A Construction for the Force, at any Point, due to Electric Point-Charges or Ideal Magnets, with an Extension to Continuous Distributions.

By Dr W. PEDDIE.

Let P_1 and P_1' be images of each other with respect to a circle of radius a. (Fig. 13.) Draw AOB the diameter perpendicular to $OP_1'P_1$. If r_1, r_1' represent respectively the distances of P_1 and P_1' from O, the centre of the circle, the condition is $r_1 r_1' = a^2$.

Let OQ_1 and OQ_1' be found by making $P_1'Q_1$ and P_1Q_1' respectively perpendicular to AP_1' and AP_1, and denote these lengths by ρ_1 and ρ_1'. We then have

$$r_1 r_1' = a^2, \quad a\rho_1 = r_1'^2.$$

Hence $r_1\rho_1 = ar_1'$, and, similarly, $r_1'\rho_1' = ar_1$. So

$$\rho_1 = \frac{a^3}{r_1^2}, \quad \rho_1' = \frac{a^3}{r_1'^2}, \quad \rho_1\rho_1' = a^2.$$

Therefore Q_1 and Q_1' are images of each other in the circle of radius a.

If now we make a similar construction with the corresponding points whose suffixes are 2 we get

$$\rho_2 = \frac{a^3}{r_2^2}, \quad \rho_2' = \frac{a^3}{r_2'^2}, \quad \rho_2\rho_2' = a^2.$$

The quantity ρ_1 represents in magnitude the force at O due to a mass or charge, of magnitude a^3, at P_1. Its direction is the direction of the force turned through a right angle. So also ρ_2 represents the force at O due to a mass or charge a^3 at P_2.

Thus, if P_1P_2 represents an ideal magnet of pole strength a^3, the line Q_1Q_2 represents the resultant force at O if it be turned left-handedly through a right angle. In the same way $Q_1'Q_2'$, turned left-handedly through a right angle, represents the force at O due to the image magnet $P_1'P_2'$. If the charges at P_1 and P_2 were unequal, the inversions would have to be performed with regard to different circles. The resultant of OQ_1 and OQ_2 represents the force, turned through a right angle, when the charges are of like sign.

The method may be extended so as to reduce the problem of the determination of the attraction of a direct system to that of the determination of the mass-vector, or the centre of inertia of the image system; or conversely.

In the usual process of electric inversion the condition $rr' = a^2$ gives the distances, measured same-wise, of a point and its electric image from the centre of the circle of inversion. In the inversion given by the condition $\rho r^2 = a^3$, we may speak of the point indicated by ρ as the force image of the point indicated by r; and of the point indicated by r as the mass image of the point indicated by ρ.

To find the attraction, at a given point, of a given distribution of matter (electricity or magnetism) we may suppose the mass to be composed of very small equal parts of magnitude a^3. Let r be the distance of one of these from the given point and find the corresponding dynamical image point, indicated by ρ, in the circle of inversion of radius a. The complex of the rs gives a complex of ρs; each of the former being associated with a mass a^3, each of the latter with a mass which we take as unity. The resultant of all the ρs, regarded as mass-vectors, represents the resultant force of attraction of the direct system at the centre of inversion. It also represents the resultant mass-vector of the image system. That is to say, if divided by the mass of the image system, it is the vector to the centre of inertia of that system.

Thus, if we know the centre of inertia, or the law of distribution of the matter, of a given system, we can find from it the attraction, at the centre of inversion, of the corresponding mass-image system. Or, if we know the resultant attraction at the centre of inversion, of a given system, we can find from it the centre of inertia of the corresponding force-image system.

To determine the relations, let V, V′ represent the corresponding small volumes in the r-system, and the ρ-system, respectively. When r is constant, ρ is constant, and we may consider similar small areas k^2r^2 and $k^2\rho^2$, so that

$$V = k^2r^2dr, \quad V' = k^2\rho^2 d\rho.$$

Substituting for r from the condition $\rho r^2 = a^3$ we get

$$V' = -2V\left(\frac{\rho}{a}\right)^{\frac{9}{2}}.$$

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Now let σ, σ' be the densities in the r-system and the ρ-system respectively. We have $V\sigma = a^3$, and so, if we put

$$\sigma = a^3 f(r, \theta, \phi) = a^3 F(\rho, \theta, \phi), \text{ say,}$$

we get
$$V' = -\frac{2}{F(\rho, \theta, \phi)}\left(\frac{\rho}{a}\right)^{\frac{9}{2}}.$$

This is the volume to be taken as of unit mass, so the density in the ρ-system is

$$\sigma' = \frac{F(\rho, \theta, \phi)}{2}\left(\frac{a}{\rho}\right)^{\frac{9}{2}},$$

or
$$\sigma' = \sigma\frac{1}{2a^3}\left(\frac{a}{\rho}\right)^{\frac{9}{2}}.$$

As an example, consider the r-system to be a spherical distribution with density varying inversely as the fifth power of the distance from an internal point. If M be the mass of the sphere, the attraction at an external point distant b from the given internal point is M/b^2. The joining line of these points gives, when multiplied by M/b^3, the mass-vector of the image distribution in magnitude and direction. If ξ be the distance of a point in the sphere from the given internal point we have

$$\xi = (r^2 + b^2 - 2br\cos\theta)^{\frac{1}{2}},$$

and $\sigma = p\xi^{-5}$.

Thus
$$\sigma' = p\left(\frac{a^6}{\rho} + b^2 - 2b\frac{a^{\frac{3}{2}}}{\sqrt{\rho}}\cos\theta\right)^{-\frac{5}{2}}\frac{1}{2a^3}\left(\frac{a}{\rho}\right)^{\frac{9}{2}},$$

where the limiting surface is given by

$$\rho = a^3/r^2 \text{ and } r^2 = b^2 + a^6 + 2ba^3\cos\theta.$$

Thus the centre of inertia of a very complicated distribution is readily got.

Conversely, if ρ corresponds to the spherical distribution, the joining line, when multiplied by M, gives the force due to the complicated r-system.

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