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COUNTEREXAMPLES IN THE THEORY OF ω-FUNCTIONS

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1. Introduction. Let ϵ stand for the set of nonnegative integers (*numbers*), V for the class of all subcollections of ϵ (*sets*), Λ for the set of isols, and Λ_R for the set of regressive isols. A *function*, f, is a mapping from a subset of ϵ into ϵ and δf and ρf denote the domain and range of f respectively. The relation of inclusion is denoted by $\subset$ and that of proper inclusion by $\subsetneqq$. The sets α and β are *recursively equivalent* (written $\alpha \simeq \beta$), if $\delta f = \alpha$ and $\rho f = \beta$ for some function f with a one-to-one partial recursive extension. We denote the recursive equivalence type of α, $\{\sigma \in V | \sigma \simeq \alpha\}$, by $\text{Req}(\alpha)$. Also R stands for $\text{Req}(\epsilon)$. The reader is assumed to be familiar with the contents of [**1**; **2**; **8**].

The concept of an ω-function was introduced in [**2**] and that of an ω-homomorphism in [**1**]. If we define an *ω-function from* a set α *into* a set β as an ω-function from α onto a subset of β, and similarly define an *ω-homomorphism from* an ω-group G *into* an ω-group H as an ω-homomorphism from G onto a subgroup of H, then the question arises as to whether the into functions behave as well as the onto ones. Gonshor, in [**7**], showed that the composition of two into ω-functions need not be an ω-function. However, his example involved the use of an r.e. set as the domain of one of the functions. The purpose of this paper is to show that even when we restrict our attention to only immune sets (or immune groups) the into ω-functions (or into ω-homomorphisms) do not behave nicely. In this paper we give an example of an ω-function which maps an immune (or regressive immune) set into itself, whose restriction to a separable subset is not an ω-function (Theorem 3). This gives rise to an example involving two into ω-functions defined on immune (or regressive immune) sets whose composition is not an ω-function (Corollary 3.1). Similar results are obtained for into ω-homomorphisms (Corollary 3.2 and Corollary 3.3). Finally, an example is given of an ω-homomorphism ϕ which maps a regressive immune group G into itself and such that the restriction of ϕ to a *gc*-subgroup H of G, where $\phi(H) = H$, is not an ω-homomorphism (Corollary 4.1).

2. Basic concepts. We recall from [**8**] that for a set α, $P(\alpha)$ is the ω-group of Gödel numbers of all finite permutations of α. If f is a finite permutation of α, we denote its Gödel number by f^*. We write our permutations in cycle notation. As in [**8**], we denote the order of an ω-group G by $o(G)$.

Remark. We need the recursive functions j, k, and l defined by

$$j(x, y) = x + \frac{(x + y)(x + y + 1)}{2}, \quad j[k(n), l(n)] = n.$$

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Notation. With every $x \in \epsilon$, we associate the transposition (i.e., a permutation which moves exactly two elements)

$$\gamma(x) = (j(x, 0), j(x, 1)),$$

its Gödel number $\gamma(x)^*$ and the group $[\gamma(x)^*]$, i.e. the cyclic subgroup of order two of the group $P(\epsilon)$ generated by $\gamma(x)^*$.

Remark. It is clear that the set of numbers moved by $\gamma(x)$ and the set of numbers moved by $\gamma(y)$ are disjoint if and only if $x \neq y$.

Notation. For any set α, we define

$$\mathscr{G}_\alpha = \bigotimes_{x \in \alpha} [\gamma(x)^*],$$

the direct product of the subgroups $[\gamma(x)^*]$ of $P(\alpha)$.

Remark. We see that the elements of $\mathscr{G}_\alpha$ are just all group products

$$\prod_{i=0}^{n} \gamma(x_i)^*, \quad \text{for } x_i \in \alpha, 0 \leqslant i \leqslant n.$$

Hence $\mathscr{G}_\alpha$ is an ω-subgroup of $P(\alpha)$ such that given any $z \in \mathscr{G}_\alpha$, we can effectively find $x_0, \ldots, x_n$ such that

$$z = \prod_{i=0}^{n} \gamma(x_i)^*.$$

The above discussion yields the following result.

THEOREM 1. *If α is a set and* $\mathrm{Req}(\alpha) = A$, *then* $o(\mathscr{G}_\alpha) = 2^A$.

COROLLARY 1.1. *Let α be a set and* $\mathrm{Req}(\alpha) = A$. *Then*
(i) $A \in \Lambda \Leftrightarrow o(\mathscr{G}_\alpha) \in \Lambda$
(ii) $A \in \Lambda_R \Leftrightarrow o(\mathscr{G}_\alpha) \in \Lambda_R$.

Proof. This follows immediately, since $o(\mathscr{G}_\alpha) = 2^A$.

Remark. Given sets α and β and a function $f : \alpha \to \beta$ we wish to define a corresponding homomorphism $\phi_f : \mathscr{G}_\alpha \to \mathscr{G}_\beta$.

Definition. Let $\alpha, \beta \subset \epsilon$ and $f : \alpha \to \beta$. Then $\phi_f : \mathscr{G}_\alpha \to \mathscr{G}_\beta$ is defined as follows. Let $z \in \mathscr{G}_\alpha$.

$$\text{If } z = 1, \text{ then } \phi_f(z) = 1, \text{ and}$$

$$\text{if } z = \prod_{i=0}^{n} \gamma(x_i)^*, \text{ then } \phi_f(z) = \prod_{i=0}^{n} \gamma(f(x_i))^*.$$

Remark. We see that if $f(x_i) = f(x_j)$ for $i \neq j$, then naturally $\prod_{i=0}^{n} \gamma(f(x_i))^*$ will collapse to fewer than n factors.

THEOREM 2. *Let $\alpha, \beta, \gamma \subset \epsilon$ and let $f : \alpha \to \beta$ and $g : \beta \to \gamma$. Then*

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(i) *f is an ω-function if and only if ϕ_f is an ω-homomorphism;*
(ii) $\phi_g \circ \phi_f = \phi_{g \circ f}$.

Proof. The proof is straightforward and is left to the reader.

3. Main results. In this section we will show that both the composition of into ω-functions, and the restriction of an ω-function which maps an immune set into itself need not be an ω-function. Since, in the case of onto ω-functions, both of the above cases yield ω-functions, we see that the into ω-functions do not behave as nicely as the onto ω-functions. By using Theorem 2 and Corollary 1.1, the construction of counterexamples for set maps will automatically give us counterexamples for ω-groups and ω-homomorphisms.

Remark. For $\alpha, \beta \subset \epsilon$, we recall that $\alpha|\beta$ means α is separable from β, i.e. there exist r.e. sets $\bar{\alpha}, \bar{\beta}$ such that $\alpha \subset \bar{\alpha}$, $\beta \subset \bar{\beta}$ and $\bar{\alpha} \cap \bar{\beta}$ is empty. We also note that for a function f and a set α, $f\,|\alpha$ means, as usual, f restricted to the set α.

Remark. For the rest of this section, unless noted otherwise, let $A = \text{Req}(\alpha)$, $B = \text{Req}(\beta)$ and $C = \text{Req}(\gamma)$.

THEOREM 3. *It is possible to choose sets α and β, with $\alpha \subset \beta$, such that*
(i) $A \in \Lambda_R$, $B = R$, *or*
(ii) $A \in \Lambda_R$, $B \in \Lambda - \Lambda_R$, *or*
(iii) $A, B \in \Lambda_R$
and such that there exists an ω-function f, where $f : \beta \to \beta$, such that $f\,|\alpha$ is not an ω-function. In cases (ii) *and* (iii) *one can choose $\alpha|\beta - \alpha$.*

Proof. (i) In [**7**], Gonshor showed that if τ is a regressive set with regressing function k, then there exist ω-functions i and f, where i is the identity map from τ into ϵ and $f : \epsilon \to \epsilon$, such that $f \circ i$ is not an ω-function. Hence, just set $\beta = \epsilon$ and $\alpha = \tau$, and we get f is an ω-function from β into β and $f\,|\alpha = f \circ i$ is not an ω-function.

(ii) By [**3**, Theorem 2.1], there exist regressive isols A and C with $A \leqq^* C$ and such that $A + C \in \Lambda - \Lambda_R$. Since $2A \in \Lambda_R$, it is clear that $A \neq C$. Let $B = A + C$ and $\alpha \in A$, $\beta \in B$, $\gamma \in C$ such that $\beta = \alpha \cup \gamma$ and $\alpha|\gamma$. Hence $\alpha \subset \beta$ and $\alpha|\beta - \alpha$. Since $A \leqq^* C$, there exists a function h, with a partial recursive extension, such that h maps α one-to-one, onto γ. Define a function f with $\delta f = \beta$ and $\rho f = \gamma$ by

$$f(x) = \begin{cases} x, & \text{if } x \in \gamma \\ h(x), & \text{if } x \in \alpha. \end{cases}$$

Because $\alpha|\gamma$, it is easy to show that f has a partial recursive extension. Also, we see that $f(i(x)) = f(x) = x$ for every $x \in \gamma$, where i is the identity function on γ. Hence f is an ω-function from β onto γ. Thus f is an ω-function from β into β. However, $f\,|\alpha = h$. It follows that if $f\,|\alpha$ is an ω-function then, since h

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is one-to-one, by [**2**, Proposition P4], $\alpha \simeq \gamma$. But $A \neq C$, hence $f|\alpha$ is not an ω-function.

(iii) Let $A \in \Lambda_R$ and $C = 2A$. Hence $C \in \Lambda_R$ and $A \neq C$. Choose $\alpha \in A$ and $\gamma \in C$ such that $\alpha|\gamma$. If we set $B = 3A$ and $\beta = \alpha \cup \gamma$, then $\beta \in B$. Also $\alpha|\beta - \alpha$ and $B \in \Lambda_R$. Since $A \leqq C$, then by [**5**, Proposition P17], $A \leqq^* C$. Now proceed as in the proof of (ii).

COROLLARY 3.1. *It is possible to choose sets α, β, γ such that*
(i) $A \in \Lambda_R$, $B = C = R$, *or*
(ii) $A, C \in \Lambda_R$, $B \in \Lambda - \Lambda_R$, *or*
(iii) $A, B, C \in \Lambda_R$,

and such that there exist ω-functions f and g, where $g : \alpha \to \beta$ and $f : \beta \to \gamma$, such that $f \circ g$ is not an ω-function.

Proof. Choose α, β, γ and f as in the theorem. In each case let g be the identity function on α. The rest is immediate.

COROLLARY 3.2. *It is possible to choose ω-groups G and H such that*
(i) $o(G) = R$ *and* $o(H) \in \Lambda_R$, *or*
(ii) $o(G) \in \Lambda - \Lambda_R$, $o(H) \in \Lambda_R$, *or*
(iii) $o(G), o(H) \in \Lambda_R$,
and such that there exists an ω-homomorphism ψ, where $\psi : G \to G$, such that $\psi|H$ is not an ω-homomorphism. Also, in cases (ii) *and* (iii) $H \leqq_{\text{rec}} G$.

Proof. Choose α, β and f as in Theorem 3.1. Let $G = \mathscr{G}_\beta$, $H = \mathscr{G}_\alpha$ and $\psi = \phi_f$. The rest of the proof follows directly from Theorem 2 and Corollary 1.1.

COROLLARY 3.3. *It is possible to choose ω-groups G_1, G_2, G_3 such that*
(i) $o(G_1) \in \Lambda$, $o(G_2) = o(G_3) = R$, *or*
(ii) $o(G_1), o(G_3) \in \Lambda_R$, $o(G_2) \in \Lambda - \Lambda_R$, *or*
(iii) $o(G_1), o(G_2), o(G_3) \in \Lambda_R$,
and such that there exist ω-homomorphisms ψ and θ such that $\psi : G_1 \to G_2$, $\theta : G_2 \to G_3$ and $\theta \circ \psi$ is not an ω-homomorphism.

Proof. Choose α, β, γ, f and g as in Corollary 3.1. Let $G_1 = \mathscr{G}_\alpha$, $G_2 = \mathscr{G}_\beta$, $G_3 = \mathscr{G}_\gamma$, $\psi = \phi_g$ and $\theta = \phi_f$. The rest of the proof follows directly from Theorem 2 and Corollary 1.1.

We can strengthen Theorem 3 (iii) as follows.

THEOREM 4. *It is possible to choose regressive immune sets α and β with $\alpha \subset \beta$ and $\alpha|\beta - \alpha$ such that there exists an ω-function f, where $f : \beta \to \beta$ and $(f|\alpha)(\alpha) = \alpha$, but $f|\alpha$ is not an ω-function.*

Proof. Let $A \in \Lambda_R$. Hence $2A, 3A \in \Lambda_R$. Choose $\alpha_3 \in A$ and let $\alpha = \{2^x | x \in \alpha_3\}$, $\alpha_1 = \{3^x | x \in \alpha_3\}$ and $\alpha_2 = \{5^x | x \in \alpha_3\}$. Thus $\alpha|\alpha_1$, $\alpha|\alpha_2$, $\alpha_1|\alpha_2$ and $\alpha|\alpha_1 \cup \alpha_2$. Put $\beta = \alpha \cup (\alpha_1 \cup \alpha_2)$. Hence $\beta \in 3A$ and β is a regressive immune set. Since $A \leqq 2A$, then by [**5**, Proposition P17], $A \leqq^* 2A$. Thus there exists a function t, with a partial recursive extension, such that t maps α one-to-one, onto $\alpha_1 \cup \alpha_2$. Therefore, define a function g from β onto $\alpha_1 \cup \alpha_2$ by

$$g(x) = \begin{cases} x, & \text{if } x \in \alpha_1 \cup \alpha_2 \\ t(x), & \text{if } x \in \alpha \end{cases}$$

and define a function h from $\alpha_1 \cup \alpha_2$ onto α by

$$h(x) = \begin{cases} s_1(x), & \text{if } x \in \alpha_1 \\ s_2(x), & \text{if } x \in \alpha_2, \end{cases}$$

where for $i = 1, 2$, s_i is a recursive equivalence from α_i onto α. Now we see that g and h have partial recursive extensions. Thus, if we set $f = h \circ g$, then f has a partial recursive extension. Furthermore, f maps β onto α and hence f maps β into itself. Suppose $y \in \alpha$. Then we have $s_1^{-1}(y) \in \beta$ and

$$f \circ s_1^{-1}(y) = h \circ g(s_1^{-1}(y)) = h(s_1^{-1}(y)) = y.$$

It follows that f is an ω-function from β into itself. Now consider $f|\alpha$. We see that $f|\alpha$ maps α onto α. We claim that $f|\alpha$ is not one-to-one. For each $y \in \alpha$, there exist $z_1 \in \alpha_1$ and $z_2 \in \alpha_2$ such that $h(z_1) = h(z_2) = y$. Hence there exist $x_1, x_2 \in \alpha$, with $x_1 \neq x_2$, such that $t(x_1) = z_1$ and $t(x_2) = z_2$. We now have $f(x_1) = h \circ g(x_1) = h(z_1) = y$ and $f(x_2) = h \circ g(x_2) = h(z_2) = y$. Thus $f|\alpha$ is not one-to-one. It follows by [**2**, Proposition P7] that $f|\alpha$ is not an ω-function, since $f|\alpha$ maps α onto α, but $f|\alpha$ is not one-to-one.

COROLLARY 4.1. *It is possible to choose regressive immune groups G and H, with $H \leqq_{gc} G$, such that there exists an ω-homomorphism ϕ from G into G, with $(\phi|H)(H) = H$, but $\phi|H$ is not an ω-homomorphism.*

Remark. Let ϕ be an ω-homomorphism from an ω-group G into G. The following are two open questions:

(1) Find a necessary and/or sufficient condition on a subgroup H of G such that $\phi|H$ is an ω-homomorphism. We see by Corollary 4.1 that even if $o(H) \in \Lambda_R$, $H \leqq_{gc} G$ and $\phi(H) = H$, $\phi|H$ may not be an ω-homomorphism.

(2) Suppose that for all ω-homomorphisms $\psi : G \to G$, $\psi(H) \subset H$. Is it then true that $\psi|H$ is an ω-homomorphism, for all ω-homomorphisms $\psi : G \to G$?

If the answer to question (2) is yes, then it will make it possible to study the analogue of fully invariant subgroups.

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