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THE C^*-ALGEBRAS OF LATTICE ATOMIC GRAPHS

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Abstract

In this article, we define lattice graphs (which generalise ultragraphs) as well as their Cuntz–Krieger families and C^*-algebras. We will give a thorough study in the special case of lattice atomic graphs.

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1. Introduction and preliminaries

Ultragraphs were first introduced by Tomforde as a generalisation of the notion of directed graphs. The C^*-algebras for ultragraphs include both the C^*-algebras of graphs (see [1, 5, 9, 10]) as well as the Exel–Laca algebra defined in [6] (see [11] and [12]).

In this paper, we define a more general notion called lattice graphs (see Definition 1.1) and define their Cuntz–Krieger families as well as their C^*-algebras (if such Cuntz–Krieger families exist). We mainly restrict our attention to what we call lattice atomic graphs (see Definition 1.1).

More precisely, we show in Section 2 that Cuntz–Krieger families exist for distributive lattice atomic graphs. We also give a graph theoretical characterisation for the existence of Cuntz–Krieger families for general lattice atomic graphs (see Proposition 2.9). In Section 3, we define the C^*-algebra of a lattice atomic graph that admits a Cuntz–Krieger family and show that, in that case, one can replace the original lattice graph with a distribution lattice graph (see Theorem 3.1).

Let us first give a definition of lattice graphs. Recall that a partially ordered set $(L, \leq)$ is a *lattice* if for any $x, y \in L$, there exists $x \wedge y, x \vee y \in L$ such that $x \wedge y \leq x, y \leq x \vee y$, and if $u, v \in L$ with $u \leq x, y \leq v$, then $u \leq x \wedge y$ and $x \vee y \leq v$. An element $0 \in L$ is the *zero* of L if $0 \leq y$ for every $y \in L$, and $x \in L \setminus \{0\}$ is an *atom* if there does not exist $y \in L \setminus \{0, x\}$ such that $y \leq x$. A lattice L is said to be *distributive* if it satisfies the following identities (for any $x, y, z \in L$):

(a) $(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z)$;
(b) $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$.

Throughout this article, we denote by L_a the set of all atoms of a lattice L (with zero).

DEFINITION 1.1. We make the following definitions.

(a) Let $\mathcal{E}^0$ be a lattice with zero. A *lattice graph* $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$ consists of a set (of vertices) $\mathcal{E}^0$, a set (of edges) $\mathcal{E}^1$, and maps $r, s : \mathcal{E}^1 \to \mathcal{E}^0 \setminus \{0\}$ (the range and source maps of each edge). An element $x \in \mathcal{E}^0$ is called a *sink* if there does not exist $e \in \mathcal{E}^1$ with $s(e) = x$.
(b) Let $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$ be a lattice graph. If $\mathcal{E}^0$ is distributive, we call $\mathcal{E}$ a *distributive lattice graph*. If $s(\mathcal{E}^1) \subseteq \mathcal{E}^0_a$, we call $\mathcal{E}$ a *lattice atomic graph*.
(c) A *standard lattice graph* is a lattice atomic graph $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$ that satisfies the following conditions:
    (i) both $\mathcal{E}^0$ and $\mathcal{E}^1$ are countable;
    (ii) $\mathcal{E}^0$ is generated by $\{x \mid x \in \mathcal{E}^0_a\} \cup \{r(e) \mid e \in \mathcal{E}^1\} \cup \{0\}$;
    (iii) if $A_x := \{a \in \mathcal{E}^0_a \mid a \leq x\}$ $(x \in \mathcal{E}^0)$, then $x \mapsto A_x$ is an injection from $\mathcal{E}^0$ to $\mathcal{P}(\mathcal{E}^0_a)$.

Ultragraphs are examples of distributive standard lattice graph. In fact, any distributive standard lattice graph comes from an ultragraph in the way as described in the following example (this fact will be proved in Corollary 2.3).

EXAMPLE 1.2. Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ be an ultragraph. If $\mathcal{E}^0 = \mathcal{G}^0$ and $\mathcal{E}^1 = \mathcal{G}^1$, then $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$ is a distributive standard lattice graph (because $\mathcal{G}^0$ is the lattice generated by the elements in G^0 and $\{r(e) \mid e \in \mathcal{G}^1\}$).

Given a standard lattice graph $(\mathcal{E}^0, \mathcal{E}^1, r, s)$, one can set $X = \mathcal{E}^0_a$ and identify $x \in \mathcal{E}^0$ with $A_x \in \mathcal{P}(X)$. However, it is not true in general that this gives a lattice homomorphism from $\mathcal{E}^0$ to $\mathcal{P}(X)$ (for example, if $\mathcal{E}^0$ is the lattice as in Example 2.7(a)). Nevertheless, this is a lattice homomorphism when $\mathcal{E}^0$ is distributive but the argument is far from obvious. In fact, this is the main difficulty in showing that any distributive standard lattice graph is an ultragraph.

2. Cuntz–Krieger families for lattice graphs

Let us begin this section with the following result concerning lattices that is crucial throughout this paper. Since we could not find this result in the literature, we give a proof here for completeness.

PROPOSITION 2.1. *For any distributive lattice L with zero, there exist a set X and an injective lattice homomorphism $\psi : L \to \mathcal{P}(X)$ such that $\psi(L_a) \subseteq X$ (here we identify elements in X as singleton subsets of X) and $\psi(0) = \emptyset$.*

PROOF. By a variant of the Birkhoff theorem (see, for example, [3, Theorem 5.7]), there is a lattice isomorphism φ_1 from L to a ring of sets $S \subseteq \mathcal{P}(Z)$. If $Y := Z \setminus \varphi_1(0)$ and $\varphi : L \to \mathcal{P}(Y)$ is defined by

$$\varphi(u) = \varphi_1(u) \setminus \varphi_1(0) \quad (u \in L),$$

then φ is an injective lattice homomorphism sending 0 to the empty set (note that any element in S contains $\varphi_1(0)$). We define an equivalent relation $\sim$ in Y by $x \sim y$ if either $x = y$ or there exists $a \in L_a$ with $x, y \in \varphi(a)$ (note that $\varphi(a) \cap \varphi(b) = \emptyset$ for any distinct $a, b \in L_a$). Let

$$X := Y/\sim$$

and $Q : Y \to X$ be the canonical quotient map. It is clear that $Q(A \cup B) = Q(A) \cup Q(B)$ and $Q(A \cap B) \subseteq Q(A) \cap Q(B)$ for any $A, B \subseteq Y$. Suppose that $u, v \in L$ and

$$z \in Q \circ \varphi(u) \cap Q \circ \varphi(v).$$

Then there exist $x \in \varphi(u)$ and $y \in \varphi(v)$ such that $z = Q(x) = Q(y)$. Consequently, either $x = y$ or there exists $a \in L_a$ with $x, y \in \varphi(a)$. In the first case, we clearly have $z \in Q(\varphi(u) \cap \varphi(v)) = Q \circ \varphi(u \wedge v)$. In the second case, $\varphi(u \wedge a) = \varphi(u) \cap \varphi(a) \neq \emptyset$ which implies $a \leq u$ (as $a \in L_a$ and φ is injective) and $a \leq v$. This shows that $x, y \in \varphi(a) \subseteq \varphi(u \wedge v)$ and $z \in Q \circ \varphi(u \wedge v)$. Thus, $\psi := Q \circ \varphi$ is a lattice homomorphism sending elements in L_a to singletons in $\mathcal{P}(X)$. It remains to show that ψ is injective. Suppose that $u, v \in L$ with

$$Q \circ \varphi(u) \subseteq Q \circ \varphi(v).$$

Then, for any $x \in \varphi(u)$, there exists $y \in \varphi(v)$ such that $x \sim y$. If $x \neq y$, then there exists $a \in L_a$ with $x, y \in \varphi(a)$. This tells us that $\varphi(a) \cap \varphi(v) \neq \emptyset$ and, as above, we have $a \leq v$ which implies $x \in \varphi(a) \subseteq \varphi(v)$. Therefore, we always have $\varphi(u) \subseteq \varphi(v)$ and so $u \leq v$ (as φ is injective). This gives the injectivity of ψ. □

REMARK 2.2. Note that although we have Proposition 2.1, a distributive lattice atomic graph is still far from being an ultragraph. For example, there is a distributive lattice atomic graph $\mathcal{E}$ with $\mathcal{E}^0$ being countable infinite and $\mathcal{E}_a^0$ being a singleton set while if $\mathcal{G}$ is an ultragraph, then $G^0 = \mathcal{G}_a^0$ is finite if and only if $\mathcal{G}^0$ is finite.

Now, we can give the converse of Example 1.2. As noted in the paragraph after that example, the difficulty is to show that $x \mapsto A_x$ is a lattice homomorphism.

COROLLARY 2.3. *Any distributive standard lattice graph $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$ can be realised as an ultragraph.*

PROOF. Let X and ψ be as in Proposition 2.1 for $L = \mathcal{E}^0$. Put $X_0 := \psi(\mathcal{E}_a^0)$ (which is a countable set). By regarding $X_0 \subseteq X$, one can define a lattice homomorphism $\psi_0 : \mathcal{E}^0 \to \mathcal{P}(X_0)$ by $\psi_0(x) := \psi(x) \cap X_0$ ($x \in \mathcal{E}^0$). It is easy to see that $\psi_0(x) = \psi(A_x)$ and so ψ_0 is injective by Definition 1.1(c)(iii). This, together with Definition 1.1(c)(ii) and the countability of $\mathcal{E}^1$, shows that $\mathcal{G} = (X_0, \mathcal{E}^1, \psi_0 \circ r, \psi_0 \circ s)$ is an ultragraph with $\mathcal{G}^0 = \mathcal{E}^0$. □

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We now define the Cuntz–Krieger family for lattice graphs and study their existence in the case of lattice atomic graphs.

DEFINITION 2.4. Let $\mathcal{E}$ be a lattice graph. A *Cuntz–Krieger $\mathcal{E}$-family* in a C^*-algebra B is a collection of partial isometries $\{\mathbf{s}_e \mid e \in \mathcal{E}^1\}$ with nonzero mutually orthogonal ranges and a collection of projections $\{\mathbf{p}_x \mid x \in \mathcal{E}^0\}$ such that for any $x, y \in \mathcal{E}^0$ and $e \in \mathcal{E}^1$:

(CK1) $\mathbf{p}_0 = 0$, $\mathbf{p}_x \mathbf{p}_y = \mathbf{p}_{x \wedge y}$ and $\mathbf{p}_{x \vee y} = \mathbf{p}_x + \mathbf{p}_y - \mathbf{p}_{x \wedge y}$;
(CK2) $\mathbf{s}_e^* \mathbf{s}_e = \mathbf{p}_{r(e)}$;
(CK3) $\mathbf{s}_e \mathbf{s}_e^* \leq \mathbf{p}_{s(e)}$;
(CK4) $\mathbf{p}_x = \sum_{f \in s^{-1}(x)} \mathbf{s}_f \mathbf{s}_f^*$ whenever $0 < |s^{-1}(x)| < +\infty$.

Using a similar argument as in [11, Theorem 2.11], we have the following.

LEMMA 2.5. *Let Y be a set and $\mathcal{E} = (\mathcal{P}(Y), \mathcal{E}^1, r, s)$ be a lattice atomic graph. There exists a Cuntz–Krieger $\mathcal{E}$-family $\{\mathbf{s}_e, \mathbf{p}_A\}$ such that $\mathbf{p}_A \neq \mathbf{p}_B$ whenever $A \neq B \in \mathcal{P}(Y)$.*

PROOF. Set $H_e := \ell_2(\mathcal{P}(Y)) \oplus \ell_2(\mathcal{E}^1)$ for each $e \in \mathcal{E}^1$. If $a \in Y$ is a sink, we put $H_a := \ell_2(\mathcal{P}(Y)) \oplus \ell_2(\mathcal{E}^1)$. If $a \in Y$ is not a sink, we put $H_a := \bigoplus_{e \in s^{-1}(a)} H_e$. Let $H_A := \bigoplus_{a \in A} H_a$ for any $A \in \mathcal{P}(Y)$. We write $H := H_Y$ and identify $H_e \subseteq H_{s(e)}$. For every $e \in \mathcal{E}^1$, we define a partial isometry $\mathbf{s}_e \in \mathcal{L}(H)$ with initial space $H_{r(e)}$ and final space H_e. For every $A \in \mathcal{P}(Y)$, we define $\mathbf{p}_A \in \mathcal{L}(H)$ to be the orthogonal projection onto H_A. It is not hard to check that $\{\mathbf{s}_e, \mathbf{p}_A\}$ is a Cuntz–Krieger $\mathcal{E}$-family satisfying the required property. □

This result, together with Proposition 2.1, gives the following corollary.

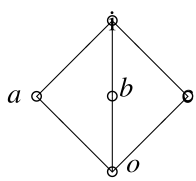
COROLLARY 2.6. *If $\mathcal{E}$ is a distributive lattice atomic graph, then there exists a Cuntz–Krieger $\mathcal{E}$-family. In fact, there exists a Cuntz–Krieger $\mathcal{E}$-family on some $\mathcal{L}(H)$ which satisfies a stronger condition than* (CK4)*:*

(CK4′) $\mathbf{p}_x = \mathrm{SOT} - \sum_{f \in s^{-1}(x)} \mathbf{s}_f \mathbf{s}_f^*$ *whenever* $s^{-1}(x) \neq \emptyset$

(where $\mathrm{SOT} - \sum$ *means the strong operator limit of the finite sums).*

However, the Cuntz–Krieger $\mathcal{E}$-family does not exist for arbitrary lattice atomic graph as can be seen in the following example.

EXAMPLE 2.7. (a) Let $\mathcal{E}^0 = \{o, a, b, c, i\}$ be a lattice represented by the following Hasse diagram:



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Put $\mathcal{E}^1 = \{\alpha, \beta, \gamma\}$, where

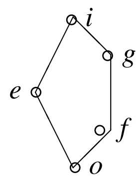
$$s(\alpha) = a, \quad s(\beta) = b, \quad s(\gamma) = c \quad \text{and} \quad r(\alpha) = r(\beta) = r(\gamma) = i.$$

Clearly, $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$ is a lattice atomic graph. Suppose that there is a Cuntz–Krieger $\mathcal{E}$-family $\{\mathbf{s}_e, \mathbf{p}_x\}$. Since $a \vee b = b \vee c = a \vee c = i$, we have, by (CK1),

$$\mathbf{p}_i = \mathbf{p}_a + \mathbf{p}_b = \mathbf{p}_b + \mathbf{p}_c = \mathbf{p}_a + \mathbf{p}_c$$

and so, $\mathbf{p}_a = \mathbf{p}_b = \mathbf{p}_c$. As $\mathbf{p}_i = 2\mathbf{p}_a$ is a projection, we must have $\mathbf{p}_a = \mathbf{p}_b = \mathbf{p}_c = 0$. Now (CK3) implies that $\mathbf{s}_\alpha \mathbf{s}_\alpha^* = \mathbf{s}_\beta \mathbf{s}_\beta^* = \mathbf{s}_\gamma \mathbf{s}_\gamma^* = 0$ which contradicts the assumption on their ranges.

(b) Let $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$ be a lattice atomic graph with $\mathcal{E}^0 = \{o, e, f, g, i\}$ being a lattice represented by the following Hasse diagram:



Suppose that $\mathcal{F}^0 = \{o, a, c, i\}$ is the sublattice of the lattice represented by the Hasse diagram in part (a) and $\varphi : \mathcal{E}^0 \to \mathcal{F}^0$ is the lattice homomorphism defined by $\varphi(o) = o$, $\varphi(i) = i$, $\varphi(e) = a$, $\varphi(f) = c = \varphi(g)$. Set $r' = \varphi \circ r$ and $s' = \varphi \circ s$. Then $\mathcal{F} = (\mathcal{F}^0, \mathcal{E}^1, r', s')$ is a lattice atomic graph such that any Cuntz–Krieger $\mathcal{F}$-family $\{\mathbf{s}_e, \mathbf{p}_y\}$ induces a Cuntz–Krieger $\mathcal{E}$-family $\{\mathbf{s}_e, \mathbf{p}_{\varphi(x)}\}$. As $\mathcal{F}^0$ is distributive, we know that a Cuntz–Krieger $\mathcal{E}$-family exists (Corollary 2.6).

It is well known that a lattice is nondistributive if and only if it has a sublattice whose Hasse diagram is isomorphic to either the diamond shape as in part (a) or the pentangonal shape as in part (b). The above example seems to indicate that the only obstruction to the existence of a Cuntz–Krieger family is the appearance of a sublattice with the Hasse diagram being of diamond shape.

In the following, we give a more thorough study of the existence of Cuntz–Krieger families which explains the difference between the diamond shape and the pentangonal shape lattices as in the above example. Let us first make the following simple observation about Cuntz–Krieger families.

LEMMA 2.8. *Let $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$ be a lattice atomic graph and $\{\mathbf{s}_e, \mathbf{p}_x\}$ be a Cuntz–Krieger $\mathcal{E}$-family on a C^*-algebra B.*

(a) *If we consider B as a C^*-subalgebra of some $\mathcal{L}(H)$, then $\mathfrak{P}_{\mathcal{E}^0} := \{\mathbf{p}_x \mid x \in \mathcal{E}^0\}$ is a distributive sublattice of the lattice of projections in $\mathcal{L}(H)$. If $\mathcal{E}$ is a lattice atomic graph, then $\mathbf{p}_{s(e)}$ is an atom in $\mathfrak{P}_{\mathcal{E}^0}$ (for every $e \in \mathcal{E}^1$).*

(b) *We have $\mathbf{s}_e \neq \mathbf{s}_f$ for every $e \neq f \in \mathcal{E}^1$.*

(c) *If $x, y \in \mathcal{E}^0_a$ with $\mathbf{p}_x = \mathbf{p}_y$ and $s^{-1}(x) \neq \emptyset$, then $x = y$.*

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PROOF. (a) It is clear that if p, q are commuting projections of $\mathcal{L}(H)$, then $p \wedge q = pq$ as well as $p \vee q = p + q - pq$. Now, this part is clear.

(b) If $e \neq f \in \mathcal{E}^1$ but $\mathbf{s}_e = \mathbf{s}_f$, then $\mathbf{s}_e\mathbf{s}_e^* = \mathbf{s}_f\mathbf{s}_f^*$ and we have $\mathbf{s}_e\mathbf{s}_e^*\mathbf{s}_f\mathbf{s}_f^* = \mathbf{s}_e\mathbf{s}_e^* \neq 0$ which contradicts the fact that $\mathbf{s}_e^*\mathbf{s}_f = 0$.

(c) For any $e \in s^{-1}(x)$, we have $0 \neq \mathbf{s}_e\mathbf{s}_e^* \leq \mathbf{p}_x$. Hence, $\mathbf{p}_{x\wedge y} = \mathbf{p}_x \neq 0$ and $x \wedge y \neq 0$ which means that $x = y$. □

PROPOSITION 2.9. *Let $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$ be a lattice atomic graph. There exists a Cuntz–Krieger $\mathcal{E}$-family if and only if there exist a distributive lattice L and a lattice homomorphism $\varphi : \mathcal{E}^0 \to L$ such that φ is injective on $\mathcal{E}^0_s := \{x \in \mathcal{E}^0_a \mid s^{-1}(x) \neq \emptyset\}$.*

PROOF. The necessity follows from Lemma 2.8. Conversely, suppose that such L and φ exist. Note that $\varphi(\mathcal{E}^0)$ is a (distributive) sublattice of L and so we can assume that φ is surjective. It is easy to see that $\varphi(0)$ is the zero of L. Moreover, $\varphi(x)$ is an atom of L for each $x \in \mathcal{E}^0_a$. Indeed, if $\varphi(y) \leq \varphi(x)$, then

$$\varphi(y) = \varphi(y \wedge x) \in \{\varphi(0), \varphi(x)\}.$$

Let $r' = \varphi \circ r$ and $s' = \varphi \circ s$. Then $\mathcal{F} = (L, \mathcal{E}^1, r', s')$ is a distributive lattice atomic graph. By Corollary 2.6, there exists a Cuntz–Krieger $\mathcal{F}$-family $\{\mathbf{s}_e, \mathbf{q}_y\}$. We claim that if $\mathbf{p}_x := \mathbf{q}_{\varphi(x)}$ $(x \in \mathcal{E}^0)$, then $\{\mathbf{s}_e, \mathbf{p}_x\}$ is a Cuntz–Krieger $\mathcal{E}$-family. The only nontrivial part is (CK4). Suppose that $x \in \mathcal{E}^0_a$ such that $0 < |s^{-1}(x)| < \infty$. For any $e \in (s')^{-1}(\varphi(x))$, we have $\varphi(s(e)) = \varphi(x)$ which means that $s(e) = x$ (because of the hypothesis and the fact that $x, s(e) \in \mathcal{E}^0_s$). Thus, $(s')^{-1}(\varphi(x)) = s^{-1}(x)$ and $\{\mathbf{s}_e, \mathbf{p}_x\}$ will satisfy (CK4). □

COROLLARY 2.10. *For any lattice K with a zero, the following two statements are equivalent.*

(a) *There exists a distributive lattice L and a lattice homomorphism $\phi : K \to L$ such that ϕ is injective on K_a.*

(b) *For any lattice atomic graph $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$ with $\mathcal{E}^0 = K$, there exists a Cuntz–Krieger $\mathcal{E}$-family.*

PROOF. (a) $\Longrightarrow$ (b) This follows directly from Proposition 2.9.
(b) $\Longrightarrow$ (a) Take any $x \in K_a$ and put $\mathcal{E}^1 := \{(a, x) \mid a \in K_a\}$. Define $r, s \mid \mathcal{E}^1 \to K$ by $s((a, x)) = a$ and $r((a, x)) = x$. Then $\mathcal{E} = (K, \mathcal{E}^1, r, s)$ is a lattice atomic graph. Let $\{\mathbf{s}_e, \mathbf{p}_x\}$ be a Cuntz–Krieger $\mathcal{E}$-family. Then $L = \{\mathbf{p}_x \mid x \in \mathcal{E}^0\}$ and $\phi(x) = \mathbf{p}_x$ will satisfy the conditions in part (a) (because of Lemma 2.8). □

3. C^*-algebras of lattice atomic graphs

Let $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$ be a lattice graph. Suppose that G is the disjoint union $\{\mathbf{s}_e \mid e \in \mathcal{E}^1\} \uplus \{\mathbf{p}_x \mid x \in \mathcal{E}^0\}$ of symbols indexed by $\mathcal{E}^1$ and $\mathcal{E}^0$ respectively, and R is the relations (CK1)–(CK4). Then the existence of a Cuntz–Krieger $\mathcal{E}$-family will imply that (G, R) is admissible in the sense of [2, Definition 1.1], and so one can construct a C^*-algebra $C^*(G, R)$ as in [2]. We denote this C^*-algebra by $C^*(\mathcal{E})$ and

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call it the *C^*-algebra of the lattice graph $\mathcal{E}$*. Note that $C^*(\mathcal{E})$ exists only when there is a Cuntz–Krieger $\mathcal{E}$-family. The canonical image of $\mathcal{E}^1 \uplus \mathcal{E}^0$ in $C^*(\mathcal{E})$ is called the *universal Cuntz–Krieger $\mathcal{E}$-family*.

In the following, we again restrict our attention to lattice atomic graphs. Our first result shows that any lattice atomic graph C^*-algebra can be realised as a distributive lattice atomic graph.

THEOREM 3.1. *Let $\mathcal{E}$ be a lattice atomic graph such that there exists a Cuntz–Krieger $\mathcal{E}$-family. Then there is a distributive lattice atomic graph $\mathcal{F}$ such that $C^*(\mathcal{E}) = C^*(\mathcal{F})$.*

PROOF. Let $\{\mathbf{s}_e, \mathbf{p}_x\}$ be the universal Cuntz–Krieger $\mathcal{E}$-family and $\mathfrak{P}_{\mathcal{E}^0}$ be the corresponding distributive lattice as given in Lemma 2.8(a). Define $\varphi : \mathcal{E}^0 \to \mathfrak{P}_{\mathcal{E}^0}$ by $\varphi(x) = \mathbf{p}_x$ as well as $r' = \varphi \circ r$ and $s' = \varphi \circ s$. Then $\mathcal{F} := (\mathfrak{P}_{\mathcal{E}^0}, \mathcal{E}^1, r', s')$ is a distributive lattice atomic graph. If $\{\mathbf{t}_e, \mathbf{q}_y\}$ is the universal Cuntz–Krieger $\mathcal{F}$-family and $\mathbf{p}'_x := \mathbf{q}_{\varphi(x)}$, then Lemma 2.8(c) and the argument of Proposition 2.9 tell us that $\{\mathbf{t}_e, \mathbf{p}'_x\}$ is a Cuntz–Krieger $\mathcal{E}$-family. This shows that there is a surjective $*$-homomorphism $\Phi : C^*(\mathcal{E}) \to C^*(\mathcal{F})$ such that

$$\Phi(\mathbf{p}_x) = \mathbf{q}_{\varphi(x)} \quad \text{and} \quad \Phi(\mathbf{s}_e) = \mathbf{t}_e.$$

Conversely, if we define $\mathbf{q}'_{\mathbf{p}_x} := \mathbf{p}_x$, then $\{\mathbf{s}_e, \mathbf{q}'_y\}$ is a Cuntz–Krieger $\mathcal{F}$-family. This induces a surjective $*$-homomorphism $\Psi : C^*(\mathcal{F}) \to C^*(\mathcal{E})$ such that $\Psi(\mathbf{q}_{\varphi(x)}) = \mathbf{p}_x$ and $\Psi(\mathbf{t}_e) = \mathbf{s}_e$. □

Note that if $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$ is a lattice atomic graph such that there exists a largest element $i \in \mathcal{E}^0$ and $\{\mathbf{s}_e, \mathbf{p}_x\}$ is the universal Cuntz–Krieger $\mathcal{E}$-family, then $\mathbf{p}_i$ is the identity. The converse is true if $\mathcal{E}$ is distributive. More generally, we have the following result which is a generalisation of the corresponding result for ultragraph C^*-algebras.

PROPOSITION 3.2. *Let $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$ be a lattice atomic graph such that there exists a Cuntz–Krieger $\mathcal{E}$-family. If $C^*(\mathcal{E})$ is unital, then $1 \in \mathfrak{P}_{\mathcal{E}^0}$ where $\mathfrak{P}_{\mathcal{E}^0}$ is the lattice as in Lemma 2.8(a) for the universal Cuntz–Krieger $\mathcal{E}$-family $\{\mathbf{s}_e, \mathbf{p}_x\}$. If, in addition, $\mathcal{E}^0$ is distributive, then the uniticity of $C^*(\mathcal{E})$ will imply that there exists $i \in \mathcal{E}^0$ with $x \leq i$ for any $x \in \mathcal{E}^0$.*

PROOF. Suppose that $C^*(\mathcal{E})$ is unital. The set $\mathfrak{F}(\mathcal{E}^0)$ of all finite subsets of $\mathcal{E}^0$ is a directed set. For any $F \in \mathfrak{F}(\mathcal{E}^0)$, we set $y_F := \bigvee\{x \mid x \in F\}$. For any $e \in \mathcal{E}^1$ and $x \in \mathcal{E}^0$, we have

$$\|\mathbf{p}_{y_F}\mathbf{s}_e - \mathbf{s}_e\| + \|\mathbf{s}_e\mathbf{p}_{y_F} - \mathbf{s}_e\| + \|\mathbf{p}_{y_F}\mathbf{p}_x - \mathbf{p}_x\| + \|\mathbf{p}_x\mathbf{p}_{y_F} - \mathbf{p}_x\| \to 0.$$

Hence, $\{\mathbf{p}_{y_F}\}$ is an approximate unit in $C^*(\mathcal{E})$ and $\|1 - \mathbf{p}_{y_F}\| \to 0$. This implies that there exists $D \in \mathfrak{F}(\mathcal{E}^0)$ with $1 = \mathbf{p}_{y_D}$. If $\mathcal{E}^0$ is distributive, the relation $\mathbf{p}_x = \mathbf{p}_{x \wedge y_D}$ ($x \in \mathcal{E}^0$) will imply that $x \leq y_D$ (note that $x \mapsto \mathbf{p}_x$ is injective because of Lemma 2.5). □

We end this paper with two examples.

EXAMPLE 3.3. (a) Let $\mathcal{E}^0 = \{0, 1, 2\}$ with the usual ordering, $\mathcal{E}^1 = \{a\}$, $r(a) = 2$ and $s(a) = 1$. Then $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$ is a distributive lattice atomic graph. If $\{\mathbf{s}_a, \mathbf{p}_1, \mathbf{p}_2\}$ is the universal Cuntz–Krieger $\mathcal{E}$-family, then $\mathbf{p}_2$ is the identity of $C^*(\mathcal{E})$ (by Proposition 3.2) and $\mathbf{p}_1 \neq \mathbf{p}_2$ (by Proposition 2.1 and Lemma 2.5). Thus, $C^*(\mathcal{E})$ is generated by a nonunitary isometry and is thus the Toeplitz algebra (see, for example, [4]).

(b) Let $\mathcal{E}^0 = \{0\} \cup [1, \infty)$ with the usual ordering, $\mathcal{E}^1 = \{a\}$ and $r(a) = 1 = s(a)$. Suppose that $\{\mathbf{s}_a, \mathbf{p}_t\}$ is the universal Cuntz–Krieger family for the distributive lattice atomic graph $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$. Note that $\mathbf{s}_a = \mathbf{p}_1$ and so $C^*(\mathcal{E})$ is generated by a increasing net of distinct nonzero projections $\{\mathbf{p}_t \mid t \in [1, \infty)\}$. Thus, there exists a locally compact Hausdorff space Ω such that $C^*(\mathcal{E}) \cong C_0(\Omega)$. For each $\omega \in \Omega$, we denote

$$S_\omega := \{t \in [1, \infty) : \omega(\mathbf{p}_t) = 1\} \neq \emptyset.$$

Note that if $\omega, \nu \in \Omega$ with $S_\omega = S_\nu$, then ω and ν agree on the generating set $\{\mathbf{p}_t\}$ which implies that $\omega = \nu$. Moreover, since $\omega(\mathbf{p}_s) = 1$ will imply $\omega(\mathbf{p}_t) = 1$ when $t \geq s$, we see that S_ω is either $[s, \infty)$ or (s, ∞) for some $s \in [1, \infty)$. We now show that both of these two types of sets are possible, that is, we have a bijection from Ω to $\{[s, \infty) \mid s \in [1, \infty)\} \cup \{(s, \infty) \mid s \in [1, \infty)\}$. In fact, for any $t \in [1, \infty)$, we can set $a_t, b_t \in \ell^\infty([1, \infty))$ by

$$b_t = \chi_{[1,t]} \quad \text{and} \quad a_t = \begin{cases} \chi_{[1,t)} & t \neq 1 \\ \chi_{\{1\}} & t = 1. \end{cases}$$

Then both $\{a_t\}$ and $\{b_t\}$ are increasing nets of nontrivial projections in $\ell^\infty([1, \infty))$ and there exist surjective $*$-homomorphisms $\Phi : C^*(\mathcal{E}) \to C^*(\{a_t\}) \subseteq \ell^\infty([1, \infty))$ and $\Psi : C^*(\mathcal{E}) \to C^*(\{b_t\})$ such that $\Phi(\mathbf{p}_t) = a_t$ and $\Psi(\mathbf{p}_t) = b_t$ $(t \in [1, \infty))$. Let $\varphi_s \in \ell^\infty([1, \infty))^*$ be defined by $\varphi_s(f) = f(s)$ $(s \in [1, \infty))$. Put $\phi_s := \varphi_s \circ \Phi \in \Omega$ and $\psi_s := \varphi_s \circ \Psi \in \Omega$. It is easy to see that $S_{\phi_s} = (s, \infty)$ and $S_{\psi_s} = [s, \infty)$. Now, we set

$$\Delta := \{\chi_{[s,\infty)} : s \in [1, \infty)\} \cup \{\chi_{(s,\infty)} : s \in [1, \infty)\} \subseteq \ell^\infty([1, \infty))$$

and equip it with the topology of pointwise convergence on $[1, \infty)$. The above tells us that $\omega \mapsto \chi_{S_\omega}$ gives a bijection from Ω to Δ. Note that $\omega(\mathbf{p}_t) = \chi_{S_\omega}(t)$ $(t \in [1, \infty))$. Thus, for $\omega_i, \omega \in \Omega$, we have $\omega_i \to \omega$ if and only if $\omega_i(\mathbf{p}_t) \to \omega(\mathbf{p}_t)$ for any $t \in [1, \infty)$ (because $\{\mathbf{p}_t\}$ generates $C^*(\mathcal{E})$ and ω_i are contractive homomorphisms). This shows that $\omega_i \to \omega$ if and only if $\chi_{S_{\omega_i}} \to \chi_{S_\omega}$ pointwisely on $[1, \infty)$. Consequently, $\Omega \cong \Delta$ as topological spaces.

REMARK 3.4. (a) Note that although Toeplitz algebra is a graph C^*-algebra, the lattice graph $\mathcal{E}$ in Example 3.3(a) is not even an ultragraph.

(b) Since the C^*-algebra in Example 3.3(b) is not separable, it is not an ultragraph C^*-algebra.

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