^\beta p(s) \Delta s + \frac{T^\beta x^\Delta(T)}{f(x(T))} + (s^\beta)^\Delta|_{s=T} F(x(T)) & \\ < -1. & \end{aligned} \tag{2.9}$$

In particular, we have $x^\Delta(t) < 0$, for $t \ge T_1$. Therefore, $x(t)$ is strictly decreasing.

Assume that $t = t_{i-1} < t_i = \sigma(t)$, *i.e.*, $t_{i-1}, t_i \in \check{\mathbb{T}}$. Then $x(\sigma(t)) < x(t)$, so

$$\begin{aligned} \int_0^1 f'(x_h(s))\, dh &= \int_0^1 f'((1-h)x(s) + h(x(\sigma(s))))\, dh \\ &= \frac{f((1-h)x(s) + h(x(\sigma(s))))|_0^1}{x(\sigma(s)) - x(s)} = \frac{f(x(\sigma(s))) - f(x(s))}{x(\sigma(s)) - x(s)}. \end{aligned} \tag{2.10}$$

If the real interval $[t_{i-1}, t_i] \subset \mathbb{T}$, then for $s \in [t_{i-1}, t_i]$ we have

$$\int_0^1 f'(x_h(s))\, dh = f'(x(s)). \tag{2.11}$$

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Let

$$y(t) := 1 + \int_{T_1}^{t} \frac{s^\beta \int_0^1 f'(x_h(s))dh[x^\Delta(s)]^2}{f(x(s))f(x(\sigma(s)))}\Delta s. \tag{2.12}$$

Hence from (2.9), we get that

$$-\frac{t^\beta x^\Delta(t)}{f(x(t))} > y(t). \tag{2.13}$$

From (2.12) and (2.13), we get that

$$y^\Delta(t) = \frac{t^\beta \int_0^1 f'(x_h(t))dh[x^\Delta(t)]^2}{f(x(t))f(x(\sigma(t)))} > y(t)\frac{\int_0^1 f'(x_h(t))dh[-x^\Delta(t)]}{f(x(\sigma(t)))}. \tag{2.14}$$

Assume that $t = t_{i-1} < t_i = \sigma(t)$. From (2.14) and (2.10), we get that

$$\frac{y(\sigma(t)) - y(t)}{y(t)(\sigma(t) - t)} > \frac{f(x(\sigma(t))) - f(x(t))}{x(\sigma(t)) - x(t)} \cdot \frac{x(t) - x(\sigma(t))}{f(x(\sigma(t)))[\sigma(t) - t]}.$$

So

$$\frac{y(\sigma(t))}{y(t)} > \frac{f(x(t))}{f(x(\sigma(t)))}.$$

That is

$$\frac{y(t_i)}{y(t_{i-1})} > \frac{f(x(t_{i-1}))}{f(x(t_i))}. \tag{2.15}$$

If the real interval $[t_{i-1}, t_i] \subset \mathbb{T}$, then for $t \in (t_{i-1}, t_i]$ it follows from (2.14) and (2.11) that

$$\frac{y'(t)}{y(t)} > \frac{f'(x(t))[-x'(t)]}{f(x(t))},$$

that is $(\ln y(t))' > -(\ln f(x(t)))'$. Integrating from t_{i-1} to t, we get that

$$\frac{y(t)}{y(t_{i-1})} > \frac{f(x(t_{i-1}))}{f(x(t))}, \quad t \in (t_{i-1}, t_i]. \tag{2.16}$$

Let $T_1 = t_{n_0}$ and let $t \in (T_1, \infty)_\mathbb{T}$. Then there is an $n > n_0$ such that $t \in (t_{n-1}, t_n]_\mathbb{T}$. From (2.16) and (2.15), we get that

$$\frac{y(t)}{y(t_{n-1})} > \frac{f(x(t_{n-1}))}{f(x(t))}, \ \frac{y(t_{n-1})}{y(t_{n-2})} > \frac{f(x(t_{n-2}))}{f(x(t_{n-1}))}, \dots, \frac{y(t_{n_0+1})}{y(t_{n_0})} > \frac{f(x(t_{n_0}))}{f(x(t_{n_0+1}))}.$$

Multiplying, we get that

$$\frac{y(t)}{y(t_{n_0})} > \frac{f(x(t_{n_0}))}{f(x(t))}.$$

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Using (2.13) again, we get

$$-\frac{t^{\beta}x^{\Delta}(t)}{f(x(t))} > y(t) > \frac{y(t_{n_0})f(x(t_{n_0}))}{f(x(t))}.$$

If we set $L := y(t_{n_0})f(x(t_{n_0}))$, we get $x^{\Delta}(t) < -\frac{L}{t^{\beta}}$. Integrating from T_1 to t and using [6, Theorem 5.68], we get that

$$x(t) - x(T_1) < -\int_{T_1}^{t} \frac{L}{s^{\beta}}\Delta s \longrightarrow -\infty, \quad \text{as } t \longrightarrow \infty.$$

Therefore $x(t) < 0$, for large t, which is a contradiction. Thus equation (1.1) is oscillatory. ■

When $\mathbb{T} = \mathbb{R}$, the following corollary is an extension of Kiguradze's theorem (the term x^{α}, $\alpha > 1$, is replaced by f satisfying (1.2)). The proof, as noted earlier, is different from that of Kiguradze [10].

Corollary 2.3 *Assume $f(x)$ satisfies* (1.2). *If there exists a real number β, $0 < \beta \leq 1$ such that $\int_1^{\infty} t^{\beta}p(t)dt = \infty$, then* (1.3) *is oscillatory.*

As a consequence of Theorem 2.2, it follows that (1.5) is oscillatory if

$$\sum^{\infty}(n+1)^{\beta}p(n) = \infty$$

for some $0 < \beta \leq 1$. We would like to show that in Theorem 2.2, the assumption that $\int_{t_1}^{\infty}(\sigma(t))^{\beta}p(t)\Delta t = \infty$ can be replaced by $\int_{t_1}^{\infty} t^{\beta}p(t)\Delta t = \infty$ (where $0 < \beta \leq 1$). This would then imply, in particular, that the condition (1.6) implies oscillation of all solutions of (1.5), which is the desired improvement of the Hooker–Patula–Mingarelli result mentioned earlier. In order to extend Theorem 2.2, we will need to restrict our attention to isolated time scales. That is, we assume that $\rho(t) < t < \sigma(t)$ for all $t > \inf \mathbb{T}$. We shall also need the additional assumption that $(\rho^{\beta}(t))^{\Delta}$ is *nonincreasing* for $0 < \beta \leq 1$. Clearly, if $\mathbb{T} = \mathbb{Z}_0$ or $\mathbb{T} = q^{\mathbb{N}_0}$, then it is easy to see that $(\rho^{\beta}(t))^{\Delta}$ is nonincreasing for $0 < \beta \leq 1$. However, the following example shows that this need not hold for arbitrary isolated time scales.

Example 2.4 Let $\mathbb{T} = \bigcup_{k=1}^{\infty}\{4k+1, 4k+2, 4k+3\}$. Then on $\mathbb{T}$ we can show that

$$(\rho^{\beta}(t))^{\Delta}|_{t=4k-1} < (\rho^{\beta}(t))^{\Delta}|_{t=4k+1}. \tag{2.17}$$

To see this, we claim that

$$\frac{\rho^{\beta}(4k+1) - \rho^{\beta}(4k-1)}{(4k+1)-(4k-1)} < \frac{\rho^{\beta}(4k+2) - \rho^{\beta}(4k+1)}{(4k+2)-(4k+1)}.$$

This is equivalent to

$$\frac{(4k-1)^{\beta} - (4k-2)^{\beta}}{2} < (4k+1)^{\beta} - (4k-1)^{\beta},$$

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which implies

$$\left(1-\frac{1}{4k}\right)^{\beta}-\left(1-\frac{2}{4k}\right)^{\beta}<2\left[\left(1+\frac{1}{4k}\right)^{\beta}-\left(1-\frac{1}{4k}\right)^{\beta}\right]. \tag{2.18}$$

By the Taylor expansion, it is easy to see that the left side of (2.18) is $\frac{\beta}{4k}+o(\frac{1}{k})$, whereas the right side is $\frac{\beta}{k}+o(\frac{1}{k})$. Therefore, (2.17) holds for large k.

We now state and prove the following theorem.

Theorem 2.5 *Assume that $\mathbb{T}$ is an isolated time scale which satisfies condition (C), and without loss of generality, assume that $\mathbb{T}=\{t_i\}_{i=1}^{\infty}$ where $0<t_1<t_2<\cdots<t_n<\dots$ with $t_n\to\infty$. Suppose further that f satisfies* (1.2). *If there exists a real number β, $0<\beta\leq 1$ such that the delta derivative $(\rho^{\beta}(t))^{\Delta}$ is nonincreasing and*

$$\int_{t_1}^{\infty} t^{\beta}p(t)\Delta t=\infty,$$

then (1.1) *is oscillatory.*

Proof Assume that (1.1) is nonoscillatory. Then without loss of generality, there is a solution $x(t)$ of (1.1) and a $T\in\mathbb{T}$ with $x(t)>0$, for all $t\in[T,\infty)_{\mathbb{T}}$. Multiplying (1.1) by $\frac{t^{\beta}}{f(x(\sigma(t)))}=\frac{(\rho(\sigma(t)))^{\beta}}{f(x(\sigma(t)))}$, integrating from T to t, and using integration by parts [5, Theorem 1.77 (v)] on the first term we get

$$\frac{\rho^{\beta}(t)x^{\Delta}(t)}{f(x(t))}-\frac{\rho^{\beta}(T)x^{\Delta}(T)}{f(x(T))}-\int_T^t\left(\frac{\rho^{\beta}(s)}{f(x(s))}\right)^{\Delta_s}x^{\Delta}(s)\Delta s+\int_T^t s^{\beta}p(s)\Delta s=0.$$

Using the quotient rule [5, Theorem 1.20] and the Pötzsche chain rule [5, Theorem 1.90], we get

$$\frac{\rho^{\beta}(t)x^{\Delta}(t)}{f(x(t))}-\frac{\rho^{\beta}(T)x^{\Delta}(T)}{f(x(T))}-\int_T^t\frac{(\rho^{\beta}(s))^{\Delta_s}x^{\Delta}(s)}{f(x(\sigma(s)))}\Delta s,$$
$$+\int_T^t\frac{\rho^{\beta}(s)\int_0^1 f'(x_h(s))dh[x^{\Delta}(s)]^2}{f(x(s))f(x(\sigma(s)))}\Delta s+\int_T^t s^{\beta}p(s)\Delta s=0,$$

where $x_h(t)=x(t)+h\mu(t)x^{\Delta}(t)=(1-h)x(t)+hx(\sigma(t))>0$.

By assumption, we have that $(\rho^{\beta}(t))^{\Delta}$ is nonincreasing. Using the second mean value theorem (Lemma 2.1) we get that

$$\int_T^t\frac{(\rho^{\beta}(s))^{\Delta}x^{\Delta}(s)}{f(x(\sigma(s)))}=(\rho^{\beta}(s))^{\Delta}|_{s=T}\Lambda(t),$$

where $m_x\leq\Lambda(t)\leq M_x$, and where m_x and M_x denote the infimum and supremum, respectively, of the function $\int_T^s\frac{x^{\Delta}(\tau)}{f(x(\sigma(\tau)))}\Delta\tau$ on $[T,t]_{\mathbb{T}}$.

As in Theorem 2.2 we have that

$$\frac{\rho^\beta(t)x^\Delta(t)}{f(x(t))} - \frac{\rho^\beta(T)x^\Delta(T)}{f(x(T))} - (\rho^\beta(s))^\Delta|_{s=T}F(x(T)) + \int_T^t \frac{\rho^\beta(s)\int_0^1 f'(x_h(s))dh[x^\Delta(s)]^2}{f(x(s))f(x(\sigma(s)))}\Delta s + \int_T^t s^\beta p(s)\Delta s \le 0.$$

Since $\int_T^\infty s^\beta p(s)\Delta s = \infty$, there exists $T_1 > T$, sufficiently large, such that for $t \ge T_1$ we have

$$\begin{aligned}(2.19)\quad \frac{\rho^\beta(t)x^\Delta(t)}{f(x(t))} + \int_{T_1}^t \frac{\rho^\beta(s)\int_0^1 f'(x_h(s)dh[x^\Delta(s)]^2}{f(x(s))f(x(\sigma(s)))}\Delta s &\le \frac{\rho^\beta(t)x^\Delta(t)}{f(x(t))} + \int_T^t \frac{\rho^\beta(s)\int_0^1 f'(x_h(s))dh[x^\Delta(s)]^2}{f(x(s))f(x(\sigma(s)))}\Delta s \\ &\le -\int_T^t s^\beta p(s)\Delta s + \frac{\rho^\beta(T)x^\Delta(T)}{f(x(T))} + (\rho^\beta(s))^\Delta|_{s=T}F(x(T)) \\ &< -1.\end{aligned}$$

In particular, we get that $x^\Delta(t) < 0$, for $t \ge T_1$. Therefore, $x(t)$ is strictly decreasing. Similar to the proof of Theorem 2.2 we let

$$(2.20)\qquad y(t) := 1 + \int_{T_1}^t \frac{\rho^\beta(s)\int_0^1 f'(x_h(s))dh[x^\Delta(s)]^2}{f(x(s))f(x(\sigma(s)))}\Delta s.$$

Then from (2.19), we get that

$$(2.21)\qquad -\frac{\rho^\beta(t)x^\Delta(t)}{f(x(t))} > y(t).$$

From (2.20) and (2.21), we get that

$$y^\Delta(t) = \frac{\rho^\beta(t)\int_0^1 f'(x_h(t))dh[x^\Delta(t)]^2}{f(x(t))f(x(\sigma(t)))} > y(t)\frac{\int_0^1 f'(x_h(t))dh[-x^\Delta(t)]}{f(x(\sigma(t)))}.$$

Again proceeding as in the proof of Theorem 2.2 we get, using (2.21),

$$-\frac{\rho^\beta(t)x^\Delta(t)}{f(x(t))} > y(t) > \frac{y(t_{n_0})f(x(t_{n_0}))}{f(x(t))},$$

where n_0 is chosen so that $t_{n_0} = T_1$. If we set $L := y(t_{n_0})f(x(t_{n_0}))$ and note that $\rho(t) < t$, we get

$$x^\Delta(t) < -\frac{L}{\rho^\beta(t)} \le -\frac{L}{t^\beta}.$$

Integrating from T_1 to t and using [6, Theorem 5.68], we get that

$$x(t) - x(T_1) < -\int_{T_1}^{t} \frac{L}{s^{\beta}} \Delta s \to -\infty, \quad \text{as } t \to \infty.$$

Therefore $x(t) < 0$, for large t, which is a contradiction. Thus all solutions of equation (1.1) are oscillatory. ∎

Remark 2.6 Under the assumptions of Theorem 2.5, note that the function $f(t) = t^{\beta}$ is concave. So when $\mathbb{T} = \mathbb{Z}_0$, by Jensen's inequality [5, Theorem 6.17], we have

$$\frac{f(n+1) + f(n-1)}{2} \leq f(n).$$

This means $(n+1)^{\beta} - n^{\beta} \leq n^{\beta} - (n-1)^{\beta}$, that is, $(\rho^{\beta}(t))^{\Delta}$ is nonincreasing.

When $\mathbb{T} = q^{\mathbb{Z}_0}, q > 1$, it is easy to see that $(\rho^{\beta}(t))^{\Delta}$ is also nonincreasing.

So we can obtain the following corollaries. Corollary 2.7 shows that with no sign assumption on $p(n)$, the condition $\sum^{\infty} np(n) = \infty$ is sufficient for the oscillation of the difference equation (1.5).

Corollary 2.7 *Assume $\mathbb{T} = \mathbb{Z}$ and there exists a real number β, $0 < \beta \leq 1$ such that $\sum n^{\beta} p(n) = \infty$, then (1.5) is oscillatory.*

Corollary 2.8 *Assume $\mathbb{T} = q^{\mathbb{Z}_0}, q > 1$ and there exists β, $0 < \beta \leq 1$ such that*

$$\int_{1}^{\infty} t^{\beta} p(t) \Delta t = \infty.$$

Then the q-difference equation $x^{\Delta\Delta}(t) + p(t)x^{\alpha}(qt) = 0$, is oscillatory.

3 Examples

Example 3.1 Consider the case when $\mathbb{T}$ is the real interval $[1, \infty)$ and suppose $f\colon \mathbb{R} \to \mathbb{R}$ is continuously differentiable and satisfies the nonlinearity condition (1.2) (as well as the conditions $f'(x) > 0$ and $xf(x) > 0$, $x \neq 0$). In [4, Example 4.4] it was shown that all solutions of (1.1) are oscillatory for the case when

$$p(t) = \frac{\lambda}{t^{1+\alpha}} + \frac{\beta \sin t}{t^{\alpha}}$$

and where λ, α, β are all positive numbers satisfying $\beta\alpha < \lambda$, $0 < \alpha < 1$.

If we apply Corollary 2.3, we conclude that $\int^{\infty} t^{\alpha} p(t)\, dt = \infty$. That is we have oscillation for *all* $\lambda > 0$ and for all $0 < \alpha \leq 1$, which improves the results of [4].

Example 3.2 Consider the difference equation

$$\Delta^2 x(n) + \left[\frac{a}{n^b(n+1)} + \frac{c(-1)^n}{n^b} \right] f(x(n+1)) = 0 \tag{3.1}$$

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for $a > 0, 0 < b \leq 1$. Under the assumption of (1.2), the equation (3.1) is oscillatory, since

$$\sum_{n=1}^{\infty} n^b \left[\frac{a}{n^b(n+1)} + \frac{c(-1)^n}{n^b} \right] = \infty.$$

As observed earlier, the result in [4] gives oscillation for only the cases $0 < b < 1$ and $0 < bc < a < c(1-b)$, since condition (A) was necessary in the proof.

Example 3.3 Consider the difference equation

$$\Delta^2 x(n) + p(n) x^{\alpha}(n+1) = 0. \tag{3.2}$$

Letting $p(n) = \frac{a}{n^2} + \frac{b(-1)^n}{n}$, $a > 0$, $b \neq 0$, we see that $p(n)$ changes sign and $\sum_{n=1}^{\infty} np(n) = \infty$. By Corollary 2.7, equation (3.2) is oscillatory.

Next, if we let $p(n) = (1-\gamma)n^{-1-\gamma} + 2t^{-\gamma}(-1)^n$, $0 < \gamma < 1$, then

$$\sum_{n=1}^{N} n^{\gamma} p(n) = \sum_{n=1}^{N} [(1-\gamma)n^{-1} - 2(-1)^n] \to \infty.$$

Therefore, if we take $\beta = \gamma$, then by Corollary 2.7, equation (3.2) is oscillatory. Notice if we take $\beta = 1$, the assumption of Corollary 2.7 will not be satisfied , since

$$\sum_{n=1}^{N} np(n) = \sum_{n=1}^{N} [(1-\gamma)n^{-\gamma} - 2n^{1-\gamma}(-1)^n],$$

and so

$$\limsup_{N\to\infty} \sum_{n=1}^{N} np(n) = \infty, \quad \liminf_{N\to\infty} \sum_{n=1}^{N} np(n) = -\infty.$$

Example 3.4 Consider the q-difference equation

$$x^{\Delta\Delta}(t) + p(t) x^{\alpha}(qt) = 0. \tag{3.3}$$

Let $p(t) = \frac{1+2(-1)^n}{t^2}$, $\beta = 1$. Then

$$\int_1^{\infty} tp(t)\Delta t = \sum_{1}^{\infty} (1 + 2(-1)^n)(q-1) = \infty.$$

By Corollary 2.8, equation (3.3) is oscillatory.

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