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NATURAL DUALITIES FOR DIHEDRAL VARIETIES

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Abstract

A strong, natural duality is established for the variety generated by a dihedral group of order $2m$ with m odd. This is the first natural duality for a non-abelian variety of groups.

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1. Introduction

Which finitely generated quasivarieties of groups admit a natural duality? The main theorem of this paper (Theorem 2) extends the list of known examples into the non-abelian realm.

For the benefit of readers not familiar with the theory of natural dualities, we begin with a brief review of what is meant by '*admitting a natural duality*' and refer to Davey [4] or the forthcoming text Clark and Davey [3] for a detailed account.

Let $\underline{\mathbf{M}}$ be a finite group and let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathscr{T}\rangle$ be a topological structure on the same underlying set, where

(a) each $g \in G$ is a homomorphism $g : \underline{\mathbf{M}}^n \to \underline{\mathbf{M}}$ for some $n \in \mathbb{N} \cup \{0\}$,
(b) each $h \in H$ is a homomorphism $h : \mathrm{dom}(h) \to \underline{\mathbf{M}}$ where $\mathrm{dom}(h)$ is a subgroup of $\underline{\mathbf{M}}^n$ for some $n \in \mathbb{N}$,
(c) each $r \in R$ is (the universe of) a subalgebra of $\underline{\mathbf{M}}^n$ for some $n \in \mathbb{N}$,
(d) $\mathscr{T}$ is the discrete topology.

Whenever (a), (b) and (c) hold, we say that the operations in G, the partial operations in H and the relations in R are *algebraic over* $\underline{\mathbf{M}}$. These compatibility conditions

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between the structure on $\underline{\mathbf{M}}$ and the structure on $\underset{\sim}{\mathbf{M}}$ guarantee that there is a naturally defined dual adjunction between the quasivariety $\mathscr{A} := \mathbb{ISP}\underline{\mathbf{M}}$ generated by $\underline{\mathbf{M}}$ and the topological quasivariety $\mathscr{X} := \mathbb{IS}_c\mathbb{P}\underset{\sim}{\mathbf{M}}$ generated by $\underset{\sim}{\mathbf{M}}$. For all $\mathbf{A} \in \mathscr{A}$ the homset $D(\mathbf{A}) := \mathscr{A}(\mathbf{A}, \underline{\mathbf{M}})$ is a closed substructure of the direct power $\underset{\sim}{\mathbf{M}}^A$ and for all $\mathbf{X} \in \mathscr{X}$ the homset $E(\mathbf{X}) := \mathscr{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$ is a subgroup of the direct power $\underline{\mathbf{M}}^X$. It follows easily that the contravariant hom-functors $\mathscr{A}(-, \underline{\mathbf{M}}) : \mathscr{A} \to \mathscr{S}$ and $\mathscr{X}(-, \underset{\sim}{\mathbf{M}}) : \mathscr{X} \to \mathscr{S}$, where $\mathscr{S}$ is the category of sets, lift to contravariant functors $D : \mathscr{A} \to \mathscr{X}$ and $E : \mathscr{X} \to \mathscr{A}$. For each $\mathbf{A} \in \mathscr{A}$ there is a natural embedding $e_\mathbf{A}$ of $\mathbf{A}$ into $ED(\mathbf{A})$ given by evaluation: for each $a \in A$ and each $x \in D(\mathbf{A}) = \mathscr{A}(\mathbf{A}, \underline{\mathbf{M}})$ define $e_\mathbf{A}(a)(x) := x(a)$. Similarly, for each $\mathbf{X} \in \mathscr{X}$ there is an embedding $\epsilon_\mathbf{X}$ of $\mathbf{X}$ into $DE(\mathbf{X})$. A simple calculation shows that $e : \mathrm{id}_\mathscr{A} \to DE$ and $\epsilon : \mathrm{id}_\mathscr{X} \to DE$ are natural transformations. If $e_\mathbf{A}$ is an isomorphism for all $\mathbf{A} \in \mathscr{A}$ we say that $\underset{\sim}{\mathbf{M}}$ yields a *(natural) duality* on $\mathscr{A}$. If, moreover, $\epsilon_\mathbf{X}$ is an isomorphism for all $\mathbf{X} \in \mathscr{X}$, we say that $\underset{\sim}{\mathbf{M}}$ yields a *full duality* on $\mathscr{X}$ (in which case $\mathscr{A}$ and $\mathscr{X}$ are dually equivalent categories). If there is some choice of G, H and R such that $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathscr{A}$ then we say that $\underline{\mathbf{M}}$ (or $\mathscr{A}$) admits a *natural duality* or, more colloquially, is *dualizable*.

The best known examples of dualizable groups are the finite cyclic groups: if $\underline{\mathbf{C}}_m = \langle C_m; \cdot, {}^{-1}, 1\rangle$ is an m–element cyclic group, then $\underset{\sim}{\mathbf{C}}_m := \langle C_m; \cdot, {}^{-1}, 1, \mathscr{T}\rangle$ yields a full duality on $\mathscr{A}_m := \mathbb{ISP}\underline{\mathbf{C}}_m$. (In this case both H and R are empty.) The class $\mathscr{A}_m$ is the variety of abelian groups of exponent m while $\mathscr{X}_m := \mathbb{IS}_c\mathbb{P}\underset{\sim}{\mathbf{M}}$ is the category of compact (totally disconnected) topological abelian groups of exponent m.

We shall refer to this as the *Pontryagin Duality* on $\mathscr{A}_m$ as it can be obtained by restricting the Pontryagin duality for the class of all abelian groups to the subvariety $\mathscr{A}_m$. The general theory of natural dualities affords several simple, direct proofs of this duality which avoid the application of Pontryagin's sledgehammer—see Davey and Werner [6] or Clark and Davey [1]. In fact, every finite abelian group $\underline{\mathbf{M}}$ is dualizable: it is shown in Davey [5] that if $G = \{\cdot, {}^{-1}, 1\} \cup \mathrm{End}\,\underline{\mathbf{M}}$ and $H = R = \emptyset$, then $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathscr{A} := \mathbb{ISP}\underline{\mathbf{M}}$ which will not in general be full.

Thus, this paper is a contribution to the solution of the following fundamental problem.

PROBLEM. *Which finite groups admit a natural duality?*

The general theory of natural dualities tells us that it order to show that $\underset{\sim}{\mathbf{M}}$ yields a full duality on $\mathscr{A}$ it is sufficient to prove the following three conditions—

(CLO) *for each $n \in \mathbb{N}$, every morphism $t : \underset{\sim}{\mathbf{M}}^n \to \underset{\sim}{\mathbf{M}}$ is an n-ary term function on $\underline{\mathbf{M}}$,*

(INJ) *$\underset{\sim}{\mathbf{M}}$ is injective in $\mathscr{X}$,*

(STR) *for every non-empty set I, for each substructure $\mathbf{X}$ of $\underset{\sim}{\mathbf{M}}^I$ and for each*

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$y \in M^I \backslash X$ there exist morphisms $\varphi, \psi : \underset{\sim}{\mathbf{M}}^I \to \underset{\sim}{\mathbf{M}}$ such that $\varphi\restriction_{\mathbf{X}} = \psi\restriction_{\mathbf{X}}$ but $\varphi(y) \neq \psi(y)$.

Together, (CLO) and (INJ) guarantee that $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathscr{A}$. The condition (CLO) is of independent algebraic interest as it asserts that the n-ary term functions on $\underline{\mathbf{M}}$ are precisely the maps $t : M^n \to M$ which preserve the operations in G, partial operations in H and relations in R. A duality which satisfies (STR) is called a *strong duality*. By Clark and Davey [1], a duality is strong if and only if it is full and (INJ) holds. Every known full duality is strong. Note that in the case of the variety $\mathscr{A}_m$, we may always choose the morphism φ in the condition (STR) to be the constant map onto 1, whence (STR) is equivalent in this case to

if $\mathbf{X}$ is a closed subgroup of $\underset{\sim}{\mathbf{C}}_m^I$ and $y \in C_m^I \backslash X$, then there is a continuous homomorphism $\psi : \underset{\sim}{\mathbf{C}}_m^I \to \underset{\sim}{\mathbf{C}}_m$ such that $\psi\restriction_{\mathbf{X}} = \underline{1}$ while $\psi(y) \neq 1$, where $\underline{1}$ is the constant map onto 1.

The conditions (INJ) and (STR) can both be reduced to the finite case (thus elimating all topological considerations) whenever H is empty and R is finite, as in the case of the finite cyclic group. The results of Clark and Davey [3] show that if $\underset{\sim}{\mathbf{M}}$ yields a strong duality on $\mathscr{A}$ and $\underline{\mathbf{M}}$ is not injective in $\mathscr{A}$, then the set H of partial operations must be non-empty. Since the dihedral group $\underline{\mathbf{D}}_m$ is not injective in the quasivariety it generates, if we wish to obtain a strong duality for the quasivariety generated by $\underset{\sim}{\mathbf{D}}_m$ we will have no choice but to include partial operations in the type of $\underset{\sim}{\mathbf{D}}_m$.

2. The dihedral groups

Let $\underline{\mathbf{D}}_m = \langle D_m; \cdot \rangle$ be the dihedral group of order $2m$ presented by $a^m = b^2 = 1$ and $ba = a^{m-1}b$. In the case that m is odd, we will establish a strong duality for $\mathscr{A} := \mathbb{ISP}\underline{\mathbf{D}}_m$, the quasivariety generated by $\underline{\mathbf{D}}_m$ (in this case, $\mathbb{ISP}\underline{\mathbf{D}}_m$ is actually the variety generated by $\underline{\mathbf{D}}_m$). Hence, we now assume that m is odd. The dual category will be the topological quasivariety $\mathscr{X} := \mathbb{IS}_c\mathbb{P}\,\underset{\sim}{\mathbf{D}}_m$ where

$$\underset{\sim}{\mathbf{D}}_m = \langle D_m; \alpha, 1; +, *; \mathscr{T} \rangle.$$

As usual, the topology, $\mathscr{T}$, is discrete. The total operations are the automorphism α of $\underline{\mathbf{D}}_m$, given by $\alpha(a) = a$ and $\alpha(b) = ab$, and the nullary operation 1. The first partial operation, $+$, is simply the partial binary map from $\underline{\mathbf{D}}_m^2$ to $\underline{\mathbf{D}}_m$ which is the restriction to $H := \{1, b\}$ of the group operation on $\underline{\mathbf{D}}_m$ (thus, $b^i + b^j := b^{i+j}$). The second partial operation is slightly more complicated. Let ϵ be the (unique) retraction of $\underline{\mathbf{D}}_m$ onto $\mathbf{H}$. Let $N \subseteq D_m$ be the kernel of ϵ in the group theoretic sense (thus, $N = \{1, a, a^2, \dots, a^{m-1}\}$), and let $K \subseteq D_m^2$ be the kernel of ϵ in the general-algebraic

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sense (thus,

$$K := \{\, (u, v) \in D_m^2 \mid \epsilon(u) = \epsilon(v) \,\} = N \times N \cup Nb \times Nb$$
$$= \{\, (a^i b^k, a^j b^k) \mid 0 \leqslant i, j \leqslant m - 1 \text{ and } 0 \leqslant k \leqslant 1 \,\}$$

is the congruence corresponding to N). Define $*$ to be the partial map from $\underline{\mathbf{D}}_m^2$ to $\underline{\mathbf{D}}_m$ whose domain is K with $a^i b^k * a^j b^k := a^{i+j} b^k$. Note that on N the operation $*$ is just the original group operation while on Nb the operation $*$ is the translation of the original group operation on N. Thus, $\langle Nb; *\rangle$ is a group and right translation by b is a group isomorphism from $\langle N; \cdot\rangle$ onto $\langle Nb; *\rangle$. Observe that α is the identity map on N and is the cycle $(b, ab, a^2b, \ldots, a^{m-1}b)$ on the other coset Nb. In the case that m is odd, α is an inner automorphism; indeed, if $m = 2k + 1$, then $\alpha(g) = a^{-k} g a^k$ for all $g \in D_m$. Each of these operations and partial operations is algebraic over $\underline{\mathbf{D}}_m$. This is obvious in each case except $*$. That $*$ is algebraic follows from the lemma below.

LEMMA 1. *Let* $\mathbf{G}$ *be a group and let* ϵ *be a retraction of* $\mathbf{G}$ *onto a subgroup* $\mathbf{H}$. *Let* N *be the kernel of* ϵ *and let*

$$K := \{(u, v) \in G^2 \mid \epsilon(u) = \epsilon(v)\} = \bigcup\{\epsilon^{-1}(h) \mid h \in H\} = \bigcup\{Nh \times Nh \mid h \in H\}$$

be the congruence corresponding to N. *Define a partial binary operation* $*$, *with domain* K, *by* $xh * yh := xyh$ *for all* $x, y \in N$ *and* $h \in H$, *or equivalently, define* $u * v := u\epsilon(u)^{-1}v = u\epsilon(v)^{-1}v$ *for all* $(u, v) \in K$.

(a) *(The restriction of)* $*$ *is a well-defined group operation on* Nh *for each* $h \in H$. *Moreover, right translation by* h *is an isomorphism of* $\langle N; \cdot\rangle$ *onto* $\langle Nh; *\rangle$.
(b) *The partial operation* $*$ *is associative wherever it is defined. It will be commutative wherever it is defined provided* $\mathbf{N}$ *is abelian.*
(c) *The map* $* : K \to G$ *is a homomorphism if and only if* $\mathbf{N}$ *is abelian.*

PROOF. For (a) we need only that $\mathbf{N}$ is a subgroup and that H is a class of representatives for the right cosets of $\mathbf{N}$. A trivial calculation establishes (b).

For (c) we need to know that $\mathbf{K}$ is a subgroup of $\underline{\mathbf{D}}_m^2$ (that is, that $\mathbf{N}$ is a normal subgroup) and that, for all $h, k \in H$, the representative of the right coset Nhk is hk, that is, that $\mathbf{H}$ is a subgroup of $\mathbf{G}$. Together this says precisely that ϵ is a retraction onto the subgroup $\mathbf{H}$.

We wish to prove that $*$ is a homomorphism, that is,

$$(1)\ (\forall w, x, y, z \in N)(\forall h, k \in H)(wh \cdot yk) * (xh \cdot zk) = (wh * xh) \cdot (yk * zk),$$

if and only if $\mathbf{N}$ is abelian. Let $w, x, y, z \in N$ and $h, k \in H$; then since $\mathbf{N}$ is normal, there exist $y', z' \in N$ such that $hy = y'h$ and $hz = z'h$. Hence

$$(wh \cdot yk) * (xh \cdot zk) = (wy'hk) * (xz'hk) = (wy'xz')hk$$

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and

$$(wh * xh) \cdot (yk * zk) = wxhyzk = wxy'hzk = (wxy'z')hk.$$

Thus (1) holds provided N is abelian, and choosing $h = k = w = z = 1$ in (1) shows that (1) implies that N is abelian.

We can now state the main result of this paper.

THEOREM 2. *The structure $\underset{\sim}{\mathbf{D}}_m$ yields a strong duality on $\mathscr{A}$, that is, the hom-functors $D : \mathscr{A} \to \mathscr{X}$ and $E : \mathscr{X} \to \mathscr{A}$ give a dual category equivalence between $\mathscr{A} := \mathbb{ISP}\underline{\mathbf{D}}_m$ and $\mathscr{X} := \mathbb{IS}_c\mathbb{P}\,\underset{\sim}{\mathbf{D}}_m$ and $\underset{\sim}{\mathbf{D}}_m$ is injective in $\mathscr{X}$.*

Unfortunately, we do not have an axiomatization of the class $\mathscr{X}$.

If $\underset{\sim}{\mathbf{D}}_m$ yields a strong duality on $\mathscr{A}$, then every closed substructure $\mathbf{X}$ of a power of $\underset{\sim}{\mathbf{D}}_m$ must (at least) be closed under every endomorphism of $\underline{\mathbf{D}}_m$ and moreover every $\mathscr{X}$–morphism from $\mathbf{X}$ to $\underset{\sim}{\mathbf{D}}_m$ must preserve the actions of the endomorphisms of $\underline{\mathbf{D}}_m$ on $\mathbf{X}$ (see Clark and Davey [1]). We begin our proof of Theorem 2 by establishing this necessary condition plus a little more. The partial operation $*$ induces a partial operation on $D(\mathbf{A})$ for all $\mathbf{A} \in \mathscr{A}$ and in particular on $D(\underline{\mathbf{D}}_m) = \operatorname{End}\underline{\mathbf{D}}_m$: if $e, f \in \operatorname{End}\underline{\mathbf{D}}_m$, then $e*f$ is defined if and only if for each $u \in G$ either $e(u), f(u) \in N$ or $e(u), f(u) \in Nb$. Denote the constant endomorphism by $\underline{1}$.

LEMMA 3. *Assume that m is odd.*

(a) *For all $k, l \in \mathbb{Z}_m$, there is an endomorphism e of $\underline{\mathbf{D}}_m$ such that $e(a) = a^k$ and $e(b) = a^l b$. Moreover, every non-constant endomorphism of $\underline{\mathbf{D}}_m$ is of this form. Thus $|\operatorname{End}\underline{\mathbf{D}}_m| = m^2 + 1$.*

(b) *$\underline{1} * \underline{1} = \underline{1}$ and, for all $e \in \operatorname{End}\underline{\mathbf{D}}_m$, the product $e * \underline{1}$ exists if and only if $e = \underline{1}$. If $e, f \in \operatorname{End}\underline{\mathbf{D}}_m$ with $e \neq \underline{1}$ and $f \neq \underline{1}$, then $e * f$ exists.*

(c) *$\langle \operatorname{End}\underline{\mathbf{D}}_m \backslash \{\underline{1}\}; * \rangle$ is an abelian group isomorphic to $\mathbb{Z}_m^2$ and is generated (as a group) by the powers (with respect to composition of maps) of the automorphism α. The retraction ϵ is an identity element for $*$ on $\operatorname{End}\underline{\mathbf{D}}_m \backslash \{\underline{1}\}$.*

PROOF. It is easily seen that $a_1 := a^k$ and $b_1 := a^l b$ satisfy the defining relations for $\underline{\mathbf{D}}_m$ and hence $e(a) := a^k$ and $e(b) := a^l b$ does determine an endomorphism of $\underline{\mathbf{D}}_m$. We must now show that when m is odd, there are no other non-constant endomorphisms. Let e be an endomorphism of $\underline{\mathbf{D}}_m$ and define $a_1 := e(a)$ and $b_1 = e(b)$. Thus

$$a_1^m = 1, \quad b_1^2 = 1 \quad \text{and} \quad b_1 a_1 = a_1^{m-1} b_1. \tag{2}$$

As m is odd, we must have $a_1 \in N$ as every element of Nb has order 2: thus $a_1 = a^k$ for some $k \in \mathbb{Z}_m$. If $b_1 \in N$, then since N contains no elements of order 2, we have

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$e(b) = b_1 = 1$. In this case (2) implies that $k = 0$ and consequently $e = \underline{1}$. If $b_1 \in Nb$, then $e(b) = a^l b$ for some $l \in \mathbb{Z}_m$, as required. Hence (a) holds.

It is clear that $\underline{1} * \underline{1}$ exists and equals $\underline{1}$. If $e \in \operatorname{End}\underline{\mathbf{D}}_m$ with $e \neq \underline{1}$, then by (a) we have $e(b) \in Nb$. Hence

$$(\underline{1}(b), e(b)) = (1, e(b)) \notin K = N^2 \cup Nb^2 = \operatorname{dom}(*)$$

and consequently $\underline{1} * e$ is not defined on $D(\underline{\mathbf{D}}_m) = \operatorname{End}\underline{\mathbf{D}}_m$. If $e, f \in \operatorname{End}\underline{\mathbf{D}}_m \backslash \{\underline{1}\}$, then $e(N) \subseteq N$, $e(Nb) \subseteq Nb$, $f(N) \subseteq N$ and $f(Nb) \subseteq Nb$ and hence, for all $u \in D_n$,

$$(e(u), f(u)) \in K = N^2 \cup Nb^2 = \operatorname{dom}(*),$$

that is, $e * f$ is defined on $D(\underline{\mathbf{D}}_m) = \operatorname{End}\underline{\mathbf{D}}_m$. Thus (b) holds.

Since, by Lemma 1, the partial operation $*$ on D_n is commutative and associative wherever it is defined, it follows that $\langle \operatorname{End}\underline{\mathbf{D}}_m \backslash \{\underline{1}\}; *\rangle$ is a commutative semigroup. It is easily seen that the map $\gamma : \operatorname{End}\underline{\mathbf{D}}_m \to \mathbb{Z}_m^2$, given by $\gamma(e) = (k, l)$ if $e(a) = a^k$ and $e(b) = a^l b$, is an isomorphism. It is clear from the definition of $*$ that ϵ is an identity element for $*$ on $\operatorname{End}\underline{\mathbf{D}}_m$. Denote powers of $e \in \operatorname{End}\underline{\mathbf{D}}_m$ with respect to composition of maps by e^s and powers with respect to $*$ by $e^{[s]}$. A simple calculation shows that $e := \mathrm{id}_{\underline{\mathbf{D}}_m}^{[m-1]} * \alpha^l * \mathrm{id}_{\underline{\mathbf{D}}_m}^{[k]}$ satisfies $e(a) = a^k$ and $e(b) = a^l b$, whence the set $\{\alpha^s \mid s = 1, \dots, m\}$ generates the group $\langle \operatorname{End}\underline{\mathbf{D}}_m \backslash \{\underline{1}\}; *\rangle$. This proves (c).

We are now ready to prove Theorem 2. We will establish this strong duality by proving the conditions (INJ), (CLO) and (STR)—see Propositions 4, 5 and 8 below. The first is of independent, group-theoretic interest.

PROPOSITION 4. *Let m be odd. A map $\varphi : D_m^n \to D_m$ is a term function on the dihedral group $\underline{\mathbf{D}}_m$ if and only if φ preserves the action of the automorphism α, the constant* 1 *and the partial operations $+$ and $*$.*

PROOF. As Kovács observes in [7], since $\mathscr{A}$ is the product variety $\mathscr{A}_m\mathscr{A}_2$, Corollary 21.13 of Neumann [8] implies that the n-generated free group in $\mathscr{A}$ is an extension of a $F_{\mathscr{A}_m}(k)$ by $F_{\mathscr{A}_2}(n)$, where $k = (n-1)|F_{\mathscr{A}_2}(n)| + 1$. Thus the n-generated free group over $\underline{\mathbf{D}}_m$ has cardinality $2^n m^{(n-1)2^n+1}$. Since the (partial) operations on $\underset{\sim}{\mathbf{D}}_m$ are algebraic over $\underline{\mathbf{D}}_m$, every n-ary $\underline{\mathbf{D}}_m$-term function belongs to $\mathscr{X}(\underset{\sim}{\mathbf{D}}_m^n, \underset{\sim}{\mathbf{D}}_m)$. Thus it suffices to show that $|\mathscr{X}(\underset{\sim}{\mathbf{D}}_m^n, \underset{\sim}{\mathbf{D}}_m)| \leqslant 2^n m^{(n-1)2^n+1}$.

Let $\varphi \in \mathscr{X}(\underset{\sim}{\mathbf{D}}_m^n, \underset{\sim}{\mathbf{D}}_m)$. Since $+$ is the original group operation on the subgroup $\{1, b\}$ of $\underline{\mathbf{D}}_m$, and since φ preserves $+$, the restriction $\varphi\restriction_{\{1,b\}^n}$ is an abelian group homomorphism. There are exactly 2^n such homomorphisms. We will show that each homomorphism from $\{1, b\}^n$ to $\{1, b\}$ has at most $m^{(n-1)2^n+1}$ extensions to a member of

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$\mathcal{X}(\underset{\sim}{\mathbf{D}}_m^n, \underset{\sim}{\mathbf{D}}_m)$. Recall our retraction ϵ of D_m onto $H = \{1, b\}$; on D_m^n it is a retraction onto $\{1, b\}^n$. Now recall our partial operation $*$; it turns each of the two cosets of N into an abelian group isomorphic to $\mathbb{Z}_m$. On D_m^n it turns each of the cosets of N^n into an abelian group isomorphic to $\mathbb{Z}_m^n$. But each such coset is equal to $\epsilon^{-1}(\boldsymbol{h})$ for some $\boldsymbol{h} \in \{1, b\}^n$. Since φ preserves $*$, there are at most m^n possibilities for $\varphi\restriction_{\epsilon^{-1}(\boldsymbol{h})}$. (Note that $\varphi(\epsilon^{-1}(\boldsymbol{h}))$ will be contained in either N or Nb, depending on whether $\varphi(\boldsymbol{h})$ equals 1 or b). This yields $|\mathcal{X}(\underset{\sim}{\mathbf{D}}_m^n, \underset{\sim}{\mathbf{D}}_m)| \leqslant 2^n m^{n2^n}$, since for each of the 2^n choices for $\boldsymbol{h} \in \{1, b\}^n$ we have at most m^n choices for $\varphi\restriction_{\epsilon^{-1}(\boldsymbol{h})}$. We now take into account the effect of the automorphism α. If $\boldsymbol{h} \neq \boldsymbol{1}$, then $\alpha(\boldsymbol{h}) \neq \boldsymbol{h}$, but $\epsilon(\alpha(\boldsymbol{h})) = \boldsymbol{h}$ whence $\alpha(\boldsymbol{h}) \in \epsilon^{-1}(\boldsymbol{h})$. As $\varphi(\alpha(\boldsymbol{h}))$ is determined by $\varphi(\boldsymbol{h})$ (since φ preserves α) and as $\alpha(\boldsymbol{h})$ is an element of order m in the abelian group on $\epsilon^{-1}(\boldsymbol{h})$ determined by $*$, there are at most m^{n-1} choices for extending φ to all of $\epsilon^{-1}(\boldsymbol{h})$ when $\boldsymbol{h} \neq \boldsymbol{1}$. Hence

$$|\mathcal{X}(\underset{\sim}{\mathbf{D}}_m^n, \underset{\sim}{\mathbf{D}}_m)| \leqslant 2^n \cdot m^{(n-1)(2^n-1)} m^n = 2^n m^{(n-1)2^n+1},$$

as required.

PROPOSITION 5. *$\underset{\sim}{\mathbf{D}}_m$ is injective in $\mathcal{X}$.*

PROOF. Let $\mathbf{X}$ be a closed substructure of $\underset{\sim}{\mathbf{D}}_m^I$ for some I, and let $\varphi \in \mathcal{X}(\mathbf{X}, \underset{\sim}{\mathbf{D}}_m)$ be a continuous structure-preserving map. By Lemma 3, the substructure $\mathbf{X}$ is closed under every endomorphism of $\underset{\sim}{\mathbf{D}}_m$ and φ preserves every endomorphism. In particular, $\mathbf{X}$ is closed under ϵ and φ preserves ϵ. We must find $\psi \in \mathcal{X}(\underset{\sim}{\mathbf{D}}_m^I, \underset{\sim}{\mathbf{D}}_m)$ with $\psi\restriction_{\mathbf{X}} = \varphi$. On $\underset{\sim}{\mathbf{D}}_m^I$, the map ϵ is a continuous retraction onto $\{1, b\}^I$. Since $\mathbf{X}$ is closed under ϵ, it follows easily that $\epsilon(X) = X \cap \{1, b\}^I$ and so is a closed subgroup of $\langle\{1, b\}^I, +\rangle$. Thus $\varphi\restriction_{\epsilon(X)}$ is a continuous $+$-homomorphism. By the Pontryagin duality for abelian groups of exponent 2, there is a continuous $+$-homomorphism $\varphi_1 : \{1, b\}^I \to \{1, b\}$ which extends $\varphi\restriction_{\epsilon(X)}$.

The set

$$\begin{aligned} X' &= X \cup \bigcup\{\alpha^l(\{1, b\}^I) \mid l \in \mathbb{Z}_m\} \\ &= X \cup \bigcup\{\{1, a^l b\}^I \mid l \in \mathbb{Z}_m\} \end{aligned}$$

is a closed substructure of $\underset{\sim}{\mathbf{D}}_m^I$. (To see that X' is closed under $*$, use the fact that if $\boldsymbol{x} * \boldsymbol{y}$ is defined and $\boldsymbol{y} \in \{1, a^l b\}^I$, then $\boldsymbol{x} * \boldsymbol{y} = \alpha^l(\boldsymbol{x})$.) Define a map $\varphi_2 : X' \to D_m$ by

$$\varphi_2\restriction_{\mathbf{X}} = \varphi, \qquad \text{and} \qquad \varphi_2\restriction_{\{1,a^l b\}^I} = \alpha^l \circ \varphi_1 \circ \alpha^{(m-l)} \qquad \text{for all } l \in \mathbb{Z}_m.$$

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For all $x \in X \cap \{1, a^l b\}$, we have

$$\begin{aligned} \alpha^l \circ \varphi_1 \circ \alpha^{(m-l)}(x) &= \alpha^l \varphi_1(\alpha^{(m-l)}(x)) \\ &= \alpha^l \varphi(\alpha^{(m-l)}(x)) \qquad \text{as } \alpha^{(m-l)}(x) \in \epsilon(X) \text{ and } \varphi_1\restriction_{\epsilon(\mathbf{X})} = \varphi \\ &= \varphi\alpha^l\alpha^{(m-l)}(x) \qquad \text{as } \varphi \text{ preserves } \alpha \\ &= \varphi(x), \end{aligned}$$

from which it follows that φ_2 is well-defined. Clearly, φ_2 is continuous and preserves the partial operation $+$ and the constant 1. We now show that φ_2 also preserves both the action of α and the partial operation $*$. If $x \in X$, then it is trivial that φ_2 preserves the action of α on x since $\varphi_2\restriction_{\mathbf{X}} = \varphi$ and φ preserves α. If $x \in \{1, a^l b\}^I$, then $\alpha(x) \in \{1, a^{l+1}b\}^I$ and hence

$$\varphi_2(\alpha(x)) = \alpha^{l+1}\varphi_1\alpha^{m-l-1}(\alpha(x)) = \alpha(\alpha^l\varphi_1\alpha^{m-l}(x)) = \alpha(\varphi_2(x)),$$

whence φ_2 preserves α. Note that if $y \in \{1, a^l b\}^I$, then $\varphi_2(y) \in \{1, a^l b\}$ since $\varphi_1(\{1, b\}^I) \subseteq \{1, b\}$. If $x \in X'$ and $y \in \{1, a^l b\}^I$ and $x * y$ is defined, then $x * y = \alpha^l(x)$ and hence

$$\begin{aligned} \varphi_2(x * y) &= \varphi_2(\alpha^l(x)) \\ &= \alpha^l(\varphi_2(x)) \qquad \text{as } \varphi_2 \text{ preserves } \alpha \\ &= \varphi_2(x) * \varphi_2(y) \qquad \text{as } \varphi_2(x) \in \{1, a^l b\}^I. \end{aligned}$$

Hence φ_2 preserves $*$ and consequently $\varphi_2 \in \mathscr{X}(\mathbf{X}', \underset{\sim}{\mathbf{D}}_m)$. That is, without loss of generality, we may assume that X contains $\{1, b\}^I$ and thus

$$Z := \{1, b\}^I \cup \{1, ab\}^I \cup \{1, a^2 b\}^I \cup \cdots \cup \{1, a^{m-1}b\}^I \subseteq X.$$

Note that $\mathbf{Z}$ is a closed substructure of $\underset{\sim}{\mathbf{D}}_m^I$ and hence, by assumption, is a closed substructure of $\mathbf{X}$.

A simple-minded attempt to define the extension ψ would proceed as follows. For any $h \in \{1, b\}^I$, the set $X \cap \epsilon^{-1}(h)$ is non-empty and so is a closed subgroup of $\epsilon^{-1}(h) = N^I h$ (under the restriction of $*$). Now $\langle \epsilon^{-1}(h); * \rangle$ is a compact topological abelian group of exponent m and since φ preserves $*$, the restriction $\varphi\restriction_{X \cap \epsilon^{-1}(h)}$ is a continuous group homomorphism with codomain $\langle N\varphi(h); * \rangle$. By the Pontryagin duality for abelian groups of exponent m, we can extend $\varphi\restriction_{X \cap \epsilon^{-1}(h)}$ to a continuous group homomorphism $\psi_h : \epsilon^{-1}(h) \to N\varphi(h)$. Doing this for each $h \in \{1, b\}^I$, we obtain an extension of φ to a map $\psi : D_m^I \to D_m$ given by $\psi\restriction_{\epsilon^{-1}(h)} = \psi_h$ for all $h \in \{1, b\}^I$.

We claim that ψ is structure preserving. Since $\mathbf{Z}$ is a substructure of $\mathbf{X}$, it is trivial that ψ preserves both $+$ and 1 and ψ preserves $*$ by construction. That ψ also preserves α follows immediately once we have established the following lemma.

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LEMMA 6. *Let* $\mathbf{X}$ *be a closed substructure of* $\underset{\sim}{\mathbf{D}}_m^I$.

(a) *For all* $x \in D_m$ *(and therefore for all* $\boldsymbol{x} \in X$*) we have* $(x, \alpha(\epsilon(x))) \in \operatorname{dom}(*)$ *and* $\alpha(x) = x * \alpha(\epsilon(x))$.

(b) *If* $\psi : X \to D_m$ *preserves* $*$ *and* ϵ *and* $\psi\restriction_{X \cap Z}$ *preserves* α, *then* ψ *preserves* α.

(c) *Let* $\mathbf{Z}$ *be a substructure of* $\mathbf{X}$. *If* $\psi : X \to D_m$ *preserves* $*$ *and* $\psi\restriction_Z$ *preserves* α, *then* ψ *preserves* α.

PROOF. The proof of (a) is a simple calculation and (b) follows easily from (a). Assume that $\mathbf{Z}$ is a substructure of $\mathbf{X}$, that ψ preserves $*$ and that $\psi\restriction_Z$ preserves α. By (b), in order to establish (c) it remains to show that ψ preserves ϵ.

Since $Z \subseteq X$, for all $\boldsymbol{h} \in \{1, b\}^I$, the set $X \cap \epsilon^{-1}(\boldsymbol{h})$ is non-empty, whence $\langle X \cap \epsilon^{-1}(\boldsymbol{h}); *\rangle$ is a group with identity element $\boldsymbol{h}$. Since ψ preserves $*$, we have $\psi(\boldsymbol{h}) \in \{1, b\}$ and $\psi(X \cap \epsilon^{-1}(\boldsymbol{h})) \subseteq N\psi(\boldsymbol{h})$. Thus

$$\epsilon(\psi(X \cap \epsilon^{-1}(\boldsymbol{h}))) \subseteq \epsilon(N\psi(\boldsymbol{h})) = \{\psi(\boldsymbol{h})\},$$

which gives $\epsilon(\psi(\boldsymbol{x})) = \psi(\epsilon(\boldsymbol{x}))$ for all $\boldsymbol{x} \in X$, as required.

Unfortunately, if I is infinite, we cannot guarantee that the extension ψ is continuous and the simple-minded approach falters. Nevertheless, the basic idea can be salvaged. Obviously, we need to invoke some kind of compactness argument. The following lemma plays a crucial role.

LEMMA 7: THE GOOD, THE BAD, BUT NO UGLY. *Let* A *and* I *be sets with* A *finite. Suppose that, for every finite* $I' \subseteq I$, *each element of* $A^{I'}$ *is labeled either 'good' or 'bad' and that if* $I'' \subseteq I'$ *and* $\boldsymbol{x} \in A^{I'}$ *is bad, then so is* $\boldsymbol{x}\restriction_{I''} \in A^{I''}$. *Then either there is a finite* $I' \subseteq I$ *such that each element of* $A^{I'}$ *is good, or there is an* $\boldsymbol{x} \in A^I$ *such that* $\boldsymbol{x}\restriction_{I'}$ *is bad for each finite* $I' \subseteq I$.

PROOF. Endow A with the discrete topology and A^I with the product topology; then A^I is a compact space with a basis of clopen sets. For finite $I' \subseteq I$, let $X(I') := \{\boldsymbol{x} \in A^I \mid \boldsymbol{x}\restriction_{I'} \text{ is bad}\}$; it is a closed set. Then by the finite intersection property, either $\bigcap\{X(I') \mid I' \subseteq I \text{ is finite}\}$ is non-empty or there are finitely many finite sets $I_1, \dots, I_k$ such that $X(I_1) \cap \dots \cap X(I_k)$ is empty. In the former case, take $\boldsymbol{x} \in \bigcap\{X(I') \mid I' \subseteq I \text{ is finite}\}$; then $\boldsymbol{x}\restriction_{I'}$ is bad for any finite $I' \subseteq I$. In the latter case, let $I' = I_1 \cup \dots \cup I_k$; then, as $X(I') \subseteq X(I_j)$ for $1 \leqslant j \leqslant k$, the set $X(I')$ is empty and so every member of $A^{I'}$ is good.

Let us apply the Good, the Bad, but no Ugly Lemma to $\{1, b\}^I$. For $\boldsymbol{h} \in \{1, b\}^I$ and finite $I' \subseteq I$, define

$$\Gamma_{I'}^{\boldsymbol{h}} := \{(\boldsymbol{x}\restriction_{I'}, \varphi(\boldsymbol{x})) \mid \boldsymbol{x} \in X \text{ and } \epsilon(\boldsymbol{x}\restriction_{I'}) = \boldsymbol{h}\restriction_{I'}\}.$$

Call $\boldsymbol{h}\restriction_{I'}$ 'good' if $\Gamma^{\boldsymbol{h}}_{I'}$ is a subset of the graph of a $*$–preserving map defined on $\epsilon^{-1}(\boldsymbol{h}\restriction_{I'}) = \epsilon^{-1}(\boldsymbol{h})\restriction_{I'}$; otherwise, call $\boldsymbol{h}\restriction_{I'}$ 'bad'. Let $I'' \subseteq I'$ and let π denote the natural restriction map from $\epsilon^{-1}(\boldsymbol{h})\restriction_{I'}$ to $\epsilon^{-1}(\boldsymbol{h})\restriction_{I''}$. If γ is an extension of $\Gamma^{\boldsymbol{h}}_{I''}$ to a $*$–preserving map on $\epsilon^{-1}(\boldsymbol{h}\restriction_{I''})$, then $\gamma \circ \pi$ is an extension of $\Gamma^{\boldsymbol{h}}_{I'}$ to a $*$–preserving map on $\epsilon^{-1}(\boldsymbol{h}\restriction_{I'})$. Hence 'badness' is hereditary in the sense required by the lemma. Thus, by the lemma, either

(a) there is a finite subset I' of I such that every member $\boldsymbol{h}'$ of $\{1, b\}^{I'}$ is good, or
(b) there exists $\boldsymbol{h} \in \{1, b\}^I$ such that $\boldsymbol{h}\restriction_{I'}$ is bad, for all finite $I' \subseteq I$.

Case (a). For each $\boldsymbol{h}' \in \{1, b\}^{I'}$, let $\psi_{\boldsymbol{h}'} : \epsilon^{-1}(\boldsymbol{h}') \to D_m$ be a $*$–preserving map which satisfies $\psi_{\boldsymbol{h}'}(\boldsymbol{x}\restriction_{I'}) = \varphi(\boldsymbol{x})$ for all $\boldsymbol{x} \in X$ with $\epsilon(\boldsymbol{x}\restriction_{I'}) = \boldsymbol{h}'$, and define $\psi' : D_m^{I'} \to D_m$ to be the union of the maps $\psi_{\boldsymbol{h}'}$ for $\boldsymbol{h}' \in \{1, b\}^{I'}$. We claim that ψ' is an $\mathscr{X}$–morphism. Let $\boldsymbol{x}'_1, \boldsymbol{x}'_2 \in \{1, b\}^{I'} = \mathrm{dom}_{\underset{\sim}{\mathbf{D}}_m^{I'}}(+)$. Thus as $Z \subseteq X$, there exist $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \{1, b\}^I \subseteq X$ with $\boldsymbol{x}_i\restriction_{I'} = \boldsymbol{x}'_i$. Thus $(\boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathrm{dom}_{\mathbf{X}}(+)$ and so $\boldsymbol{x}_1 + \boldsymbol{x}_2 \in X$ and $(\boldsymbol{x}_1 + \boldsymbol{x}_2)\restriction_{I'} = \boldsymbol{x}'_1 + \boldsymbol{x}'_2$. Define $\boldsymbol{h}' := \epsilon(\boldsymbol{x}_1 + \boldsymbol{x}_2)$ and $\boldsymbol{h}'_i := \epsilon(\boldsymbol{x}'_i)$. Thus

$$\begin{aligned}
\psi'(\boldsymbol{x}'_1 + \boldsymbol{x}'_2) &= \psi_{\boldsymbol{h}'}(\boldsymbol{x}'_1 + \boldsymbol{x}'_2) && \text{(definition of } \psi') \\
&= \psi_{\boldsymbol{h}'}((\boldsymbol{x}_1 + \boldsymbol{x}_2)\restriction_{I'}) \\
&= \varphi(\boldsymbol{x}_1 + \boldsymbol{x}_2) && \text{(definition of } \psi_{\boldsymbol{h}'}) \\
&= \varphi(\boldsymbol{x}_1) + \varphi(\boldsymbol{x}_2) && \text{(as } \varphi \text{ preserves } +) \\
&= \psi_{\boldsymbol{h}'_1}(\boldsymbol{x}_1\restriction_{I'}) + \psi_{\boldsymbol{h}'_2}(\boldsymbol{x}_2\restriction_{I'}) && \text{(definition of } \psi_{\boldsymbol{h}'_i}) \\
&= \psi_{\boldsymbol{h}'_1}(\boldsymbol{x}'_1) + \psi_{\boldsymbol{h}'_2}(\boldsymbol{x}'_2) \\
&= \psi'(\boldsymbol{x}'_1) + \psi'(\boldsymbol{x}'_2),
\end{aligned}$$

whence ψ' preserves $+$. By Lemma 6 applied to ψ', it remains to show that $\psi'\restriction_{\mathbf{Z}'}$ preserves α, where $Z' := \bigcup\{\{1, a^l b\}^{I'} \mid l \in \mathbb{Z}_m\}$. Let $\boldsymbol{x}' \in Z'$ and define $h' := \epsilon(\boldsymbol{x}') = \epsilon(\alpha(\boldsymbol{x}'))$. Let $\boldsymbol{x} \in Z$ with $\boldsymbol{x}\restriction_{I'} = \boldsymbol{x}'$. Note that $\boldsymbol{x} \in X$ as $Z \subseteq X$. Now

$$\begin{aligned}
\psi'(\alpha(\boldsymbol{x}')) &= \psi_{\boldsymbol{h}'}(\alpha(\boldsymbol{x})\restriction_{I'}) && \text{(definition of } \psi') \\
&= \varphi(\alpha(\boldsymbol{x})) && \text{(definition of } \psi_{\boldsymbol{h}'}) \\
&= \alpha(\varphi(\boldsymbol{x})) && \text{(as } \varphi \text{ preserves } \alpha) \\
&= \alpha(\psi_{\boldsymbol{h}'}(\boldsymbol{x}\restriction_{I'})) && \text{(definition of } \psi_{\boldsymbol{h}'}) \\
&= \alpha(\psi_{\boldsymbol{h}'}(\boldsymbol{x}')) \\
&= \alpha(\psi'(\boldsymbol{x}')),
\end{aligned}$$

and consequently ψ' preserves α on $\mathbf{Z}'$, as required. Thus $\psi' : \underset{\sim}{\mathbf{D}}_m^{I'} \to \underset{\sim}{\mathbf{D}}_m$ is an $\mathscr{X}$–morphism, as claimed.

Finally, let $\pi_{I'} : \underset{\sim}{\mathbf{D}}_m^I \to (\underset{\sim}{\mathbf{D}}_m^{I'})$ denote the restriction map. Then the map $\psi' \circ \pi_{I'} : \underset{\sim}{\mathbf{D}}_m^I \to \underset{\sim}{\mathbf{D}}_m$ is an $\mathscr{X}$–morphism which extends φ since $\psi(\boldsymbol{x}) = \psi'(\boldsymbol{x}\restriction_{I'}) = \psi_{\epsilon(\boldsymbol{x}\restriction_{I'})}(\boldsymbol{x}\restriction_{I'}) = \varphi(\boldsymbol{x})$ for all $\boldsymbol{x} \in X$.

Case (b). Assume that $\boldsymbol{h} \in \{1, b\}^I$ with $\boldsymbol{h}\restriction_{I'}$ bad, for all finite $I' \subseteq I$, that is, for every finite subset I' of I, the set

$$\Gamma_{I'}^{\boldsymbol{h}} := \{ (\boldsymbol{x}\restriction_{I'}, \varphi(\boldsymbol{x})) \mid \boldsymbol{x} \in X \text{ and } \epsilon(\boldsymbol{x}\restriction_{I'}) = \boldsymbol{h}\restriction_{I'} \}.$$

is not a subset of a $*$–preserving map defined on $\epsilon^{-1}(\boldsymbol{h}\restriction_{I'})$. Define $X^0 := X \cup \epsilon^{-1}(\boldsymbol{h})$. Then X^0 is closed under $*$ and, as in the simple-minded approach, we may apply the Pontryagin duality for abelian groups of exponent m to extend φ to a continuous $*$–preserving map $\varphi^0 : X^0 \to D_m$. Since $\epsilon(\boldsymbol{x}\restriction_{I'}) = \epsilon(\boldsymbol{x})\restriction_{I'}$ and since $\epsilon^{-1}(\boldsymbol{h}) \subseteq X^0$, we have

$$\begin{aligned} \Gamma_{I'}^{\boldsymbol{h}} &= \{ (\boldsymbol{x}\restriction_{I'}, \varphi(\boldsymbol{x})) \mid \boldsymbol{x} \in X \text{ and } \epsilon(\boldsymbol{x})\restriction_{I'} = \boldsymbol{h}\restriction_{I'} \} \\ &\subseteq \{ (\boldsymbol{x}\restriction_{I'}, \varphi^0(\boldsymbol{x})) \mid \boldsymbol{x} \in X \text{ and } \epsilon(\boldsymbol{x})\restriction_{I'} = \boldsymbol{h}\restriction_{I'} \} \\ &= \{ (\boldsymbol{x}\restriction_{I'}, \varphi^0(\boldsymbol{x})) \mid \boldsymbol{x} \in \epsilon^{-1}(\boldsymbol{h}) \}. \end{aligned}$$

The set $\Gamma^0 := \{ (\boldsymbol{x}\restriction_{I'}, \varphi^0(\boldsymbol{x})) \mid \boldsymbol{x} \in \epsilon^{-1}(\boldsymbol{h}) \}$ is easily seen to be a $*$–closed subset of $D_m^{I'} \times D_m$. Hence, if Γ_0 were the graph of a map, then it would be the graph of a $*$–preserving map defined on $\epsilon^{-1}(\boldsymbol{h})\restriction_{I'}$, contradicting the fact that $\boldsymbol{h}\restriction_{I'}$ is bad. Thus there exist $\boldsymbol{y}, \boldsymbol{z} \in X$ (depending on I') such that $\boldsymbol{y}\restriction_{I'} = \boldsymbol{z}\restriction_{I'}$ but $\varphi^0(\boldsymbol{y}) \neq \varphi^0(\boldsymbol{z})$. This means that the continuous map φ^0 does not depend on any finite subset of I, a contradiction to the fact that every continuous map from a closed subspace of any power of $\underset{\sim}{\mathbf{D}}_m$ into $\underset{\sim}{\mathbf{D}}_m$ depends on only finitely many components. Hence, Case (b) cannot occur.

PROPOSITION 8. *If* $\mathbf{X}$ *is a closed substructure of* $\underset{\sim}{\mathbf{D}}_m^I$ *for some set* I *and* $\boldsymbol{y} \in D_m^I \backslash X$, *then there exists a continuous morphism* $\psi : \underset{\sim}{\mathbf{D}}_m^I \to \underset{\sim}{\mathbf{D}}_m$ *such that* $\psi\restriction_X = \underline{1}$ *but* $\psi(\boldsymbol{y}) \neq 1$, *where* $\underline{1}$ *is the constant map onto 1.*

PROOF. Let $\mathbf{X}$ be a substructure of $\underset{\sim}{\mathbf{D}}_m^I$ and let $\boldsymbol{y} \notin X$.
If $\boldsymbol{y} \in Z$, say $\boldsymbol{y} \in \{1, a^l b\}^I$, then define $\boldsymbol{y}_1 \in \{1, b\}^I$ by

$$\boldsymbol{y}_1(i) = \begin{cases} 1 & \text{if } \boldsymbol{y}(i) = 1, \\ b & \text{if } \boldsymbol{y}(i) = a^l b. \end{cases}$$

Note that $\alpha^l(\boldsymbol{y}_1) = \boldsymbol{y}$ whence $\boldsymbol{y}_1 \in \{1, b\}^I \backslash X$ (as X is closed under α). Since the Pontryagin duality for $\mathscr{A}_2$ is as strong, there exists a continuous group homomorphism $\varphi_1 : \{1, b\}^I \to \{1, b\}$ such that $\varphi_1\restriction_{\epsilon(\mathbf{X})} = \underline{1}$ but $\varphi_1(\boldsymbol{y}_1) \neq 1$. As in the proof of Proposition 5, there exists an extension $\psi : \underset{\sim}{\mathbf{D}}_m^I \to \underset{\sim}{\mathbf{D}}_m$ with $\psi\restriction_{\mathbf{X}} = \underline{1}$ and $\psi\restriction_{\{1,b\}^I} = \varphi_1$. Suppose that $\psi(\boldsymbol{y}) = 1$; then

$$\varphi_1(\boldsymbol{y}_1) = \psi(\boldsymbol{y}_1) = \psi(\alpha^{m-l}(\boldsymbol{y})) = \alpha^{m-l}(\psi(\boldsymbol{y})) = \alpha^{m-l}(1) = 1,$$

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a contradiction. Hence $\psi(\boldsymbol{y}) \neq 1$, as required.

If $\boldsymbol{y} \notin Z$, then, as in the proof of Proposition 5, we may assume that $Z \subseteq X$. Since $\mathbf{X}$ is a closed subspace of $\underset{\sim}{\mathbf{D}}_m^I$ and $\boldsymbol{y} \notin X$, there exists a finite subset I' of I such that $\boldsymbol{x}\restriction_{I'} \neq \boldsymbol{y}\restriction_{I'}$ for all $\boldsymbol{x} \in X$. Let $\boldsymbol{h} := \epsilon(\boldsymbol{y})$, $\boldsymbol{y}' := \boldsymbol{y}\restriction_{I'}$ and $\boldsymbol{h}' := \boldsymbol{h}\restriction_{I'} = \epsilon(\boldsymbol{y}')$. Since the Pontryagin duality for $\mathscr{A}_m$ is strong, there exists a $*$–preserving map $\psi_{\boldsymbol{h}'} : \epsilon^{-1}(\boldsymbol{h}') \to \underset{\sim}{\mathbf{D}}_m$ such that $\psi_{\boldsymbol{h}'}(\boldsymbol{x}\restriction_{I'}) = 1$ for all $\boldsymbol{x} \in X$ with $\epsilon(\boldsymbol{x}\restriction_{I'}) = \boldsymbol{h}'$ while $\psi_{h'}(\boldsymbol{y}\restriction_{I'}) \neq 1$. Define $\psi' : D_m^{I'} \to D_m$ by

$$\psi'(z) = \begin{cases} 1 & \text{if } z \notin \epsilon^{-1}(\boldsymbol{h}'), \\ \psi_{\boldsymbol{h}'}(z) & \text{if } z \in \epsilon^{-1}(\boldsymbol{h}'), \end{cases}$$

and let $\psi := \psi' \circ \pi_{I'}$. Clearly, $\psi : \underset{\sim}{\mathbf{D}}_m^I \to \underset{\sim}{\mathbf{D}}_m$ preserves $*$ and $\psi\restriction_{\mathbf{Z}}$ preserves α (since $Z \subseteq X$ and $\psi(X) = \{1\}$). Thus, Lemma 6(c), with $\mathbf{X} = \underset{\sim}{\mathbf{D}}_m^I$, implies that ψ preserves α. Finally, $\{1, b\}^I \subseteq Z \subseteq X$ implies that ψ takes the constant value 1 on $\{1, b\}$ and so preserves $+$. Thus ψ is the required $\mathscr{X}$–morphism.

This concludes the proof of Theorem 2. We close the paper with an interesting special case of the problem stated in the introduction.

PROBLEM. Does every finite metacyclic group admit a duality? In particular, does every dihedral group of order $2m$, with m even, admit a duality. Indeed, does $\underline{\mathbf{D}}_4$ admit a duality?

NOTE ADDED IN PROOF. Cs. Szabo and the second author have shown that no finite, non-abelian nilpotent group admits a duality.

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