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ON GENERATORS AND DISTURBANCES OF DYNAMICAL SYSTEMS IN THE CONTEXT OF CHAOTIC POINTS

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Abstract

We analyse local aspects of chaos for nonautonomous periodic dynamical systems in the context of generating autonomous dynamical systems and the possibility of disturbing them.

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1. Introduction and preliminaries

Chaos appeared in the mathematical vocabulary in 1975 in the paper of Li and Yorke [9]. Since then, several different definitions of chaos have been proposed (for a survey, see [8], or see the monograph [12] for functions on the unit interval). In [10], the first author of this paper introduced the idea of a chaotic point depending on the existence of a homoclinic point. In this paper, we generalise chaotic points and homoclinic points for nonautonomous dynamical systems. We also consider the relationship between properties of nonautonomous dynamical systems and the associated autonomous dynamical systems. These results will be applied to chaotic points. The last section of the paper, containing the most important results, treats the possibility of modifying (more precisely, disturbing) a dynamical system in such a way that a certain point becomes a chaotic point of this system.

Throughout the paper, we restrict our considerations to continuous functions having as a domain and range topological manifolds that are metric spaces. We use $\mathbb{N}_0$, $\mathbb{N}_+$, $\mathbb{R}_0$ and $\mathbb{R}_+$ to denote the set of all nonnegative integers, positive integers, nonnegative reals and positive reals, respectively. Let (X, ρ) be a metric space and let $f : X \to X$. If $A \subset X$, then the symbols $\mathrm{Int}(A)$, $\mathrm{cl}(A)$ and $\mathrm{Fr}(A)$ stand for the interior, the closure and the boundary of the set A, respectively, and $B(x_0, r)$ is an open ball with the centre at x_0 and radius r.

By $\mathrm{L}(x, y)$, we denote an arc with endpoints x and y, that is, a space homeomorphic to the closed interval $[0, 1]$ via a homeomorphism $h : [0, 1] \to \mathrm{L}(x, y)$, where $h(0) = x$ and $h(1) = y$. A subarc $\mathrm{K}(a, b)$ of an arc $\mathrm{L}(x, y)$ is denoted by $\mathrm{L}_{\mathrm{K}(x,y)}(a, b)$.

The symbol $f_{|A}$ stands for a restriction of the function f to a set $A \subset X$.

If $\{K_n\}_{n=0}^{\infty}$ is a sequence of nonempty subsets of X, we say that $\{K_n\}_{n=0}^{\infty}$ has the extension property if, for any $i, j \in \mathbb{N}_0$ and any continuous function $g : A \to K_j$, where $A \subset K_i$ is a closed set, there exists a continuous function $g_* : K_i \to K_j$ that is an extension of g, that is, $g_{*|A} = g$.

Let $\{A_s\}_{s\in S}$ be a cover of X and let $f_s : A_s \to X$ for $s \in S$. If all the functions f_s are compatible (that is, $f_{s_1}(A_{s_1} \cap A_{s_2}) = f_{s_2}(A_{s_1} \cap A_{s_2})$ for $s_1, s_2 \in S$), then the symbol $\nabla_{s\in S} f_s$ denotes a function defined in the following way: $\nabla_{s\in S} f_s(x) = f_s(x)$, for $x \in A_s$.

We understand an m-dimensional manifold M to be a compact metric space such that each point x_0 has an open neighbourhood U homeomorphic with some open set V in $\mathbb{H}^m = \{(x_1, \dots, x_m) \in \mathbb{R}^m : x_m \geq 0\}$ (compare [7]). The symbol $\mathbb{B}(M)$ denotes the set of all closed submanifolds K of X such that $\dim(K) = \dim(M)$.

We focus our attention on the most important notions connected with dynamical systems. (Entropy and strong entropy point are quite complex and in this paper they play a complementary role, so we will only give suitable references.)

A nonautonomous dynamical system is a pair $(X, (f_{1,\infty}))$, where X is a nonempty set and $(f_{1,\infty})$ is a sequence of continuous self-maps $\{f_n\}_{n=1}^{\infty}$ defined on X. For simplicity, we denote it by $(f_{1,\infty})$ and say briefly that $(f_{1,\infty})$ is a dynamical system on X. If $f_n = f$ for all $n \in \mathbb{N}_+$, the system is an autonomous dynamical system denoted by (f).

We say that a dynamical system $(f_{1,\infty})$ is periodic if there exists a positive integer k, called a period of the system, such that $f_n = f_{n \pmod k}$ if $n \pmod k \neq 0$ and $f_n = f_k$ otherwise. By $\mathcal{P}(f_{1,\infty})$, we denote the set of all periods of the system $(f_{1,\infty})$. If $k \in \mathcal{P}(f_{1,\infty})$, then we sometimes write $(f_1, \dots, f_k)$ instead of $(f_{1,\infty})$. Each autonomous dynamical system is a periodic dynamical system, whose periods are all positive integers.

If $(f_{1,\infty}) = \{f_i\}_{i=1}^{\infty}$, then f_1^0 is the identity function and we write $f_1^m = f_m \circ \cdots \circ f_1$.

Let $(f_{1,\infty})$ be a dynamical system, and let $x_0 \in X$ and $n \in \mathbb{N}_+$. We say that x_0 is an n-periodic point of the system $(f_{1,\infty})$ if $f_1^{kn}(x_0) = x_0$ for $k \in \mathbb{N}_+$. By $\mathrm{Per}_n(f_{1,\infty})$, we denote a set of all n-periodic points of the system $(f_{1,\infty})$. Put $\mathrm{Per}(f_{1,\infty}) = \bigcup_{n=1}^{\infty} \mathrm{Per}_n(f_{1,\infty})$.

We say that x_0 is a fixed point of the system $(f_{1,\infty})$ if $f_i(x_0) = x_0$ for $i \in \mathbb{N}_+$. The set of all fixed points of the system $(f_{1,\infty})$ is denoted by $\mathrm{Fix}(f_{1,\infty})$.

We say that a dynamical system $(f_{1,\infty})$ has the local homeomorphism property at x_0 if there exists an open set U such that $x_0 \in U$ and $(f_i)_{|U} : U \to U$ is a homeomorphism for any $i \in \mathbb{N}_+$. In the case of a finite sequence of functions $\{f_i\}_{i=1}^{k}$, we can also define the local homeomorphism property at x_0, related to the periodic system $(f_1, \dots, f_k)$. In particular, we can consider the local homeomorphism property at x_0 of a function f, related to the autonomous system (f).

Note that if functions $f, g : X \to X$ have the local homeomorphism property at x_0, the composition $g \circ f$ may not have this property at x_0.

Let $k \in \mathbb{N}_+$ and let $\{f_i\}_{i=1}^{k}$ be a sequence of continuous functions that has the local homeomorphism property at x_0. Then the function $g = f_k \circ f_{k-1} \circ \cdots \circ f_1$ has the local homeomorphism property at x_0. This remark leads to the following lemma.

LEMMA 1.1. *If a dynamical system $(f_{1,\infty})$ has the local homeomorphism property at x_0, then, for any $k \in \mathbb{N}_+$, the function f_1^k has the local homeomorphism property at x_0.*

2. Generated system and chaotic points

We investigate chaotic points in both autonomous and nonautonomous systems. First, we introduce the notion of an autonomous dynamical system generated by a periodic nonautonomous system.

Let $(f_{1,\infty})$ be a periodic nonautonomous dynamical system. Then each system (ψ) such that $\psi = f_1^k$, $k \in \mathcal{P}(f_{1,\infty})$, is called an autonomous dynamical system generated by $(f_{1,\infty})$. The system $(f_{1,\infty})$ is called a (periodic) generator of (ψ). By $\mathrm{PG}(\psi)$, we denote the set of all periodic generators of the system (ψ). Note that $\mathrm{PG}(\psi) \neq \emptyset$.

For a periodic dynamical system $(f_{1,\infty})$, there may be different generated systems with different properties according to the chosen period. For example, assume the definition of a strong entropy point introduced in [11] and let us consider a function constructed in the following way. Choose sequences $\{a_1^n\}_{n=1}^{\infty}$ and $\{a_2^n\}_{n=1}^{\infty}$ such that

$$\tfrac{1}{2} \le a_1^1 < a_2^1 < a_1^2 < a_2^2 < \cdots \to 1.$$

Let $c_1^i < \cdots < c_{2i}^i$ be elements of the interval (a_1^i, a_2^i), $i \in \mathbb{N}_+$. Put

$$f(a_1^i) = f(c_{2k}^i) = 1 - a_1^i, \quad f(a_2^i) = f(c_{2k-1}^i) = 1 - a_2^i \quad \text{for all } k = 1, \dots, i.$$

Let f be a linear function on each of the intervals

$$[a_1^i, c_1^i], \quad [c_{2k-1}^i, c_{2k}^i], \quad [c_{2k}^i, c_{2k+1}^i], \quad [c_{2i}^i, a_2^i] \quad \text{for all } k = 1, \dots, i.$$

Finally, let $f(x) = -x + 1$ for $x \notin \bigcup_{i=1}^{\infty}[a_1^i, a_2^i]$. Consider a nonautonomous dynamical system $(f_{1,\infty}) = \{f_i\}_{i=1}^{\infty}$ such that $f_i = f$ for each $i \in \mathbb{N}_+$. Of course, $1, 2 \in \mathcal{P}(f_{1,\infty})$. Put $\psi_1 = f$ and $\psi_2 = f^2$. Then both (ψ_1) and (ψ_2) are dynamical systems generated by $(f_{1,\infty})$. It is easy to see that $x_0 = 0$ is a strong entropy point of (ψ_2) but is not a strong entropy point of (ψ_1).

The above example is somewhat surprising in view of the following proposition, which follows immediately from [5, Lemma 4.3]. We use h to denote the entropy of a dynamical system (see [5]).

PROPOSITION 2.1. *Let $(f_{1,\infty}) \in \mathrm{PG}(\psi)$. Then $h(f_{1,\infty}) > 0$ if and only if $h(\psi) > 0$.*

The notion of chaotic points for autonomous systems was introduced in [10]. In a similar way, we will introduce this notion for nonautonomous systems below.

First, we define the unstable manifold for a dynamical system $(f_{1,\infty})$ by

$$\mathcal{W}(x_0, f_{1,\infty}) = \bigcap_{n=1}^{\infty} \bigcup_{m=0}^{\infty} f_1^m\Big(B\Big(x_0, \frac{1}{n}\Big)\Big).$$

LEMMA 2.2. *Let $(f_{1,\infty})$ be a dynamical system. Then $t \in \mathcal{W}(x_0, f_{1,\infty})$ if and only if there exist sequences $\{y_n\}_{n=1}^{\infty} \subset X$ and $\{k_n\}_{n=1}^{\infty} \subset \mathbb{N}_0$ such that $y_n \to x_0$ and $f_1^{k_n}(y_n) = t$.*

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PROOF. *Necessity.* Let $t \in \mathcal{W}(x_0, f_{1,\infty})$. Then $t \in \bigcup_{k=0}^{\infty} f_1^k(B(x_0, 1/n))$ for any $n \in \mathbb{N}_+$. Fix $n \in \mathbb{N}_+$. There exists $k_m \in \mathbb{N}_0$ such that

$$t \in f_1^{k_m}\Big(B\Big(x_0, \frac{1}{n}\Big)\Big).$$

Therefore one can find $y_m \in B(x_0, 1/n)$ for which

$$t = f_1^{k_m}(y_m).$$

Since n was chosen arbitrarily, it is possible to define sequences $\{k_m\}_{m=1}^{\infty} \subset \mathbb{N}_0$ and $\{y_m\}_{m=1}^{\infty} \subset X$, convergent to x_0, such that $f_1^{k_m}(y_m) = t$.

Sufficiency. Let $\{y_m\}_{m=1}^{\infty} \subset X$ and $\{k_m\}_{m=1}^{\infty} \subset \mathbb{N}_0$ be sequences such that $y_m \to x_0$ and $f_1^{k_m}(y_m) = t$ for some $t \in X$. Fix an arbitrary $n \in \mathbb{N}_+$. Since $y_m \to x_0$, there exists m_0 such that $y_{m_0} \in B(x_0, 1/n)$ and $f_1^{k_{m_0}}(y_{m_0}) = t$. Therefore

$$t \in f_1^{k_{m_0}}\Big(B\Big(x_0, \frac{1}{n}\Big)\Big) \subset \bigcup_{k=0}^{\infty} f_1^k\Big(B\Big(x_0, \frac{1}{n}\Big)\Big).$$

Since n was chosen arbitrarily,

$$t \in \bigcap_{n=1}^{\infty} \bigcup_{k=0}^{\infty} f_1^k\Big(B\Big(x_0, \frac{1}{n}\Big)\Big) = \mathcal{W}(x_0, f_{1,\infty}). \qquad \square$$

In many papers, the unstable manifold of an autonomous dynamical system is defined for periodic points (for example, [12, Definition 4.13] or [10]). For this reason, our considerations of the unstable manifold will involve the periodic points.

Let $x_0 \in \mathrm{Per}(f_{1,\infty})$. A point t is called an $(x_0, f_{1,\infty})$-homoclinic point if $x_0 \neq t$ in $\mathcal{W}(x_0, f_{1,\infty})$ and x_0 is a limit of a sequence $\{f_1^{m_k}(t)\}_{k=0}^{\infty}$ for some sequence of positive integers $\{m_k\}_{k=0}^{\infty}$. We say that a point x_0 is a chaotic point of a system $(f_{1,\infty})$ if, for each neighbourhood of x_0, there exists an $(x_0, f_{1,\infty})$-homoclinic point. Let $\mathrm{H}_0(f_{1,\infty})$ denote the set of all chaotic points of $(f_{1,\infty})$.

In the introduction, we mentioned that, even in the case of autonomous dynamical systems, there exist many different (inequivalent) definitions of chaos. Similarly, there are many inequivalent definitions of a homoclinic point (see [3, 10, 12]). Many mathematicians take the view that the basic criterion for a chaotic dynamical system is positive entropy of the system. (Some heuristic justification can be found in [2] and [13].) In this context, it is interesting that if a function $f : [0, 1] \to [0, 1]$ is continuous with a positive entropy, then there exists a chaotic point of f. In addition, if there exists a positive integer p and a point $z \in \mathrm{Per}_p(f)$ that is a chaotic point of (f^p), then the entropy of f is positive.

THEOREM 2.3. *Let $\psi : X \to X$ be an arbitrary continuous function and let t be an (x_0, ψ)-homoclinic point. Then t is an $(x_0, f_{1,\infty})$-homoclinic point for any $(f_{1,\infty})$ in* $\mathrm{PG}(\psi)$.

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Proof. Let $\psi : X \to X$ be an arbitrary continuous function and let t be an (x_0, ψ)-homoclinic point. Then x_0 is a periodic point of the dynamical system (ψ), so there exists $p \in \mathbb{N}_+$ such that $x_0 \in \mathrm{Per}_p(\psi)$. Let $(f_{1,\infty})$ be an arbitrary dynamical system belonging to $\mathrm{PG}(\psi)$. We will prove that $x_0 \in \mathrm{Per}(f_{1,\infty})$. Since $(f_{1,\infty}) \in \mathrm{PG}(\psi)$, there exists $k \in \mathcal{P}(f_{1,\infty})$ such that $\psi = f_1^k$. Then $x_0 \in \mathrm{Per}_p(f_1^k)$, so $(f_1^k)^{np}(x_0) = x_0$ for any $n \in \mathbb{N}_+$ and

$$(f_1^k)^{np}(x_0) = f_1^{nkp}(x_0) = x_0.$$

Thus $x_0 \in \mathrm{Per}_{kp}(f_{1,\infty})$ and $x_0 \in \mathrm{Per}(f_{1,\infty})$.

Since t is an (x_0, ψ)-homoclinic point, $x_0 \neq t$ for $t \in \mathcal{W}(x_0, \psi)$ and x_0 is the limit of the sequence $\{(\psi)^{m_w}(t)\}_{w=0}^{\infty}$ for some sequence $\{m_w\}_{w=0}^{\infty}$ of positive integers. We will show that t is an $(x_0, f_{1,\infty})$-homoclinic point. We have already proved that $x_0 \in \mathrm{Per}(f_{1,\infty})$.

Obviously, $x_0 \neq t$. We already know that $t \in \mathcal{W}(x_0, \psi)$ and so, according to Lemma 2.2, there exist sequences $\{y_n\}_{n=1}^{\infty} \subset X$ and $\{k_n\}_{n=1}^{\infty} \subset \mathbb{N}_0$ such that $y_n \to x_0$ and $(\psi)^{k_n}(y_n) = t$ for any $n \in \mathbb{N}_+$. To prove that $t \in \mathcal{W}(x_0, f_{1,\infty})$, it is enough to find a sequence $\{z_n\}_{i=1}^{\infty} \subset \mathbb{N}_0$ such that $f_1^{z_n}(y_n) = t$ for any $n \in \mathbb{N}_+$. Put $z_n = k \cdot k_n$. Then

$$f_1^{z_n}(y_n) = f_1^{k \cdot k_n}(y_n) = (\psi)^{k_n}(y_n) = t.$$

Define $\{n_w\}_{w=1}^{\infty}$ by $n_w = k \cdot m_w$. Then $f_1^{n_w} = (f_1^k)^{m_w} = (\psi)^{m_w}$, so $\{n_w\}_{w=1}^{\infty}$ is a sequence of positive integers such that $f_1^{n_w}(t) = (\psi)^{m_w}(t) \to x_0$ as $w \to \infty$. □

It is a simple matter to show that the converse theorem is not true. As a consequence of the above theorem, we obtain the following result.

Theorem 2.4. *Let ψ be a continuous function such that x_0 is a chaotic point of the system (ψ). For any $(f_{1,\infty}) \in \mathrm{PG}(\psi)$, the point x_0 is a chaotic point of the system $(f_{1,\infty})$.*

The main result of this paper (included in the next part) will be connected with the possibility of modifying a dynamical system in such a way that some point becomes a chaotic point. This is especially interesting when the point is not a chaotic point. In this context, the following theorem appears to be of interest.

Theorem 2.5. *If a continuous function $f : [0, 1] \to [0, 1]$ has the local homeomorphism property at x_0, then $x_0 \notin \mathrm{H}_0(f)$.*

Proof. To obtain a contradiction, we suppose that there exist a continuous function $f : [0, 1] \to [0, 1]$ and a point $x_0 \in [0, 1]$ such that f has the local homeomorphism property at x_0 and

$$x_0 \in \mathrm{H}_0(f). \tag{2.1}$$

According to our assumptions, there exists a neighbourhood U of the point x_0 such that $f_{|\mathrm{cl}(U)} : \mathrm{cl}(U) \to \mathrm{cl}(U)$ is a homeomorphism. By [1, Corollary 4.2.5],

$$h(f_{|\mathrm{cl}(U)}) = 0. \tag{2.2}$$

We will show that $x_0 \in \mathrm{H}_0(f_{|\mathrm{cl}(U)})$. From (2.1), we see that $x_0 \in \mathrm{Per}(f_{|\mathrm{cl}(U)})$.

Let V be a neighbourhood of the point x_0 in the space $\mathrm{cl}(U)$ with the induced topology, that is, $V = \mathrm{cl}(U) \cap V_1$, where V_1 is open in X. We have $x_0 \in V$. Since $x_0 \in \mathrm{H}_0(f)$, there exists an (x_0, f)-homoclinic point $t \in V$. We will show that t is also an $(x_0, f_{|\mathrm{cl}(U)})$-homoclinic point.

We know that $x \neq t$ and that there exist sequences $\{y_n\}_{n=1}^{\infty} \subset X$ and $\{k_n\}_{n=1}^{\infty} \subset \mathbb{N}_0$ such that $y_n \to x_0$ and $f^{k_n}(y_n) = t$ for all $n \in \mathbb{N}_+$. One can assume that $y_n \in U \subset \mathrm{cl}(U)$. By Lemma 2.2, $t \in \mathcal{W}(x_0, f_{|\mathrm{cl}(U)})$. Moreover, t is an (x_0, f)-homoclinic point. Thus $x_0 = \lim_{k\to\infty} f^{m_k}(t)$ for some sequence $\{m_k\}_{k=1}^{\infty} \subset \mathbb{N}_+$. Then there exists $s \in \mathbb{N}_+$ such that $f^{m_k}(t) \in U \subset \mathrm{cl}(U)$ for $k \geq s$. Therefore $f^{m_k}(t) = f^{m_k}_{|\mathrm{cl}(U)}(t)$ for $k \geq s$ and so $x_0 = \lim_{k\to\infty} f^{m_k}_{|\mathrm{cl}(U)}(t)$ for $k \geq s$.

Thus t is an $(x_0, f_{|\mathrm{cl}(U)})$-homoclinic point and $x_0 \in \mathrm{H}_0(f_{|\mathrm{cl}(U)})$. From [12, Theorem 4.24], $h(f_{|\mathrm{cl}(U)}) > 0$. But this contradicts (2.2). □

3. Disruption of dynamical systems

Many papers consider modification of dynamical systems For example, periodic disturbance of $(f_{1,\infty})$ appears in [6]. Our definition follows this idea.

Let $\varepsilon > 0$ and let ρ_u be the metric of uniform convergence. We say that a family Φ of functions ε-disrupts a periodic nonautonomous dynamical system $(f_1, \ldots, f_k)$ to a system with some property **P** if the system $(f_1, \ldots, f_{k-1}, \varphi \circ f_k)$ has property **P** and $\rho_u(f_1^k, \varphi) < \varepsilon$ for each $\varphi \in \Phi$. In the case of an autonomous dynamical system (f), the family Φ of functions ε-disrupts (f) to a system with some property **P** if the system $(\varphi \circ f)$ has property **P** and $\rho_u(f, \varphi) < \varepsilon$ for each $\varphi \in \Phi$.

By [10, Theorem 3], a continuous function can be distorted in such a way that a certain fixed point becomes a full chaotic point. Consequently, it seems interesting to discuss an analogous problem, where the starting point is a nonautonomous periodic dynamical system and a periodic point. The theorem below presents a result of this type.

Theorem 3.1. *Let $(f_{1,\infty})$ be a nonautonomous periodic dynamical system on X such that $(f_{1,\infty})$ has the local homeomorphism property at $x_0 \in \mathrm{Per}(f_{1,\infty})$. For any $\varepsilon > 0$, there exist a dynamical system (ψ) generated by $(f_{1,\infty})$ and an uncountable family Φ of continuous functions that ε-disrupts (ψ) to a system such that the point x_0 is a chaotic point. As a consequence, x_0 is a chaotic point of each system $(\tau_{1,\infty}) \in \mathrm{PG}(\varphi \circ \psi)$, where $\varphi \in \Phi$ (and x_0 is a chaotic point of some disruption of $(f_{1,\infty})$ by Φ).*

Proof. Let $(f_{1,\infty})$ be a nonautonomous periodic dynamical system. Then there exists a positive integer m such that $f_n = f_{n \pmod m}$ if $n \pmod m \neq 0$ and $f_n = f_m$ otherwise. By our assumption, $x_0 \in \mathrm{Per}(f_{1,\infty})$ and there is a positive integer n that is a period of x_0.

Set $k = n \cdot m$. Then $k \in \mathcal{P}(f_{1,\infty})$ and $f_1^k(x_0) = x_0$. Obviously, f_1^k is continuous. By Lemma 1.1, we conclude that f_1^k has the local homeomorphism property at x_0.

Put $\psi = f_1^k$, so that the system (ψ) is generated by $(f_{1,\infty})$. Consider $\varepsilon > 0$. Let W be a neighbourhood of x_0 such that $f_i(W) = W$ and $(f_i)_{|W}$ is a homeomorphism. Of course, $\psi(W) = W$ and $(\psi)_{|W}$ is a homeomorphism.

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Since X is manifold, there exists a base $\mathbb{B}(x_0) = \{K_n\}_{n=0}^{\infty}$ at x_0 such that $K_n \in \mathbb{B}(X)$ for $n \in \mathbb{N}_0$ and the following conditions are fulfilled:

[M1] $x_0 \in \mathrm{Int}(K_n)$ for $n \in \mathbb{N}_+$;
[M2] $K_{n+1} \subset \mathrm{Int}(K_n)$ for $n \in \mathbb{N}_0$;
[M3] $\lim_{n\to\infty}\mathrm{diam}(K_n) = 0$; and
[M4] the sequence $\{K_n\}_{n=0}^{\infty}$ has the extension property.

Consequently, $x_0 \in \mathrm{Int}(K_n)$ for all $n \in \mathbb{N}_+$ and there exist $w \geq 2$ and $n_0 \in \mathbb{N}_0$ such that $K_{n_0} \subset B(x_0, \varepsilon/w) \subset W$. It is evident that there is a nonnegative integer n_1 such that

$$K_{n_1} \subset K_{n_0} \quad \text{and} \quad \psi(K_{n_1}) \subset \mathrm{Int}(K_{n_0}). \tag{3.1}$$

Obviously, $x_0 \in \psi(K_{n_1}) \cap \mathrm{Int}(K_{n_1})$. We will prove that

$$\text{there exists an arc } \mathfrak{L} = \mathrm{L}(x_0, a_0) \subset \psi(K_{n_1}) \cap \mathrm{Int}(K_{n_1}). \tag{3.2}$$

Choose a positive integer $n_* > n_1$ such that $K_{n_*} \in \mathbb{B}(x_0)$ and $\psi(K_{n_*}) \subset \mathrm{Int}(K_{n_1})$. From [M2], $K_{n_*} \subset \mathrm{Int}(K_{n_1}) \subset K_{n_1}$. Moreover, $\psi(K_{n_*}) \subset \psi(K_{n_1})$. It follows that $\psi(K_{n_*}) \subset \psi(K_{n_1}) \cap \mathrm{Int}(K_{n_1})$ and it is easy to see that $K_{n_*}\setminus\{x_0\} \neq \emptyset$, so let $z_1 \in K_{n_*}\setminus\{x_0\}$. Put $a_0 = \psi(z_1)$. Then

$$a_0 = \psi(z_1) \in \psi(K_{n_*}) \subset \mathrm{Int}(K_{n_1}) \subset W.$$

Since $z_1 \neq x_0$ and $z_1, x_0 \in W$ and because $(\psi)_{|W}$ is an injective function,

$$\psi(z_1) \neq \psi(x_0) \quad \text{and} \quad a_0 \neq x_0.$$

Thus there are two distinct points $a_0, x_0 \in \psi(K_{n_*}) \subset \psi(K_{n_1}) \cap \mathrm{Int}(K_{n_1})$. Moreover, since K_{n_*} is arcwise connected and $(\psi)_{|W}$ is a homeomorphism, the set $(\psi)_{|W}(K_{n_*})$ is also arcwise connected. Therefore (3.2) is true.

Let $\{x_n\}_{n=1}^{\infty} \subset \mathfrak{L}$ with $x_1 \neq a_0$ be a sequence that converges to x_0 and is such that $\mathrm{L}_{\mathfrak{L}}(x_0, x_{n+1}) \subset \mathrm{L}_{\mathfrak{L}}(x_0, x_n)$ for $n \in \mathbb{N}_+$. Let $h_n : \mathrm{L}_{\mathfrak{L}}(x_{n+1}, x_n) \to \mathrm{L}_{\mathfrak{L}}(x_n, x_{n-1})$ be a homeomorphism with $h_n(x_{n+1}) = x_n$ and $h_n(x_n) = x_{n-1}$ for $n \geq 2$. Finally, let $h_1 : \mathrm{L}_{\mathfrak{L}}(x_2, x_1) \to \mathrm{L}_{\mathfrak{L}}(x_1, a_0)$ be a homeomorphism with $h_1(x_2) = x_1$ and $h_1(x_1) = a_0$.

Next, we will consider an arbitrary uncountable family H of homeomorphisms $h : \mathrm{L}_{\mathfrak{L}}(x_1, a_0) \to \mathfrak{L}$ such that $h(x_1) = a_0$ and $h(a_0) = x_0$. Fix $h \in \mathrm{H}$ and define a function $\varphi^*_{(h)} : \mathfrak{L} \to \mathfrak{L}$ by

$$\varphi^*_{(h)}(x) = \begin{cases} \nabla_{n=1}^{\infty} h_n(x) & \text{for all } x \in \mathfrak{L}\setminus(\mathrm{L}_{\mathfrak{L}}(x_1, a_0) \cup \{x_0\}), \\ h(x) & \text{for all } x \in \mathrm{L}_{\mathfrak{L}}(x_1, a_0), \\ x_0 & \text{for } x = x_0. \end{cases}$$

Applying [4, Theorem 2.1.13] yields, immediately, that $\varphi^*_{(h)}$ is continuous. By induction, for any positive integers k, n and for any point x_n,

$$(\varphi^*_{(h)})^k(x_n) = \begin{cases} x_{n-k} & \text{for all } k < n, \\ a_0 & \text{for } k = n, \\ x_0 & \text{for all } k > n. \end{cases} \tag{3.3}$$

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Clearly, $(((\psi)_{|W})^{-1})_{|\mathfrak{L}}$ is a homeomorphism. Put $g = (((\psi)_{|W})^{-1})_{|\mathfrak{L}}$. Then

$$g : \mathfrak{L} \xrightarrow{\text{onto}} g(\mathfrak{L}) \subset K_{n_1} \subset K_{n_0} \subset B\Big(x_0, \frac{\varepsilon}{w}\Big) \subset W.$$

Obviously, $g(\mathfrak{L})$ is an arc because g is a homeomorphism. Denote the set $g(\mathfrak{L})$ by the symbol $\mathfrak{L}_1$. Of course, $g(x_0) = x_0$.

Let $\{y_n\}_{n=0}^{\infty}$ be a sequence consisting of elements of $\mathfrak{L}_1$ such that $\psi(y_n) = x_n$ for all $n \in \mathbb{N}_+$ and $\psi(y_0) = a_0$. We see at once that

$$y_n \longrightarrow x_0. \tag{3.4}$$

Also $g \circ \varphi^*_{(h)} : \mathfrak{L} \to \mathfrak{L}_1 \subset K_{n_0}$ and, by (3.2), $\mathfrak{L} \subset \operatorname{Int} K_{n_1}$. Define $\widehat{\varphi_{(h)}} : \mathfrak{L} \cup \operatorname{Fr} K_{n_1} \to K_{n_0}$ by

$$\widehat{\varphi_{(h)}}(x) = \begin{cases} g \circ \varphi^*_{(h)}(x) & \text{for all } x \in \mathfrak{L}, \\ \psi(x) & \text{for all } x \in \operatorname{Fr} K_{n_1}. \end{cases}$$

Since $\mathfrak{L} \cup \operatorname{Fr} K_{n_1} \subset K_{n_1}$ is a closed set and the sequence $\{K_n\}$ has the extension property, there exists a continuous function $\overline{\varphi_{(h)}} : K_{n_1} \to K_{n_0}$ that is an extension of the function $\widehat{\varphi_{(h)}}$. So we can define a continuous function $\varphi_{(h)} : X \to X$ by

$$\varphi_{(h)}(x) = \begin{cases} \overline{\varphi_{(h)}}(x) & \text{for all } x \in K_{n_1}, \\ \psi(x) & \text{for all } x \in X \backslash K_{n_1}. \end{cases}$$

Since $(\varphi_{(h)})_{|K_{n_1}} : K_{n_1} \to K_{n_0} \subset B(x_0, \varepsilon/w)$, by (3.1), $\psi(K_{n_1}) \subset B(x_0, \varepsilon/w)$ and it follows that $\rho_u((\psi)_{|K_{n_1}}, (\varphi_{(h)})_{|K_{n_1}}) < \varepsilon$. We also have $(\varphi_{(h)})_{|X\backslash K_{n_1}} = (\psi)_{|X\backslash K_{n_1}}$. Therefore

$$\rho_u((\psi)_{|X\backslash K_{n_1}}, (\varphi_{(h)})_{|X\backslash K_{n_1}}) = 0.$$

Consequently, the condition $\rho_u(\psi, \varphi_{(h)}) < \varepsilon$ is fulfilled.

Observe that $(\varphi_{(h)})_{|\mathfrak{L}} = g \circ \varphi^*_{(h)}$. Put $\xi_{(h)} = \varphi_{(h)} \circ \psi$ and consider the autonomous dynamical system $(\xi_{(h)})$. An easy computation shows that $\xi^p_{(h)} = (\varphi_{(h)} \circ \psi)^p$ for $p \geq 1$. It is evident that $\psi \circ g(x) = x$ for all $x \in \mathfrak{L}$.

Let $x \in \mathfrak{L}_1$. Then

$$\psi(x) \in \mathfrak{L}, \quad \varphi^*_{(h)}(\psi(x)) \in \mathfrak{L} \quad \text{and} \quad \varphi_{(h)}(\psi(x)) = g(\varphi^*_{(h)}(\psi(x))) \in \mathfrak{L}_1.$$

Consequently, $\xi^p_{(h)}(x) = g \circ (\varphi^*_{(h)})^p \circ \psi(x)$ for $x \in \mathfrak{L}_1$ and $p \geq 1$. Of course, $x_0 \in \operatorname{Fix}(\xi_{(h)})$.

Now we will prove that

$$x_0 \in \mathrm{H}_0(\xi_{(h)}). \tag{3.5}$$

First, we will show that all elements of the sequence $\{y_n\}_{n=1}^{\infty}$ are $(x_0, \xi_{(h)})$-homoclinic. Let $n_0 \in \mathbb{N}_+$. Obviously, $x_0 \neq y_{n_0}$. We are going to prove that $y_{n_0} \in \mathcal{W}(x_0, \xi_{(h)})$. For this reason, put $z_n = y_{n_0+n}$. By (3.4), we infer that $z_n \to x_0$. By (3.3),

$$\xi^{k_n}_{(h)}(z_n) = \xi^n_{(h)}(y_{n_0+n}) = g \circ (\varphi^*_{(h)})^n \circ \psi(y_{n_0+n}) = g \circ (\varphi^*_{(h)})^n(x_{n_0+n}) = g(x_{n_0}) = y_{n_0}$$

for all $n \in \mathbb{N}_+$. This finishes the proof of (3.5).

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In the next step, we will prove that there exists a sequence $\{m_l\} \in \mathbb{N}_+$ such that $\xi_{(h)}^{m_l}(y_{n_0}) \to x_0$ as $l \to \infty$. Let $\{m_l\}$ be a sequence such that $m_l > n_0$ for all $l \in \mathbb{N}_+$. Then

$$\xi_{(h)}^{m_l}(y_{n_0}) = g \circ (\varphi_{(h)}^*)^{m_l} \circ \psi(y_{n_0}) = g \circ (\varphi_{(h)}^*)^{m_l}(x_{n_0}) = g(x_0) = x_0,$$

which gives $\xi_{(h)}^{m_l}(y_{n_0}) \to x_0$ as $l \to \infty$. Consequently, each point from the sequence $\{y_n\}_{n=1}^{\infty}$ is $(x_0, \xi_{(h)})$-homoclinic and it follows easily that $x_0 \in \mathrm{H}_0(\xi_{(h)})$.

Let $\Phi = \{\varphi_{(h)} : h \in \mathrm{H}\}$. We will prove that

$$\Phi \text{ is an uncountable family.} \tag{3.6}$$

Let $h, k : \mathrm{L}_{\mathfrak{L}}(x_1, a_0) \to \mathfrak{L}$ be two distinct functions from the family H. Then there exists $t \in \mathrm{L}_{\mathfrak{L}}(x_1, a_0)$ such that $h(t) \neq k(t)$. Thus $\varphi_{(h)}^*(t) \neq \varphi_{(k)}^*(t)$. Since g is injective,

$$g \circ \varphi_{(h)}^*(t) \neq g \circ \varphi_{(k)}^*(t),$$

which gives $\widehat{\varphi_{(h)}}(t) \neq \widehat{\varphi_{(k)}}(t)$, so $\varphi_{(h)}(t) \neq \varphi_{(k)}(t)$. As a result, we can find t such that the functions $\varphi_{(h)}, \varphi_{(k)}$ assume different values at this point. This proves that the functions $\varphi_{(h)}$ and $\varphi_{(k)}$ are different. Since H is an uncountable family, we get (3.6).

Combining the previous considerations with (3.6), we deduce that Φ is a family of continuous functions that disrupts the system (ψ) to a system such that x_0 is a chaotic point.

Put $(\zeta_{1,\infty}) = (f_1, \dots, f_{k-1}, \varphi_{(h)} \circ f_k)$. Then $\zeta_1^k = \varphi_{(h)} \circ f_1^k = \varphi_{(h)} \circ \psi$. For any $x \in \mathfrak{L}_1$, we have $\zeta_1^{pk}(x) = (\zeta_1^k)^p(x) = g \circ (\varphi_{(h)}^*)^p \circ \psi(x)$ for $p \geq 1$.

Now we will prove that $x_0 \in \mathrm{H}_0(\zeta_{1,\infty})$. First, we show that all points from the sequence $\{y_n\}_{n=1}^{\infty} \subset \mathfrak{L}_1$ are $(x_0, \zeta_{1,\infty})$-homoclinic. Let $n_0 \in \mathbb{N}_+$. We will prove that $y_{n_0} \in \mathcal{W}(x_0, \zeta_{1,\infty})$, that is, there exist a sequence $\{z_n\} \subset \mathfrak{L}$ and a sequence of positive integers $\{k_n\}_{n=1}^{\infty}$ such that $k_n \to \infty$, $z_n \to x_0$ and $\zeta_1^{k_n}(z_n) = y_{n_0}$. In order to satisfy these conditions, we can put $k_n = kn$ and $z_n = y_{n_0+n}$ so that $z_n \to x_0$ as $n \to \infty$ by (3.4). By (3.3),

$$\begin{aligned}\zeta_1^{k_n}(z_n) &= \zeta_1^{kn}(y_{n_0+n}) = (\zeta_1^k)^n(y_{n_0+n}) \\ &= g \circ (\varphi_{(h)}^*)^n \circ \psi(y_{n_0+n}) = g \circ (\varphi_{(h)}^*)^n(x_{n_0+n}) = g(x_{n_0}) = y_{n_0}\end{aligned}$$

for all $n \in \mathbb{N}_+$. As a result, $y_{n_0} \in \mathcal{W}(x_0, \zeta_{1,\infty})$.

Finally, we show that there exists a sequence $\{m_l\}_{l=1}^{\infty} \in \mathbb{N}_+$ such that $\zeta_1^{m_l}(y_{n_0}) \to x_0$ as $l \to \infty$. Let $\{m_l\}_{l=1}^{\infty}$ be a sequence such that $m_l = k \cdot w_l$ for $l \in \mathbb{N}_+$ and $w_l > n_0$ for $w_l \in \mathbb{N}_+$. Then

$$\zeta_1^{m_l}(y_{n_0}) = (\zeta_1^k)^{w_l}(y_{n_0}) = g \circ (\varphi_{(h)}^*)^{w_l} \circ \psi(y_{n_0}) = g \circ (\varphi_{(h)}^*)^{w_l}(x_{n_0}) = g(x_0) = x_0.$$

This gives $\zeta_1^{m_l}(y_{n_0}) \to x_0$ as $l \to \infty$. Consequently, any point from the sequence $\{y_n\}_{n=1}^{\infty}$ is $(x_0, \zeta_{1,\infty})$-homoclinic and $x_0 \in \mathrm{H}_0(\zeta_{1,\infty})$. □

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