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JOINT BROWDER SPECTRA AND TENSOR PRODUCTS

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There exists in the literature several notions of joint spectra which can be generalized to joint Browder spectra. The purpose of this note is to show that various notions of joint Browder spectra coincide for a special class of operators.

We first consider various notions of joint spectra existing in the literature (see, for example, [*6*], [*7*] and [*14*]).

DEFINITION 1. Let $A = (A_1, \ldots, A_n)$ be an n-tuple of commuting operators (bounded linear transformations) on a complex Banach space X .

(a) The *joint spectrum* $\sigma(A)$ of A is defined as

$$\sigma(A) = \sigma^l(A) \cup \sigma^r(A) ,$$

where the *left* (*right*) joint spectrum $\sigma^l(A)$ $(\sigma^r(A))$ is defined as the set of all points $z = (z_1, \ldots, z_n)$ in $\mathbf{C}^n$ (the n-fold Cartesian product of the set of all complex numbers $\mathbb{C}$) such that $\{A_j-z_j\}_{1\le j\le n}$ generates a proper left (right) ideal in the algebra $B(X)$ of all operators on X .

(b) The *commutant joint spectrum* $\sigma^1(A)$ is defined as the set of all

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$z = (z_1, \ldots, z_n)$ in $\mathbb{C}^n$ such that the set $\{A_j - z_j\}_{1 \le j \le n}$ is contained in a proper (two sided) ideal of A', where A' is the set of all elements of $B(X)$ that commute with $A_1, \ldots, A_n$.

(c) The *double commutant spectrum* $\sigma^2(A)$ of A is the set of all $z = (z_1, \ldots, z_n)$ in $\mathbb{C}^n$ such that the closed ideal generated by the set $\{A_j - z_j\}_{1 \le j \le n}$ is a proper ideal in A'', where $A'' = (A')'$.

(d) The *Taylor joint spectrum* $\sigma^T(A)$ of A is defined to be the set of all complex n-tuple $z = (z_1, \ldots, z_n)$ for which $A - z = (A_1 - z_1, \ldots, A_n - z_n)$ is singular.

The n-tuple A is said to be non-singular if the Koszul complex $E(X, A) = \left\{E_k^n(X), d_k^n\right\}_{k \in J}$ is exact, where J is the set of all integers, E^n is the complex algebra with identity e, generated by indeterminants $e_1, \ldots, e_n$, $E_k^n(X) = E_k^n \otimes X$ $\left(E_k^n\right.$ is generated by $e_{j_1} \wedge \ldots \wedge e_{j_k}$ with $\left.1 \le j_1 < \ldots < j_k \le n\right)$, and

$$d_k^n\left(x \otimes e_{j_1} \wedge \ldots \wedge e_{j_k}\right) = \sum_{i=1}^{k} (-1)^{i+1} A_{j_i} x \otimes e_{j_1} \wedge \ldots \wedge \hat{e}_{j_i} \wedge \ldots \wedge e_{j_k} .$$

See [14].

(e) The *polynomial spectrum* $\sigma^P(A)$ of A is defined as follows:

$$\sigma^P(A) = \{z = (z_1, \ldots, z_n) : p(z) \in \sigma(p(A)) \text{ for all polynomials } p : \mathbb{C}^n \to \mathbb{C}\} .$$

Consult [1] for details.

The discussions above lead to the following inclusion relation between the various kinds of spectra

$$\sigma(A) \subseteq \sigma^T(A) \subseteq \sigma^1(A) \subseteq \sigma^2(A) \subseteq \sigma^P(A) .$$

The first containment is a direct consequence of Proposition 2.10 [15] for

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operators on a Hilbert space. The fact that $\sigma^T(A) \subseteq \sigma^1(A) \subseteq \sigma^2(A)$ is discussed in [14] and it is not hard to see that $\sigma^2(A) \subseteq \sigma^P(A)$. See [1].

Recall that all these notions of spectra coincide in case of a single operator. However, we will show below that there exists a special class of operators for which the notions of joint spectra discussed in (b), (c), (d) and (e) coincide. See Theorem 2 below.

Let $X_1, \ldots, X_n$ be Banach spaces, and let B_i be an operator on X_i for all i, $1 \le i \le n$. Let $Y = X_1 \otimes \ldots \otimes X_n$ be the completion of the tensor product with respect to some cross-norm. Then

$$A_1 = B_1 \otimes I_2 \otimes \ldots \otimes I_n , \quad A_2 = I_1 \otimes B_2 \otimes I_3 \otimes \ldots \otimes I_n , \ldots ,$$

$$A_k = I_1 \otimes \ldots \otimes B_k \otimes I_{k+1} \otimes \ldots \otimes I_n, \ldots, A_n = I_1 \otimes \ldots \otimes B_n$$

are commuting operators on Y. It is proved in [10] that

(1) $$\sigma^2(A_1, \ldots, A_n) = \prod_{k=1}^{n} \sigma(B_k) .$$

Recently Ceausescu and Vasilescu [5] proved that (1) holds for the Taylor joint spectrum (that is, $\sigma^T(A_1, \ldots, A_n) = \prod_{k=1}^{n} \sigma(B_k)$) in the case that the X_i's are complex Hilbert spaces. Combining all these results one concludes easily that the Taylor joint spectrum, the commutant joint spectrum, the double commutant joint spectrum and the polynomial joint spectrum all coincide for these classes of operators. Thus we have explicitly the following

THEOREM 2. *Let $A = (A_1, \ldots, A_n)$ be the n-tuple of operators $A_k = I_1 \otimes \ldots \otimes I_{k-1} \otimes B_k \otimes I_{k+1} \otimes \ldots \otimes I_n$, $k = 1, \ldots, n$, on the Hilbert space $H = H_1 \otimes \ldots \otimes H_n$ as described above. Then*

$$\sigma^j(A) = \prod_{k=1}^{n} \sigma(B) ,$$

for $j = 1, 2, P, T$.

Proof. The result follows easily from the following observation and the fact that $\sigma(A_k) = \sigma(B_k)$ for $k = 1, \ldots, n$ (see Proposition 3)

$$\sigma^T(A) \subseteq \sigma^1(A) \subseteq \sigma^2(A) \subseteq \sigma^P(A) \subseteq \prod_{k=1}^{n} \sigma(A_k) = \prod_{k=1}^{n} \sigma(B_k) = \sigma^T(A) .$$

REMARK 1. Note that Theorem 2 is true for Banach space operators for $j = 1, 2, P$. However, for $j = T$, the result is still unknown for Banach spaces.

REMARK 2. It is not hard to see that Theorem 2 is not valid for the joint spectrum $\sigma(A)$ in Definition 1 (a). Indeed, consider the operators $A_1 = U \otimes 1$ and $A_2 = 1 \otimes U^*$, where U is the unilateral shift defined by $Ue_n = e_{n+1}$ and $\{e_n\}_0^\infty$ is an orthonormal basis of a Hilbert space. Thus we have

$$\sigma(A_1, A_2) = \sigma^l(A_1, A_2) \cup \sigma^r(A_1, A_2)$$
$$= (\partial D \times D) \cup (D \times \partial D) \neq D^2 = \sigma(U) \times \sigma(U^*) ,$$

where D is the closed unit disc and ∂D its boundary; see [9].

Furthermore, it is also clear that all the above notions of joint spectra coincide in case of operators in finite dimensional spaces.

PROPOSITION 3. *Consider the operator* $B \otimes 1$, *where* B *is an operator on a Hilbert space and* 1 *is the identity operator. Then*

$$\sigma_b(B \otimes 1) = \sigma_e(B \otimes 1) = \sigma_\omega(B \otimes 1) = \sigma(B \otimes 1) = \sigma(B) ,$$

where $\sigma_\omega(S)$ *is the Weyl spectrum of an operator* S *defined by* $\sigma_\omega(S) = \bigcap_K \sigma(S+K)$ *and* K *is a compact operator. For other notions of spectra see Definition 4.*

Proof. From Lemma 4.2 [2] it follows that $\sigma(B \otimes 1) = \sigma(B)$ and $\sigma_e(B \otimes 1) = \sigma(B)$. Hence we obtain

$$\sigma(B) = \sigma_e(B \otimes 1) \subseteq \sigma_\omega(B \otimes 1) \subseteq \sigma_b(B \otimes 1) \subseteq \sigma(B \otimes 1) = \sigma(B) .$$

This proves the proposition.

However, the situation in case of a pair of operators is different (see [12] and [13]). This will be the subject of our discussion for the rest of the sections of this paper.

Let us first introduce various notions of joint essential spectra and joint Browder spectra.

Let $A = (A_1, \ldots, A_n)$ be an n-tuple of elements in $B(X)$. Let A_e^1 be the almost commutant of A (commute modulo the compact operators), A_e^2 be the almost double commutant of A.

The reader is referred to [3], [12] and [13] for details.

DEFINITION 4. $z = (z_1, \ldots, z_n) \in \rho_b^j(A)$ if $z \in \rho_b^j(A)$ in A_e^j and there exists a deleted neighborhood N of z such that if $\lambda = (\lambda_1, \ldots, \lambda_n) \in N$, then $\lambda \in \rho^j(A)$ for $j = 1, 2, T$. Now we say that $z \in \sigma_b^j(A)$, the *joint Browder spectrum* of A if $z \in' \rho_b^j$ for $j = 1, 2, T$. Here $\in'$ means "does not belong to". For the definition of *polynomial joint Browder spectrum* see [4]. Moreover, it follows from Definition 4 and Theorem 8 in [4] that

$$\sigma_b^j(A) = \sigma_e^j(A) \cup \{\text{accumulation points of } \sigma^j(A)\} ,$$

for $j = 1, 2, P, T$. Here $\sigma_e^j(A)$ is the joint essential spectrum of A for $j = 1, 2, P, T$. See [4], [6], [8] and [12].

This together with the fact that any accumulation point of a set G is also an accumulation point of a set containing G and that $\sigma_e^T(A) \subseteq \sigma_e^1(A) \subseteq \sigma_e^2(A) \subseteq \sigma_e^P(A)$ imply that $\sigma_b^T(A) \subseteq \sigma_b^1(A) \subseteq \sigma_b^2(A) \subseteq \sigma_b^P(A)$.

Next we shall prove a theorem on joint Browder spectra (Theorem 7) which is analogous to Theorem 2 on joint spectra. First of all note that Schechter and Snow [12] showed that if $A_1 = B \otimes 1$ and $A_2 = 1 \otimes C$ are operators on the Banach space $X_1 \otimes X_2$, then

$$\{\sigma_e(B) \times \sigma(C)\} \cup \{\sigma(B) \times \sigma_e(C)\} \subseteq \sigma_e^2(A_1, A_2) .$$

We prove analogous results for various notions of joint essential spectra of operators on Hilbert spaces as given in Theorem 6. This generalizes the results of Schechter and Snow to various notions of joint essential spectra, in particular to Taylor joint essential spectra which is a smaller set than that of the double commutant joint essential spectra, for tensor products of operators on Hilbert spaces.

LEMMA 5. *If* B *and* C *are operators on Hilbert spaces* H_1 *and* H_2 *, then*

$$\left\{\sigma_e^l(B) \times \sigma(C)\right\} \cup \left\{\sigma(B) \times \sigma_e^l(C)\right\} \subseteq \sigma_e^j(A) \quad \textit{for} \quad j = 1, 2, P, T ,$$

where $A = (A_1, A_2)$ *and* $A_1 = B \otimes 1$ *and* $A_2 = 1 \otimes C$ *as defined above. Here* $\sigma_e^l(T)$ *denotes the left joint essential spectrum of an operator* T (see [8]).

Proof. It is enough to show that $\sigma_e^l(B) \times \sigma(C) \subseteq \sigma_e^T(A)$. In other words it is sufficient to show that if $0 \in \sigma_e^l(B)$ and $0 \in \sigma(C)$, then $(0, 0) \in \sigma_e^T(A)$. If $0 \in \sigma_e^l(B)$ and $0 \in \sigma(C)$, then there exists an orthonormal sequence $\{e_n\}$ [11] and a sequence $\{x_n\}$ of unit vectors such that $\|Be_n\| \to 0$, and either $\|Cx_n\| \to 0$ or $\|C^*x_n\| \to 0$. Set $z_k = e_k \otimes x_k$ and

$$\zeta_k = \begin{pmatrix} z_k \\ 0 \end{pmatrix} \quad \text{if} \quad \|C^*x_k\| \to 0 ,$$

and

$$\zeta'_k = \begin{pmatrix} 0 \\ z_k \end{pmatrix} \quad \text{if} \quad \|Cx_k\| \to 0 .$$

Clearly the sequences ζ_k and ζ'_k are orthonormal and hence weakly tend to zero. Furthermore, we know that A is Fredholm if and only if the matrix

$$V = \begin{pmatrix} A_1 & A_2 \\ -A_2^* & A_1^* \end{pmatrix}$$

is Fredholm. This fact together with the definitions of ζ_k and ζ'_k

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implies that either $\|V\zeta_k\| \to 0$ or $\|V^*\zeta_k'\| \to 0$. This means V is not Fredholm [11] and hence A is not Fredholm. Therefore, $(0, 0) \in \sigma_e^T(A)$. Since $\sigma_e^T(A) \subseteq \sigma_e^1(A) \subseteq \sigma_e^2(A) \subseteq \sigma_e^P(A)$, the result is proved.

THEOREM 6. *If B, C and A are as given in Lemma 5, then*

$$\{\sigma_e(B) \times \sigma(C)\} \cup \{\sigma(B) \times \sigma_e(C)\} \subseteq \sigma_e^j(A) \quad \textit{for} \quad j = 1, 2, P, T .$$

Proof. By similar techniques as given in the above lemma one can show that

$$\left\{\sigma_e^r(B) \times \sigma(C)\right\} \cup \left\{\sigma(B) \times \sigma_e^r(C)\right\} \subseteq \sigma_e^j(A) \quad \text{for} \quad j = 1, 2, P, T .$$

This fact together with Lemma 5 proves the results. Here $\sigma_e^r(T)$ denotes the right joint essential spectrum of an operator T (see [8]).

The following theorem was proved by Snow [13] for $\sigma_b^2(A)$, where $A = (A_1, A_2)$, $A_1 = B \otimes 1$, $A_2 = 1 \otimes C$ and B, C are operators on Banach spaces X_1 and X_2 . However, we show below that various notions of joint Browder spectrum coincide for operators A_1 and A_2 on Hilbert space $H_1 \otimes H_2$. This is in some sense a generalization of Snow's result to various notions of joint Browder spectrum for tensor products of operators on Hilbert spaces.

THEOREM 7. *If A_1, A_2 and A are as defined in Lemma 5, then*

$$\sigma_b^j(A) = \{\sigma_b(B) \times \sigma(C)\} \cup \{\sigma(B) \times \sigma_b(C)\} \quad \textit{for} \quad j = 1, 2 \ \textit{and} \ T . \tag{2}$$

Proof. We first show the equality of (2) for Taylor joint spectrum $\sigma^T(A)$. From Theorem 2 we have $\sigma^T(A) = \sigma(B) \times \sigma(C)$. Let $z_1 \in \sigma_b(A)$ and $z_2 \in \sigma(C)$. If $z_1 \in \sigma_e(B)$, then it follows from Theorem 6 that $(z_1, z_2) \in \sigma_e^T(A) \subseteq \sigma_b^T(A)$. If $z_1 \in \sigma(B)\backslash\sigma_e(B)$, then z_1 is not an isolated point and hence, by Theorem 2, (z_1, z_2) is not an isolated point of $\sigma^T(A)$. This implies that $(z_1, z_2) \in \sigma_b^T(A)$, and hence

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$\sigma_b(B) \times \sigma(C) \subseteq \sigma_b^T(A)$. Similarly, one shows that $\sigma(B) \times \sigma_b(C) \subseteq \sigma_b^T(A)$. Thus we have

$$\{\sigma_b(B) \times \sigma(C)\} \cup \{\sigma(B) \times \sigma_b(C)\} \subseteq \sigma_b^T(A) .$$

Conversely, suppose $z = (z_1, z_2) \in \sigma^T(A)$ but $z \in' \{\sigma_b(B) \times \sigma(C)\} \cup \{\sigma(B) \times \sigma_b(C)\}$. Then $z_1 \in \sigma(B)\backslash\sigma_b(B)$ and $z_2 \in \sigma(C)\backslash\sigma_b(C)$. Hence by Lemma 17 of [12] we obtain that $z \in \rho_e^2(A)$. But $\rho_e^2(A) \subseteq \rho_e^T(A)$. Thus by Theorem 2 we conclude that z is an isolated point of $\sigma^T(A)$ and hence $z \in' \sigma_b^T(A)$. This implies that $\sigma_b^T(A) \subseteq \{\sigma_b(B) \times \sigma(C)\} \cup \{\sigma(B) \times \sigma_b(C)\}$. Thus

$$\{\sigma_b(B) \times \sigma(C)\} \cup \{\sigma(B) \times \sigma_b(C)\} \subseteq \sigma_b^T(A) \subseteq \sigma_b^1(A) \subseteq \sigma_b^2(A)$$
$$= \{\sigma_b(B) \times \sigma(C)\} \cup \{\sigma(B) \times \sigma_b(C)\} .$$

This proves the theorem.

Thus the theorem illustrates the fact that analogous to joint spectra (Theorem 2) the various notions of joint Browder spectra $\{\sigma_b^j(A),\ j = 1, 2, T\}$ also coincide for these special class of operators on Hilbert spaces.

THEOREM 8. *If B and C are, respectively, operators on Banach spaces X_1 and X_2 with $A_1 = B \otimes 1$, $A_2 = 1 \otimes C$ and $A = (A_1, A_2)$, then*

$$\sigma_b^1(A) = \sigma_b^2(A) = \{\sigma_b(B) \times \sigma(C)\} \cup \{\sigma(B) \times \sigma_b(C)\} .$$

We leave the proof to the reader.

REMARK 3. Whether Theorems 7 and 8 are true for polynomial joint Browder spectrum is an open question.

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