ON THE GROWTH OF LINEAR RECURRENCES IN FUNCTION FIELDS

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Abstract

Let \((G_n)_{n=0}^{\infty}\) be a nondegenerate linear recurrence sequence whose power sum representation is given by \(G_n = a_1(n)\alpha_1^n + \cdots + a_t(n)\alpha_t^n\). We prove a function field analogue of the well-known result in the number field case that, under some nonrestrictive conditions, \(|G_n| \geq (\max_{j=1, \ldots, t} |\alpha_j|)^n(1-\varepsilon)\) for \(n\) large enough.


Keywords and phrases: function fields, linear recurrences, S-units.

1. Introduction

Let \((G_n)_{n=0}^{\infty}\) be a nondegenerate linear recurrence sequence with power sum representation \(G_n = a_1(n)\alpha_1^n + \cdots + a_t(n)\alpha_t^n\). This expression makes sense for a sequence \((G_n)_{n=0}^{\infty}\) taking values in any field \(K\); the characteristic roots \(\alpha_i\) as well as the coefficients of the polynomials \(a_i\) then lie in a finite extension \(L\) of \(K\). In this paper \(K\) is either a number field or a function field in one variable of characteristic zero. The nondegenerate condition means in the number field case that no ratio \(\alpha_i/\alpha_j\) for \(i \neq j\) is a root of unity and in the function field case that no ratio \(\alpha_i/\alpha_j\) for \(i \neq j\) is contained in the field of constants. In the number field case it is well known that, if \(\max_{j=1, \ldots, t} |\alpha_j| > 1\), then for any \(\varepsilon > 0\) the inequality

\[ |G_n| \geq (\max_{j=1, \ldots, t} |\alpha_j|)^n(1-\varepsilon) \]  \hspace{1cm} (1.1)

is satisfied for every sufficiently large \(n\).

The purpose of this paper is to prove an analogous result in the case of a function field in one variable of characteristic zero. Thus we answer, in the setting we are supported by Austrian Science Fund (FWF): I4406.

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working in, Open Question 3 in [11]; we are grateful to Shparlinski for bringing this paper to our attention. Firstly, we will prove a theorem which states an inequality for an arbitrary valuation in the splitting field \(L\) of the characteristic polynomial belonging to the linear recurrence sequence. Secondly, we will derive a corollary for the special case of polynomial power sums. In this special case the inequality takes a form very similar to (1.1). At this point we make the following observation: Theorem 1 in [6] already implies that under some nondegeneracy conditions the degree of polynomials in a linear recurrence sequence with polynomial roots cannot be bounded and therefore must grow to infinity as \(n\) does. But this theorem does not say how fast it must grow. In the present paper we will give a bound depending on \(n\) for the minimal possible degree of \(G_n\).

The number field case is often mentioned (see [3, 8] or more recently [1, 2]), but it is not that easy to access a proof of it (see [10] or the formulation in [9]). So we give a complete proof based on results of Evertse and Schmidt in the Appendix. By doing so we contribute to the goal that well-known facts should be fully accessible with proof following van der Poorten’s wise statement in [9] that ‘all too frequently, the well known is [often] not generally known, let alone known well’.

2. Results and notations

Throughout the paper we denote by \(K\) a function field in one variable over \(\mathbb{C}\). By \(L\) we usually denote a finite algebraic extension of \(K\). For the convenience of the reader we give a short summary of the notion of valuations that can also be found in [4]. For \(c \in \mathbb{C}\) and \(f(x) \in \mathbb{C}(x)\), where \(\mathbb{C}(x)\) is the rational function field over \(\mathbb{C}\), we denote by \(\nu_c(f)\) the unique integer such that \(f(x) = (x - c)^{\nu_c(f)} p(x)/q(x)\) with \(p(x), q(x) \in \mathbb{C}[x]\) such that \(p(c)q(c) \neq 0\). Further, we write \(\nu_{\infty}(f) = \deg q - \deg p\) if \(f(x) = p(x)/q(x)\). These functions \(\nu : \mathbb{C}(x) \to \mathbb{Z}\) are up to equivalence all valuations in \(\mathbb{C}(x)\). If \(\nu_c(f) > 0\), then \(c\) is called a zero of \(f\), and if \(\nu_c(f) < 0\), then \(c\) is called a pole of \(f\), where \(c \in \mathbb{C} \cup \{\infty\}\). For a finite extension \(K\) of \(\mathbb{C}(x)\) each valuation in \(\mathbb{C}(x)\) can be extended to no more than \([K : \mathbb{C}(x)]\) valuations in \(K\). This again gives up to equivalence all valuations in \(K\). Both in \(\mathbb{C}(x)\) as well as in \(K\) the sum formula

\[\sum_{\nu} \nu(f) = 0\]

holds, where the sum is taken over all valuations in the relevant function field. Valuations have the properties \(\nu(fg) = \nu(f) + \nu(g)\) and \(\nu(f + g) \geq \min(\nu(f), \nu(g))\) for all \(f, g \in K\). Each valuation in a function field corresponds to a place and vice versa. The places can be thought of as the equivalence classes of valuations. For more information about valuations and places we refer to [13].

For any power sum \(G_n = a_1(n)\alpha_1^n + \cdots + a_t(n)\alpha_t^n\) with \(a_j(n) = \sum_{k=0}^{m_j} a_{jk} n^k\) and any valuation \(\mu\) (in a function field \(L/K\) containing the \(\alpha_j\) and the coefficients of the \(a_j\)) we
Our main result is now the following theorem which gives a bound in the other direction.

**Theorem 2.1.** Let \((G_n)_{n=0}^\infty\) be a nondegenerate linear recurrence sequence taking values in \(K\) with power sum representation \(G_n = \alpha_1 a_1(n) + \cdots + \alpha_t a_t(n)\). Let \(L = K(\alpha_1, \ldots, \alpha_t)\) be the splitting field of the characteristic polynomial of that sequence and let \(\mu\) be a valuation on \(L\). Then there is an effectively computable constant \(C\), independent of \(n\), such that, for every sufficiently large \(n\), the inequality

\[
\mu(G_n) \leq C + n \cdot \min_{j=1, \ldots, t} \mu(\alpha_j)
\]

holds.

For the special case of a linear recurrence sequence of complex polynomials having complex polynomials as characteristic roots we get the following lower bound for the degree of the \(n\)th member of the sequence.

**Corollary 2.2.** Let \((G_n)_{n=0}^\infty\) be a nondegenerate linear recurrence sequence of polynomials in \(\mathbb{C}[x]\) with power sum representation \(G_n = \alpha_1 a_1(n) + \cdots + \alpha_t a_t(n)\) such that \(\alpha_1, \ldots, \alpha_t \in \mathbb{C}[x]\). Then there is an effectively computable constant \(C\), independent of \(n\), such that, for every sufficiently large \(n\), the inequality

\[
\deg G_n \geq n \cdot \max_{j=1, \ldots, t} \deg \alpha_j - C
\]

holds.

In the case of a binary recurrence sequence of polynomials, that is, \(t = 2\) in Corollary 2.2, one can use Mason’s function field abc theorem (see [7]) to show that the number of distinct zeros of \(G_n\) must go to infinity as \(n\) does. By considering this in slightly more detail, the number of distinct zeros of \(G_n\) can be bounded above (trivially) and below (by means of the abc theorem) both by linear polynomials in \(n\).

It would be interesting to prove a function field variant of Corollary 3.1 in [1]. However, because of Lemma 3.1, which is based on Dirichlet’s classical approximation theorem, we are not (yet) able to prove such a statement.

In the proof given in the next section we will apply the following special case of Theorem 1 in [6].
Lemma 2.3. Let $K$ be as above and $L$ be a finite extension of $K$ of genus $g$. Furthermore, let $\alpha_1, \ldots, \alpha_d \in L^*$ with $d \geq 2$ be such that $\alpha_i/\alpha_j \notin \mathbb{C}^*$ for each pair of subscripts $i,j$ with $1 \leq i < j \leq d$. Moreover, for every $i = 1, \ldots, d$, let $\pi_{il}, \ldots, \pi_{ir_i} \in L$ be $r_i$ linearly independent elements over $\mathbb{C}$. Put

$$q = \sum_{i=1}^{d} r_i.$$ 

Then, for every $n \in \mathbb{N}$ such that

$$\{\pi_{il}\alpha_i^l : l = 1, \ldots, r_i, i = 1, \ldots, d\}$$

is linearly dependent over $\mathbb{C}$, but no proper subset of this set is linearly dependent over $\mathbb{C}$, we have

$$n \leq C = C(q, g, \pi_{il}, \alpha_i : l = 1, \ldots, r_i, i = 1, \ldots, d).$$

The proof will also make use of height functions in function fields. Let us therefore define the height of an element $f \in L^*$ by

$$\mathcal{H}(f) := -\sum_{\nu} \min(0, \nu(f)) = \sum_{\nu} \max(0, \nu(f)),$$

where the sum is taken over all valuations in the function field $L/\mathbb{C}$. For every $z \in L \setminus \mathbb{C}$,

$$\mathcal{H}(z) = \sum_{\nu} \max(0, \nu(z)) = \sum_{\nu} \max(0, \nu_P(z))$$

$$= \deg \sum_{\nu} \max(0, \nu_P(z)) P = \deg(z) = [L : \mathbb{C}(z)] = \deg_{\mathbb{C}}(z),$$

by Theorem 1.4.11 in [13], where we have used the fact that all places have degree one since we are working over $\mathbb{C}$ (instead of the height, one can use $\deg_{\mathbb{C}}(z) = [L : \mathbb{C}(z)]$ as in [14]). Additionally, we define $\mathcal{H}(0) = \infty$. This height function satisfies some basic properties that are listed in the next lemma which is proven in [5].

Lemma 2.4. Let $\mathcal{H}$ denote the height on $L/\mathbb{C}$ as above. Then, for $f, g \in L^*$:

(a) $\mathcal{H}(f) \geq 0$ and $\mathcal{H}(f) = \mathcal{H}(1/f)$;
(b) $\mathcal{H}(f) - \mathcal{H}(g) \leq \mathcal{H}(f + g) + \mathcal{H}(g)$;
(c) $\mathcal{H}(f) - \mathcal{H}(g) \leq \mathcal{H}(fg) \leq \mathcal{H}(f) + \mathcal{H}(g)$;
(d) $\mathcal{H}(f^n) = |n| \cdot \mathcal{H}(f)$;
(e) $\mathcal{H}(f) = 0 \iff f \in \mathbb{C}^*$;
(f) $\mathcal{H}(A(f)) = \deg A \cdot \mathcal{H}(f)$ for any $A \in \mathbb{C}[T] \setminus \{0\}$.

We will also use the following function field analogue of the Schmidt subspace theorem.

Proposition 2.5 (Zannier [14]). Let $F/\mathbb{C}$ be a function field in one variable and of genus $g$. Let $\varphi_1, \ldots, \varphi_n \in F$ be linearly independent over $\mathbb{C}$ and let $r \in \{0, 1, \ldots, n\}$. 
Let $S$ be a finite set of places of $F$ containing all the poles of $\varphi_1, \ldots, \varphi_n$ and all the zeros of $\varphi_1, \ldots, \varphi_r$. Put $\sigma = \sum_{i=1}^n \varphi_i$. Then
\[
\sum_{v \in S} (\nu(v) - \min_{i=1, \ldots, n} \nu(\varphi_i)) \leq \left(\frac{n}{2}\right)(|S| + 2q - 2) + \sum_{i=r+1}^n \mathcal{H}(\varphi_i).
\]

3. Proofs

**Proof of Theorem 2.1.** Denote the coefficients of the polynomial $a_j(n) \in \mathbb{L}[n]$ by $a_{j_0}, a_{j_1}, \ldots, a_{j_{m_j}}$, where $m_j$ is the degree of $a_j(n)$. So
\[
a_j(n) = \sum_{k=0}^{m_j} a_{jk} n^k.
\]

First assume that the recurrence sequence is of the shape $G_n = a_1(n)\alpha_1^n$. Using Lemma 2.4,
\[
\mu(G_n) = \mu(a_1(n)) + n\mu(\alpha_1) \leq \mathcal{H}(a_1(n)) + n\mu(\alpha_1)
\]
\[
\leq \sum_{k=0}^{m_1} \mathcal{H}(a_{1k} n^k) + n\mu(\alpha_1) = \sum_{k=0}^{m_1} \mathcal{H}(a_{1k}) + n\mu(\alpha_1).
\]

Thus from now on we can assume that $t \geq 2$. Let $\pi_{j_1}, \ldots, \pi_{j_t}$ be a maximal $\mathbb{C}$-linear independent subset of $a_{j_0}, a_{j_1}, \ldots, a_{j_{m_j}}$, Then we can write the sequence as
\[
G_n = \sum_{j=1}^t \left( \sum_{i=1}^{k_j} \big(b_j(n)\pi_{ji}\big) \alpha_j^n \right)
\]
with polynomials $b_{ji}(n) \in \mathbb{C}[n]$. Since $a_j(n)$ is not the zero polynomial, there is for each $j$ at least one index $i$ such that $b_{ji}(n)$ is not the zero polynomial. Without loss of generality we can assume that no $b_{ji}(n)$ is the zero polynomial since otherwise we can throw out all zero polynomials and renumber the remaining terms. It does not matter whether all $\pi_{ji}$ occur in the sum or not. Moreover, we assume that $n$ is large enough such that $b_{ji}(n) \neq 0$ for all $j, i$.

Consider as a next step the set
\[
M := \{\pi_{ji}\alpha_j^n : i = 1, \ldots, k_j, j = 1, \ldots, t\}.
\]

We intend to apply Lemma 2.3. If $M$ is linearly dependent over $\mathbb{C}$, then we choose a minimal linearly dependent subset $\bar{M}$ of $M$, that is, a linearly dependent subset $\bar{M}$ with the property that no proper subset of $\bar{M}$ is linearly dependent. Let $\bar{G}_n$ be the linear recurrence sequence associated with this subset $\bar{M}$, that is,
\[
\bar{G}_n = \sum_{j=1}^s \left( \sum_{i=1}^{k_j} \big(b_j(n)\pi_{ji}\big) \alpha_j^n \right)
\]
for $s \leq t$ and after a suitable renumbering of the summands. Since $\pi_{j_1}, \ldots, \pi_{j_t}$ are $\mathbb{C}$-linearly independent we have $s \geq 2$. Applying Lemma 2.3 to

$$\tilde{M} := \{\pi_{ji}a^n_j : i = 1, \ldots, k_j, j = 1, \ldots, s\}$$

gives an upper bound for $n$. Thus for $n$ large enough this subset $\tilde{M}$ of $M$ cannot be linearly dependent. Because of the fact that there are only finitely many subsets of $M$, for $n$ large enough the set $M$ must be linearly independent.

We assume from here on that $n$ is large enough such that $M$ is linearly independent. For each fixed $n$ we have $b_{ji}(n) \in \mathbb{C}^*$. Thus the set

$$M' := \{b_{ji}(n)\pi_{ji}a^n_j : i = 1, \ldots, k_j, j = 1, \ldots, t\}.$$

is linearly independent over $\mathbb{C}$ and contains for each $j = 1, \ldots, t$ at least one element. Let $S$ be a finite set of places of $L$ containing all zeros and poles of $\alpha_j$ for $j = 1, \ldots, t$ and of the nonzero $a_{ji}$ for $j = 1, \ldots, t$ and $i = 1, \ldots, m_j$ as well as $\mu$ and the places lying over $\infty$. Now applying Proposition 2.5 yields

$$\sum_{v \in S} \left(\nu(G_n) - \min_{j=1,\ldots,t} \nu(b_{ji}(n)\pi_{ji}a^n_j)\right) \leq \left(\sum_{j=1}^t k_j\right) \frac{1}{2} \left(|S| + 2\alpha - 2\right) =: C_1$$

and, since each summand in the sum on the left-hand side is nonnegative,

$$\mu(G_n) - \min_{j=1,\ldots,t} \mu(b_{ji}(n)\pi_{ji}a^n_j) \leq C_1.$$ 

Therefore for all $j_0 = 1, \ldots, t$ and $i_0 = 1, \ldots, k_{j_0}$,

$$\mu(G_n) \leq C_1 + \min_{j=1,\ldots,t} \mu(b_{ji}(n)\pi_{ji}a^n_j)$$

$$\leq C_1 + \mu(b_{j_0i_0}(n)\pi_{j_0i_0}a^n_{j_0})$$

$$= C_1 + \mu(\pi_{j_0i_0}) + n\mu(\alpha_{j_0})$$

$$\leq C_1 + \max_{j=1,\ldots,t} \mu(\alpha_{j_0}) + n\mu(\alpha_{j_0})$$

$$\leq C_1 + \max_{j=1,\ldots,t} \mathcal{H}(a_{ji}) + n\mu(\alpha_{j_0})$$

$$= C_2 + n\mu(\alpha_{j_0}).$$

Since this holds for all $j_0 = 1, \ldots, t$,

$$\mu(G_n) \leq C_2 + n \cdot \min_{j=1,\ldots,t} \mu(\alpha_j). \quad \square$$

**Proof of Corollary 2.2.** We can apply Theorem 2.1 with $L = K = \mathbb{C}(x)$ and $\mu = \nu_\infty$. This yields

$$-\deg G_n = \nu_\infty(G_n) \leq C + n \cdot \min_{j=1,\ldots,t} \nu_\infty(\alpha_j) = C - n \cdot \max_{j=1,\ldots,t} \deg \alpha_j$$

which immediately implies the inequality in question. \quad \square
Appendix A. The number field case

In this appendix we will give a proof of the following theorem.

**Theorem A.1.** Let \((G_n)_{n=0}^{\infty}\) be a nondegenerate linear recurrence sequence taking values in a number field \(K\) and let \(G_n = a_1(n)\alpha_1^n + \cdots + a_t(n)\alpha_t^n\) with algebraic integers \(\alpha_1, \ldots, \alpha_t\) be its power sum representation satisfying \(\max_{j=1, \ldots, t} |\alpha_j| > 1\). Denote by \(|\cdot|\) the usual absolute value on \(\mathbb{C}\). Then, for any \(\varepsilon > 0\), the inequality

\[ |G_n| \geq (\max_{j=1, \ldots, t} |\alpha_j|)^n(1-\varepsilon) \]

is satisfied for every sufficiently large \(n\).

Note that this result is not effective in the sense that we do not give a bound \(n_0\) such that the inequality is satisfied for all \(n\) greater than \(n_0\). If we were to look more precisely at the limitations placed on \(n\) in the proof given below, it would be possible to give an (admittedly) rather complicated upper bound on the number of exceptions. This bound would have the following form: if \(n/\log n > B_1\), then there are at most \(B_2\) values of \(n\) for which the inequality is not valid. Since the explicit constants are not so enlightening we will not calculate them in detail.

From here on \(K\) will denote a number field. In the proof we will need three auxiliary results which are listed below. The first one is a result of Schmidt.

**Lemma A.2 (Schmidt [12]).** Suppose that \((G_n)_{n \in \mathbb{Z}}\) is a nondegenerate linear recurrence sequence of complex numbers, whose characteristic polynomial has \(k\) distinct roots of multiplicity at most \(a\). Then the number of solutions \(n \in \mathbb{Z}\) of the equation

\[ G_n = 0 \]

can be bounded above by

\[ c(k, a) = e^{(7k^a)a^a}. \]

The second is a result of Evertse. We use the notation

\[ ||x|| = \max_{k=0, \ldots, t} \max_{i=1, \ldots, D} |\sigma_i(x_k)| \]

with \(\{\sigma_1, \ldots, \sigma_D\}\) the set of all embeddings of \(K\) in \(\mathbb{C}\) and \(x = (x_0, x_1, \ldots, x_t)\). Moreover, we denote by \(O_K\) the ring of integers in \(K\).

**Lemma A.3 (Evertse [3]).** Let \(t\) be a nonnegative integer and \(S\) a finite set of places in \(K\), containing all infinite places. Then for every \(\varepsilon > 0\) a constant \(C\) exists, depending only on \(\varepsilon, S, K, t\) such that for each nonempty subset \(T\) of \(S\) and every vector \(x = (x_0, x_1, \ldots, x_t) \in O_K^{t+1}\) with

\[ x_{i_0} + x_{i_1} + \cdots + x_{i_s} \neq 0 \]
for each nonempty subset \( \{i_0, i_1, \ldots, i_s\} \) of \( \{0, 1, \ldots, t\} \),

\[
\left( \prod_{k=0}^t \prod_{v \in S} \|x_k\|_v \right) \prod_{v \in T} \|x_0 + x_1 + \cdots + x_t\|_v \geq C \left( \prod_{v \in T} \max_{k=0,\ldots,j} \|x_k\|_v \right) \|x\|^{-\varepsilon}.
\]

Furthermore, we will need the following lemma which also can be found in [3].

**Lemma A.4.** Suppose \( K \) is a number field of degree \( D \), let \( f(X) \in K[X] \) be a polynomial of degree \( m \) and \( T \) a nonempty set of primes on \( K \). Then there exists a positive constant \( c \), depending only on \( K, f \) such that for all \( r \in \mathbb{Z} \) with \( r \neq 0 \) and \( f(r) \neq 0 \),

\[
c^{-1}|r|^{-Dm} \leq \left( \prod_{v} \max(1, \|f(r)\|_v) \right)^{-1} \leq \prod_{v} \|f(r)\|_v \leq \prod_{v} \max(1, \|f(r)\|_v) \leq c|r|^{-Dm}.
\]

**Proof of Theorem A.1.** Since the characteristic roots \( \alpha_j \) of \( G_n \) are algebraic integers we can find a nonzero integer \( z \) such that \( z \alpha_j(n) \alpha_j^n \) are algebraic integers for all \( j = 1, \ldots, t \) and all \( n \in \mathbb{N} \). Set \( L = K(\alpha_1, \ldots, \alpha_t) \), the splitting field of the characteristic polynomial of the sequence \( G_n \). Choose \( S \) as a finite set of places in \( L \) containing all infinite places as well as all places such that \( \alpha_1, \ldots, \alpha_t \) are \( S \)-units. Let \( \mu \) be such that \( \|\|_\mu = |\cdot| \) is the usual absolute value on \( \mathbb{C} \). In particular, \( \mu \in S \). Further, define \( T = \{\mu\} \).

As \( G_n \) is nondegenerate, the sequence \( \widetilde{G_n} = zG_n \) is also nondegenerate. Therefore by Lemma A.2, for \( n \) large enough,

\[
z \alpha_{j_1}(n) \alpha_{j_1}^n + \cdots + z \alpha_{j_s}(n) \alpha_{j_s}^n \neq 0
\]

for each non-empty subset \( \{j_1, \ldots, j_s\} \) of \( \{1, \ldots, t\} \). Thus we can apply Lemma A.3 and get

\[
\left( \prod_{j=1}^t \prod_{v \in S} \|z \alpha_{j}(n) \alpha_{j}^n\|_v \right) |zG_n| \geq C \max_{j=1,\ldots,t} |z \alpha_{j}(n) \alpha_{j}^n| \||z\|^{-\varepsilon}
\]

for \( x = (\alpha_1(n) \alpha_1^n, \ldots, \alpha_t(n) \alpha_t^n) \). Without loss of generality, we can assume that \( |\alpha_1| = \max_{j=1,\ldots,t} |\alpha_j| \). Since \( z \) is a fixed integer and the \( \alpha_j \) are \( S \)-units, we can rewrite this as

\[
\left( \prod_{j=1}^t \prod_{v \in S} \|\alpha_{j}(n)\|_v \right) |G_n| \geq C_1 \max_{j=1,\ldots,t} |\alpha_{j}(n) \alpha_{j}^n| \||x|^{-\varepsilon}
\]

\[
\geq C_1 |\alpha_1(n) \alpha_1^n| \||x|^{-\varepsilon} = C_1 |\alpha_1(n)| |\alpha_1|^n \||x|^{-\varepsilon}. \tag{A.1}
\]

In preparation for the next step, note that there exists a positive constant \( A \) such that

\[
\max_{j=1,\ldots,t} |\sigma_j(\alpha_j)| \leq A \cdot |\alpha_1|.
\]
We decompose $\epsilon = \gamma \cdot \delta$ with small $\delta$ and $A^\gamma \leq |\alpha_1|$. This gives the estimates

$$||x|| = \max_{j=1,\ldots,t} |\sigma_i(a_j(n))a_j^n| = \max_{j=1,\ldots,t} |\sigma_i(a_j(n))\sigma_i(\alpha_j)^n|$$

$$\leq \max_{j=1,\ldots,t} |\sigma_i(a_j(n))| \cdot \max_{j=1,\ldots,t} |\sigma_i(\alpha_j)|^n$$

$$\leq C_2 n^m \cdot \max_{j=1,\ldots,t} |\sigma_i(\alpha_j)|^n \leq C_2 n^m A^n |\alpha_1|^n,$$

with $m = \max_{j=1,\ldots,d} \deg a_j$, and

$$||x||^e \leq C_3 n^{me} A^{\gamma n_0} |\alpha_1|^{me} \leq C_3 n^{me} |\alpha_1|^{n(\epsilon + \delta)}.$$

Now we insert this into inequality (A.1), giving

$$\left( \prod_{j=1}^t \prod_{v \in S} ||a_j(n)||_{v} \right) |G_n| \geq C_4 |a_1(n)| |\alpha_1|^n n^{-me} |\alpha_1|^{-n(\epsilon + \delta)} \geq C_5 n^{-me} |\alpha_1|^{n(1-\epsilon-\delta)}.$$

Applying Lemma A.4 to the product in the brackets on the left hand side gives the bound

$$\prod_{j=1}^t \prod_{v \in S} ||a_j(n)||_{v} \leq \prod_{j=1}^t c_{6j}^{(j)} n^{D_m} \leq C_7 n^{D_m}.$$

Altogether, for $n$ large enough,

$$|G_n| \geq C_8 n^{-D_{me}} |\alpha_1|^{n(1-\epsilon-\delta)}.$$

Hence, recalling that $|\alpha_1| = \max_{j=1,\ldots,d} |\alpha_j|$, for $n$ large enough,

$$|G_n| \geq (\max_{j=1,\ldots,d} |\alpha_j|)^n(1-\tilde{\epsilon}).$$

This proves the theorem. □

References


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