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A NOTE ON JEŚMANOWICZ' CONJECTURE CONCERNING PRIMITIVE PYTHAGOREAN TRIPLES

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Abstract

Let a, b, c be a primitive Pythagorean triple and set $a = m^2 - n^2, b = 2mn, c = m^2 + n^2$, where m and n are positive integers with $m > n$, $\gcd(m, n) = 1$ and $m \not\equiv n \pmod 2$. In 1956, Jeśmanowicz conjectured that the only positive integer solution to the Diophantine equation $(m^2 - n^2)^x + (2mn)^y = (m^2 + n^2)^z$ is $(x, y, z) = (2, 2, 2)$. We use biquadratic character theory to investigate the case with $(m, n) \equiv (2, 3) \pmod 4$. We show that Jeśmanowicz' conjecture is true in this case if $m + n \not\equiv 1 \pmod{16}$ or $y > 1$. Finally, using these results together with Laurent's refinement of Baker's theorem, we show that Jeśmanowicz' conjecture is true if $(m, n) \equiv (2, 3) \pmod 4$ and $n < 100$.

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1. Introduction

Let a, b, c be positive integers satisfying $a^2 + b^2 = c^2$. Clearly, the Diophantine equation

$$a^x + b^x = c^z \tag{1.1}$$

has the positive integer solution $(x, y, z) = (2, 2, 2)$. In 1956, Sierpiński [14] and Jeśmanowicz [6] showed that this is the only solution of the equation (1.1) for $(a, b, c) = (3, 4, 5), (5, 12, 13), (7, 24, 25), (9, 40, 41)$ and $(11, 60, 61)$. Based on these results, Jeśmanowicz conjectured that the only positive integer solution to equation (1.1) is $(x, y, z) = (2, 2, 2)$. If (a, b, c) is a primitive Pythagorean triple with $2 \mid b$, it is well known that a, b, c can be written as

$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2,$$

where m, n are positive integers such that $\gcd(m, n) = 1$, $m > n, m \not\equiv n \pmod 2$. Then (1.1) can be written as

$$(m^2 - n^2)^x + (2mn)^y = (m^2 + n^2)^z. \tag{1.2}$$

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Many special cases of Jeśmanowicz' conjecture have been settled for primitive Pythagorean triples. In 1959, Lu [9] proved that, if $n = 1$, then (1.2) has only the positive integer solution $(x, y, z) = (2, 2, 2)$. In 1965, Dem̒janenko [2] extended the results of [14] and [6] by proving the following proposition.

Proposition 1.1. *If $m - n = 1$, then (1.2) has only the positive integer solution $(x, y, z) = (2, 2, 2)$.*

The results of Lu and Dem̒janenko were extended by Miyazaki [10] in 2013 and further extended by Miyazaki *et al.* [12] in 2014. Further results concerning Jeśmanowicz' conjecture have been obtained under the condition that $2 \,\|\, mn$. For example, Le [8] showed that if $2 \,\|\, mn$ and $m^2 + n^2$ is a power of an odd prime q, then Jeśmanowicz' conjecture is true. Using the theory of linear forms in two logarithms, Guo and Le [5] proved that if $n = 3, m \equiv 2 \pmod 4$ and $m > 6\,000$, then Jeśmanowicz' conjecture is true. Takakuwa [15] further proved that if $n = 3, 7, 11, 15$ and $m \equiv 2 \pmod 4$, then Jeśmanowicz' conjecture is true. Under the assumption that m or n has no prime factor of the type $4k + 1$, Deng and Cohen [4] and Deng [3] proved amongst other results that Jeśmanowicz' conjecture is true if $2 \,\|\, m$ or $2 \,\|\, n$ and some other conditions concerning the prime factors of $m \pm n$ are satisfied. Again, under the condition that $2 \,\|\, mn$, Cao [1] proved the following two propositions.

Proposition 1.2. *If $(m, n) \equiv (2, 1) \pmod 4$, or $(m, n) \equiv (2, 3) \pmod 4$ and $m + n$ has a prime factor of the form $4k + 3$, then (1.2) has only the positive integer solution $(x, y, z) = (2, 2, 2)$.*

Proposition 1.3. *If $(m, n) \equiv (1, 6)$ or $(5, 2) \pmod 8$, or $(m, n) \equiv (3, 2) \pmod 4$ and $m + n$ has a prime factor of the form $4k + 3$, then (1.2) has only the positive integer solution $(x, y, z) = (2, 2, 2)$.*

In 2014, Terai [16] proved that if $n = 2$, then Jeśmanowicz' conjecture is true without any assumption on m. Recently, Miyazaki and Terai [11] extended this result.

In this paper, using biquadratic character theory, we first show that, for some $(m, n) \equiv (2, 3) \pmod 4$, the condition that $m + n$ has a prime factor of the form $4k + 3$ in Proposition 1.2 can be removed. Then, using the above results together with Laurent's refinement of Baker's theorem on linear forms in two logarithms, we will extend Takakuwa's result in [15]. The following results will be proved.

Theorem 1.4. *If $(m, n) \equiv (2, 3) \pmod 8$ or $(m, n) \equiv (6, 7) \pmod 8$, then (1.2) has only the positive integer solution $(x, y, z) = (2, 2, 2)$.*

Theorem 1.5. *If one of the following conditions is satisfied, then (1.2) has only the positive integer solution $(x, y, z) = (2, 2, 2)$:*

(i) $y > 1$ *and* $(m, n) \equiv (2, 7) \pmod 8$;

(ii) $(m, n) \equiv (2, 7) \pmod{16}$ *or* $(m, n) \equiv (10, 15) \pmod{16}$.

Theorem 1.6. *If one of the following conditions is satisfied, then (1.2) has only the positive integer solution $(x, y, z) = (2, 2, 2)$:*

(i) *$y > 1$ and $(m, n) \equiv (6, 3) \pmod 8$;*
(ii) *$(m, n) \equiv (6, 3) \pmod{16}$ or $(m, n) \equiv (14, 11) \pmod{16}$.*

THEOREM 1.7. *If $(m, n) \equiv (2, 3) \pmod 4$ and $n < 100$, then (1.2) has only the positive integer solution $(x, y, z) = (2, 2, 2)$.*

When $y = 1$, (1.2) becomes

$$(m^2 - n^2)^x + 2mn = (m^2 + n^2)^z. \tag{1.3}$$

For every given n, using the lower bound for linear forms in two logarithms due to Laurent [7], we obtain an upper bound for the corresponding m. Then by considering the congruences

$$(m^2 - n^2)^x + 2mn \equiv (m^2 + n^2)^z \pmod N \tag{1.4}$$

for a suitable choice of the modulus N in the relevant range of x, z, m, we prove that (1.3) has no solution. It is worth noting that checking the congruence (1.4) involves much less computation than checking the equation (1.3).

2. Preliminaries

In order to prove that equation (1.3) has no solution, we need three lemmas. We first recall some notation. Let α_1 and α_2 be real algebraic numbers with $\alpha_1 \geq 1$ and $\alpha_2 \geq 1$. For an algebraic number α of degree n, let

$$h(\alpha) = \frac{1}{n}\Big(\log|\alpha_0| + \sum_{j=1}^{n} \log\max\{1, |\alpha^{(j)}|\}\Big)$$

denote the *logarithmic height* of α, where α_0 is the leading coefficient of the minimal polynomial of α (over $\mathbb{Z}$) and $(\alpha^{(j)})_{1\leq j\leq n}$ are the conjugates of α. Let A_1 and A_2 be real numbers greater than 1 with

$$\log A_i \geq \max\Big\{h(\alpha_i), \frac{|\log\alpha_i|}{D}, \frac{1}{D}\Big\}$$

for $i \in \{1, 2\}$, where D is the degree of the number field $Q(\alpha_1, \alpha_2)$ over Q. Let b_1 and b_2 be positive integers. Define

$$b' = \frac{b_1}{D\log A_2} + \frac{b_2}{D\log A_1},$$
$$\Lambda = b_2\log\alpha_2 - b_1\log\alpha_1.$$

The following lemma is obtained from a result due to Laurent [7, Corollary 2] by taking $m = 10$ and $C_2 = 25.2$.

LEMMA 2.1 [16, Proposition 2]. *Let Λ be given as above with $\alpha_1 \geq 1$ and $\alpha_2 \geq 1$ and suppose that α_1 and α_2 are multiplicatively independent. Then*

$$\log|\Lambda| \geq -25.2D^4\Big(\max\Big\{\log b' + 0.38, \frac{10}{D}\Big\}\Big)^2 \log A_1 \log A_2.$$

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When $n = 2$, using Lemma 2.1, Terai [16, Lemmas 3 and 4] found upper and lower bounds for x in equation (1.3). In a similar may, we have the following generalisation.

LEMMA 2.2. *If x, z satisfy equation (1.3), then, for any positive integer n,*

$$x < 2\,521 \log c.$$

PROOF. Let $\Lambda = z \log c - x \log a$. Since

$$0 < \Lambda = \log \frac{c^z}{a^x} = \log\left(1 + \frac{b}{a^x}\right) < \frac{b}{a^x},$$

$$\log \Lambda < \log b - x \log a. \tag{2.1}$$

On the other hand, from Lemma 2.1,

$$\log |\Lambda| \geq -25.2(\max\{\log b' + 0.38, 10\})^2 \log a \log c, \tag{2.2}$$

where $b' = x/\log c + z/\log a$. Since

$$\begin{aligned} a^{x+1} - c^z = a(c^z - b) - c^z &\geq (m^2 - n^2 - 1)(m^2 + n^2)^2 - (m^2 - n^2)2mn \\ &= (m^2 - n^2)[(m^2 + n^2)^2 - 2mn] - (m^2 + n^2)^2 \\ &\geq (m + n)(m^4 + n^4) - 2(m^4 + n^4) > 0, \end{aligned}$$

we have $a^{x+1} > c^z$ and $b' < (2x + 1)/\log c$. Let $M = x/\log c$. From (2.1) and (2.2),

$$x \log a < \log b + 25.2\left(\max\left\{\log\left(2M + \frac{1}{\log c}\right) + 0.38, 10\right\}\right)^2 \log a \log c.$$

Since $\log c = \log(m^2 + n^2) > \log 13 > 2$ and $b < c$,

$$M < 1 + 25.2\left(\max\left\{\log\left(2M + \frac{1}{2}\right) + 0.38, 10\right\}\right)^2,$$

and we thus obtain $M < 2\,521$. This completes the proof of Lemma 2.2. □

LEMMA 2.3. *If x, z satisfy equation (1.3), then, for any positive integer n,*

$$x > \frac{m^2 - n^2}{2n^2} \log c.$$

PROOF. Clearly, (1.3) implies $x > z$. Let $\Lambda_0 = \log c - \log a$. Then

$$x\Lambda_0 - \Lambda = x(\log c - \log a) - z \log c - x \log a = (x - z) \log c \geq \log c.$$

From

$$\Lambda_0 = \log\left(\frac{c}{a}\right) = \log\left(\frac{m^2 + n^2}{m^2 - n^2}\right) = \log\left(1 + \frac{2n^2}{m^2 - n^2}\right) < \frac{2n^2}{m^2 - n^2},$$

$$x \geq \frac{\log c}{\Lambda_0} + \frac{\Lambda}{\Lambda_0} > \frac{\log c}{\Lambda_0} > \frac{m^2 - n^2}{2n^2} \log c.$$

This completes the proof of Lemma 2.3. □

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3. Proof of the main results

Proof of Theorem 1.4. From $(m, n) \equiv (2, 3) \pmod 8$, taking (1.2) modulo 4, we have $(-1)^x \equiv 1 \pmod 4$, and hence $x \equiv 0 \pmod 2$. Taking (1.2) modulo $m + n$ gives $(-2n^2)^y \equiv (2n^2)^z \pmod{m + n}$, and thus

$$(-1)^y = \left(\frac{-2n^2}{m+n}\right)^y = \left(\frac{2n^2}{m+n}\right)^z = (-1)^z,$$

which implies $y \equiv z \pmod 2$. There are two cases to consider.

Case 1. Suppose $y > 1$. In this case, from (1.2), $1 \equiv 3^x \equiv 5^z \pmod 8$. Therefore, $y \equiv z \equiv 0 \pmod 2$. From the proof of [5, Lemma 2] (page 226, lines 3–13), we see that $x = y = z = 2$.

Case 2. Suppose $y = 1$. In this case, we use (1.3) and work over the Gaussian integers.

Taking (1.3) modulo $n - mi$, where $i = \sqrt{-1}$, gives $(-2n^2)^x \equiv -2m^2 i \pmod{n - mi}$. Since $n \equiv 3 \pmod 8$ and $m \equiv 2 \pmod 8$, $n - mi$ is an odd primary number, and so $m = 2m_1$ and

$$\left(\frac{-2n^2}{n-mi}\right)_4^x = \left(\frac{-8m_1^2 i}{n-mi}\right)_4, \tag{3.1}$$

where $(\frac{*}{*})_4$ denotes the biquadratic residue symbol (see, for example, [13]). By the properties of the biquadratic residue symbol and biquadratic reciprocity,

$$\begin{aligned}\left(\frac{-2n^2}{n-mi}\right)_4 &= \left(\frac{-2}{n-mi}\right)_4\left(\frac{-n}{n-mi}\right)_4\left(\frac{-n}{n-mi}\right)_4\\ &= \left(\frac{-2}{n-mi}\right)_4\left(\frac{mi}{n}\right)_4\left(\frac{mi}{n}\right)_4 = \left(\frac{-2}{n-mi}\right)_4\\ &= \left(\frac{i}{n-mi}\right)_4\left(\frac{1+i}{n-mi}\right)_4^2 = i^{(m^2+n^2-1)/4} \cdot i^{(m++n-m^2-1)/2} = -i\end{aligned}$$

and

$$\left(\frac{-8m_1^2 i}{n-mi}\right)_4 = \left(\frac{2i}{n-mi}\right)_4\left(\frac{m_1^2}{n-mi}\right)_4 = \left(\frac{2i}{n-mi}\right)_4 = \left(\frac{1+i}{n-mi}\right)_4^2 = 1.$$

Thus, (3.1) implies $(-i)^x = 1$, and hence $x \equiv 0 \pmod 4$.

Next taking (1.3) modulo $m + n$ gives $-2n^2 \equiv (2n^2)^z \pmod{m + n}$, and so

$$(2n^2)^{z-1} \equiv -1 \pmod{m+n}. \tag{3.2}$$

If $m + n$ has a prime factor p with $p \equiv -1 \pmod 4$, taking (3.2) modulo p gives

$$1 = \left(\frac{2n^2}{p}\right)^{z-1} = \left(\frac{-1}{p}\right) = -1,$$

which is a contradiction. Thus, $m + n$ must have a prime factor p with $p \equiv 5 \pmod 8$ and taking (3.2) modulo p gives

$$(2n^2)^{z-1} \equiv -1 \pmod p. \tag{3.3}$$

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From $p \equiv 5 \pmod 8$, we get $(-1)^{(p-1)/4} \equiv -1 \pmod 2$, and hence -1 is a nonbiquadratic residue modulo p. Since $(n^2)^{z-1} = (n^4)^{(z-1)/2}$ is a biquadratic residue modulo p, 2^{z-1} must be a nonbiquadratic residue modulo p by (3.3). Hence, $4 \nmid z-1$ and $z \equiv 3 \pmod 4$.

Finally, we show that (1.3) has no solution. Taking (1.3) modulo 16,

$$(-5)^x + 12 \equiv 13^z \pmod{16}. \tag{3.4}$$

From $x \equiv 0 \pmod 4$ and $z \equiv 3 \pmod 4$, we obtain $(-5)^x \equiv 1 \pmod{16}$ and $13^z \equiv 5 \pmod{16}$. Therefore, (3.4) cannot hold and (1.3) has no solution. This implies $y > 1$ and, similarly to the proof of Case 1, we deduce that Jeśmanowicz' conjecture is true.

For $(m, n) \equiv (6, 7) \pmod 8$, we similarly prove that (1.2) has only the positive integer solution $(x, y, z) = (2, 2, 2)$. □

Proof of Theorem 1.5. From $(m, n) \equiv (2, 7) \pmod 8$, taking (1.2) modulo 4 gives $x \equiv 0 \pmod 2$ and taking (1.2) modulo $m-n$ gives $(2n^2)^y \equiv (2n^2)^z \pmod{m-n}$. Thus,

$$(-1)^y = \left(\frac{2n^2}{m-n}\right)^y = \left(\frac{2n^2}{m-n}\right)^z = (-1)^z,$$

which implies $y \equiv z \pmod 2$. There are two cases to consider.

Case 1. Suppose $y > 1$. In this case, similarly to the proof of Theorem 1.4, we have $x = y = z = 2$.

Case 2. Suppose $y = 1$. In this case, $y \equiv z \equiv 1 \pmod 2$. Similarly to the proof of Theorem 1.4,

$$\left(\frac{-2n^2}{n-mi}\right)_4^x = \left(\frac{-8m_1^2 i}{n-mi}\right)_4,$$

where $m = 2m_1$ and

$$\left(\frac{-2n^2}{n-mi}\right)_4 = i^{(m^2+n^2-1)/4} \cdot i^{(m++n-m^2-1)/2} = i^3 \cdot i^{4l+2} = i,$$
$$\left(\frac{-8m_1^2 i}{n-mi}\right)_4 = \left(\frac{1+i}{n-mi}\right)_4^2 = i^{(m++n-m^2-1)/2} = i^{4l+2} = -1.$$

Then from $i^x = -1$ we get $x \equiv 2 \pmod 4$.

Next, we prove that if $(m, n) \equiv (2, 7) \pmod{16}$, then equation (1.3) has no solution. Taking (1.3) modulo $m+n$, we have $(2n^2)^{z-1} \equiv -1 \pmod{m+n}$. Since $m+n \equiv 9 \pmod{16}$, $m+n$ must have a prime factor p such that $p \not\equiv 1 \pmod{16}$, that is, $p \equiv 3,\ 5, 7, 9, 11, 13, 15 \pmod{16}$. But $(2n^2)^{z-1} \equiv -1 \pmod{m+n}$ so $p \not\equiv 3 \pmod 4$, and therefore $p \equiv 5, 9, 13 \pmod{16}$.

If $p \equiv 5, 13 \pmod{16}$, that is, $p \equiv 5 \pmod 8$, then from $(-1)^{(p-1)/4} \equiv -1 \pmod 2$ we deduce that -1 is a nonbiquadratic residue modulo p, and therefore $4 \nmid z-1$, which implies $z \equiv 3 \pmod 4$.

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If $p \equiv 9 \pmod{16}$, from $(2n^2/p) = 1$ we see that there is an integer h satisfying $2n^2 \equiv h^2 \pmod{p}$. Hence,

$$h^{2(z-1)}) \equiv -1 \pmod{p}. \tag{3.5}$$

Because $(-1)^{(p-1)/8} \equiv -1 \pmod{2}$, -1 is a nonbiquadratic residue modulo p, and therefore $8 \nmid 2(z-1)$, which implies $z \equiv 3 \pmod{4}$. Working modulo 16, since $(m, n) \equiv (2, 7) \pmod{16}$,

$$(m^2 - n^2)^x \equiv (4-1)^{4x_1+2} \equiv 9 \pmod{16},$$
$$2mn \equiv 2 \cdot 2 \cdot 7 \equiv 12 \pmod{16},$$
$$(m^2 + n^2)^z \equiv (4+1)^{4z_1+3} \equiv 13 \pmod{16}.$$

Then (3.5) gives $5 \equiv 13 \pmod{16}$, which is a contradiction. Thus, if $(m, n) \equiv (2, 7) \pmod{16}$, we must have $y > 1$ and, from Case 1, Jeśmanowicz' conjecture is true when $(m, n) \equiv (2, 7) \pmod{16}$. When $(m, n) \equiv (10, 15) \pmod{16}$, we can similarly get the same conclusion. □

Proof of Theorem 1.6. The proof of Theorem 1.6 is very similar to that of Theorem 1.5. We omit the details. □

Proof of Theorem 1.7. By Theorems 1.5 and 1.6 and the first part of Proposition 1.2, we need only prove that (1.3) has no solution when $y = 1$, $(m, n) \equiv (2, 15), (10, 7) \pmod{16}$ and $(m, n) \equiv (6, 11), (14, 3) \pmod{16}$. To this end, as pointed out in Section 1, we consider the congruence (1.4).

We first give upper and lower bounds for m (for given n). From Lemmas 2.2 and 2.3, $(m^2 - n^2)/2n^2 < 2\,521$, and we deduce that $m < 71n$. On the other hand, since $m \equiv 2 \pmod{4}, n \equiv 3 \pmod{4}$, by Proposition 1.1, $m \geq n + 3$ if $(m, n) \equiv (2, 15)$, $(10, 7) \pmod{16}$ or $m \geq n + 11$ if $(m, n) \equiv (6, 11), (14, 3) \pmod{16}$. Therefore, $m^2 - n^2$, $2mn$ and $m^2 + n^2$ are all bounded above and below for given n.

Next, we consider the moduli N. Let $x \equiv \alpha \pmod{72}, z \equiv \beta \pmod{72}$. From $x \equiv 0 \pmod{2}$ and $z \equiv 1 \pmod{2}$, we may suppose that $\alpha \in \{2, 4, \ldots, 72\}$ and $\beta \in \{1, 3, \ldots, 71\}$. Since $\gcd(m, n) = 1$, we have $m^2 + n^2 \not\equiv 0 \pmod{3}$. In addition, if $3 \mid mn$, then $m^2 - n^2 \not\equiv 0 \pmod{3}$. Put $N_1 = 2^4 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 37 \cdot 73$. It is easy to prove that

$$(m^2 - n^2)^x \equiv (m^2 - n^2)^\alpha \pmod{3^3 N_1},$$
$$(m^2 + n^2)^z \equiv (m^2 + n^2)^\beta \pmod{3^3 N_1}.$$

Thus, if (1.3) has the solution (x, z) with $x \equiv \alpha \pmod{72}$ and $z \equiv \beta \pmod{72}$, then

$$(m^2 - n^2)^\alpha + 2mn \equiv (m^2 + n^2)^\beta \pmod{3^3 N_1} \tag{3.6}$$

when $3 \mid mn$. If $3 \nmid mn$, that is, $3 \nmid n$ and $3 \nmid m$, we have $m^2 - n^2 \equiv 0 \pmod{3}$, and therefore $(m^2 - n^2)^x \equiv (m^2 - n^2)^\alpha \pmod{3^3}$ when $\alpha \geq 3$. So, in this case, (3.6) still holds. If $3 \nmid n, 3 \nmid m$ and $\alpha = 2$, then $(m^2 - n^2)^x \equiv (m^2 - n^2)^\alpha \pmod{3^2}$. In this case,

$$(m^2 - n^2)^\alpha + 2mn \equiv (m^2 + n^2)^\beta \pmod{3^2 N_1}. \tag{3.7}$$

Finally, with computer assistance, in the range $2 \leq \alpha \leq 72, 1 \leq \beta \leq 71$, we checked (3.7) in the following four cases using Maple 18:

(1) $n \in \{7, 23, 39, 55, 71, 87\}$ with $m \equiv 10 \pmod{16}, n + 3 \le m < 71n$;
(2) $n \in \{15, 31, 47, 63, 79, 95\}$ with $m \equiv 2 \pmod{16}, n + 3 \le m < 71n$;
(3) $n \in \{3, 19, 35, 51, 67, 83, 99\}$ with $m \equiv 14 \pmod{16}, n + 11 \le m < 71n$;
(4) $n \in \{11, 27, 43, 59, 75, 91\}$ with $m \equiv 6 \pmod{16}, n + 11 \le m < 71n$.

By Proposition 1.1, in Cases (1) and (2), $n + 3 \le m$, and, in Cases (3) and (4), $n + 11 \le m$. In Cases (1) to (3), no solutions (m, n, α, β) to (3.7) were found. In Case (4), (3.7) has the following three solutions when $n = 91$:

$$(m, n, \alpha, \beta) = (2\,027, 91, 14, 5), (4\,198, 91, 22, 37), (4\,198, 91, 58, 37).$$

Although all three solutions satisfy $\alpha > 2$, none of them is a solution to congruence (3.6). Hence, (1.3) has no solutions. This completes the proof of Theorem 1.7. □

Remark 3.1. We used a 64-bit computer with frequency 2.93 GHz to check congruences (3.6) and (3.7) by Maple 18. For each $n \le 99$ the computation takes less than 1 second. Equation (1.3) can be checked by Maple 18 for $n = 3, 7, 11, \ldots, 99$. But the computation takes much more time. For example, we checked (1.3) for $n = 3$ by Maple 18 in the range of $6 \le m \le 210, 1 \le z < x \le 2\,521 \log c$ with $m \equiv 14 \pmod{16}$ (cf. case (3) in the proof of Theorem 1.7), $x \equiv 0 \pmod 2, z \equiv 1 \pmod 2$, and though we took Lemma 2.3 and [12, Lemma 12(i)] into consideration, the computation still took 625 seconds.

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