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ON SEMIPRIME AMPLE JORDAN RINGS

$J \subseteq H$ WITH CHAIN CONDITION[(1)]

BY
DANIEL J. BRITTEN

The purpose of this paper is to point out that the arguments of [2] with slight modification extend the main result of [2] to the case of H satisfying either ACC or DCC on quadratic ideals and they extend [6, Theorem 2] to R being semiprime. Thus we obtain

THEOREM 1. *Let R be a semiprime associative ring with involution $*$ and J a closed ample quadratic Jordan subring of $H(R)$ satisfying either ACC or DCC on quadratic ideals. Then R is Goldie. In this case, J has a Jordan ring of quotients J' which is a closed ample quadratic Jordan subring of $H(R')$ where R' is the associative ring of quotients of R.*

In what follows, R denotes an associative ring with involution $*$; $H=H(R)=\{x \in R : x^*=x\}$: $T=T(R)=\{x+x^* : x \in R\}$; and $N=N(R)=\{xx^* : x \in R\}$. J will denote a special quadratic Jordan ring; that is, J will be an additive subgroup of $(R, +)$, closed under the quadratic operator $yU_x = xyx$ and the binary composition x^2. Following Montgomery [5], special quadratic Jordan subring J of H is called ample provided $N \cup T \subseteq J \subseteq H$.

Montgomery [5] and [6] removes the 2-torsion free assumption of Britten [1] and enlargens the class of Jordan subrings to be considered from H to ample special quadratic Jordan subrings J. (If $\frac{1}{2} \in R$, then $H=T$ and so the only ample subring is H.) Thus for this paper, [2], [5] and [6] are the basic references.

In [5, Theorem 4.5], Montgomery assumes in addition to J being ample that $xJx^* \subseteq J$ for all $x \in R$. This property is also valid for the Jordan ring of quotients obtained from this theorem and so we shall say that J is closed ample if it is ample and $xJX^* \subseteq J$ for all $x \in R$.

Finally if $S \subseteq R$, $A_\ell(S)$ denotes $\{x \in R : xS=0\}$ and $A_r(S)$ denotes $\{x \in R : Sx=0\}$, then we have

LEMMA 1. *Let R be a semiprime ring with $*$. Then*
(i) *if B is a left ideal of R and $A=A_\ell(B)$ then $BA=0$;*
(ii) *If B is a $*$-ideal (i.e. $B^*=B$) of R then $A=A_\ell(B)$ is a $*$-ideal and $B \cap (T \cup N)=0$ implies $B=0$.*

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(i) is easily shown and (ii) is [5, Lemma 3.1]. We shall also have need of [5, Theorem 4.5] which we formulate as

THEOREM 2. *Let R be an associative ring with involution $*$ and a ring of quotients R' (R' has a natural involution extending $*$). Let J be a closed ample Jordan subring of $H(R)$ such that every regular element of J is regular in R. Then J has a Jordan ring of quotients J' which is a closed ample Jordan subring of $H(R')$. When $J=H(R)$, $J'=H(R')$.*

Proof. Since we are asking that J' be closed ample and Montgomery asked that it only be ample, we must check that $xJ'x^* \subseteq J'$ for all $x \in R'$. (Here we are using $*$ to denote the natural involution on R' which extends the involution on R given to us in the statement of the theorem.) Thus for us the ring of quotients J' exists, is an ample Jordan subring of $H(R')$ and $J'=\{aU_b^{-1}: a, b \in J, b \text{ regular in } J\}$.

Let $x \in R'$ and $y \in J'$. Express x as $c^{-1}d$ and y as $b^{-1}ab^{-1}$ where $d \in R$, $a \in J$ and c, b are regular elements in N. Then $xyx^*=c^{-1}db^{-1}ab^{-1}d^*c^{-1}$. By the common multiple property of R, we may express db^{-1} as $b_1^{-1}d_1$, with $b_1 \in N$. Thus

$$\begin{aligned} xyx^* &= c^{-1}b_1^{-1}d_1ad_1^*b_1^{-1}c^{-1} \\ &= (d_1ad_1^*)U_{cU_{b_1}}^{-1}U_{b_1}. \end{aligned}$$

Set $s=(d_1ad_1^*)U_{cU_b}^{-1} \in J'$ and $t=b_1 \in J$. We have now reduced the problem to showing that $sU_t \in J'$ for $s \in J'$ and $t \in J$. This was done by Montgomery in her proof of [5, Theorem 4.5].

Let R be a semiprime ring with $*$ and J an ample quadratic Jordan subring of H satisfy either ACC or DCC on quadratic ideals.

For any subset $S \subseteq R$, we have $B=A_\ell(S)=A_\ell A_r A_\ell(S)$. In particular, if S is a $*$-ideal then B is a $*$-ideal and B is the left annihilator of the $*$-ideal $A=A_r(B)$. Let us gather together all such proper $*$-ideals B into the set $\mathfrak{B}=\{B: B=A_\ell(A) \text{ for } *\text{-ideal } A=A_r(B) \text{ with } B \neq R\}$.

Since $B \cap J$ is a quadratic ideal of J for all $B \in \mathfrak{B}$, if J satisfies ACC on quadratic ideals $\mathfrak{B}$ contains elements B_i such that $B_i \cap J$ is not a proper subset of $B \cap J$ regardless of the choice of $B \in \mathfrak{B}$. That is, $\emptyset \neq \beta_1=\{B_i \in \mathfrak{B}: B_i \cap J \text{ is maximal}\}$.

On the other hand, if J satisfies DCC on quadratic ideals, $\mathfrak{B}$ contains elements B_i such that $A_i \cap J$ is minimal where $A_i=A_r(B_i)$. That is, $\emptyset \neq \beta_2=\{B_i \in \mathfrak{B}: A_i \cap J \text{ is minimal where } A_i=A_r(B_i)\}$.

Now define

$$\beta=\begin{cases} \beta_1 & \text{if } J \text{ satisfies } ACC \\ \beta_2 & \text{if } J \text{ satisfies } DCC \end{cases}$$

LEMMA 2. *Let R be a semiprime ring with $*$ and J an ample quadratic Jordan subring of H satisfying either ACC or DCC on quadratic ideals. Then if β is as* *above then*

(i) $B_i \in \beta$ *is a *-prime ideal* (*i.e.* R/B_i *is a *-prime ring*);

(ii) *if* B_i *and* B_j *are distinct elements of* β *then* $B_i \not\subseteq B_j$ *but* $A_\ell(B_i) \subseteq B_j$;

(iii) β *is a finite set*;

(iv) *the elements of* β *intersect at zero.*

Proof. (i) Let C and D be nonzero *-ideals of R such that $CD \subseteq B_i \in \beta$. Let $A_i = A_r(B_i)$.

First suppose that J satisfies ACC on quadratic ideals. If $DA_i = 0$, then $D \subseteq B_i$. Therefore, we may assume that $A_iD + DA_i \neq 0$. By Lemma 1 (i), $B_i \cup C \subseteq A_\ell(A_iD + DA_i)$ and $A_\ell(A_iD + DA_i) \neq R$ be semi-primeness of R. But $B_i \cap J \subseteq A_\ell(A_iD + DA_i) \cap J$ and by maximality of $B_i \cap J$ we must have $B_i \cap J = A_\ell(A_iD + DA_i) \cap J$ and hence $C \cap J \subseteq B_i \cap J$. Thus $(A_iC + CA_i) \cap J \subseteq A_i \cap B_i = 0$, by semi-primeness, and hence, by Lemma 1 (ii), $CA_i = 0$ so that $C \subseteq B$.

Now suppose J satisfies DCC on quadratic ideals. If either $(A_iD + DA_i) \cap J$ or $(A_iC + CA_i) \cap J$ is zero, then we are done by Lemma 1 (ii). Thus we assume that neither is zero. By the minimality of $A_i \cap J$ we have

$$0 \neq (A_iD + DA_i) \cap J \subseteq A_r(C + B_i) \cap J = A_i \cap J$$

$$0 \neq (A_iC + CA_i) \cap J \subseteq A_r(D + B_i) \cap J = A_i \cap J$$

Thus $A_i \cap J \subseteq A_r(C + D + B_i)$ and hence $Q = (A_iD + DA_i) \cap J$ is nonzero, by Lemma 1 (ii), and is such that $JU_Q = 0$, contrary to the semi-primeness of J given to us by [3, Theorem 2].

(ii) follows from the *-primeness of the elements B_i of β. (See [2, Lemma 4]).

(iii) follows by showing that $\sum A_i$ is a direct sum and using the chain condition on J in the presence of $0 \neq (N \cup T) \cap A_i \subseteq J$. (See [2, Lemma 5]).

(iv) Let $\{B_1, \ldots, B_k\} = \beta$ in accordance with (iii) and corresponding to each $B_i \in \beta$, let $A_i = A_r(B_i)$. Let $I = \bigcap_1^k B_i$ and suppose $I \neq 0$. Then $IA_i = 0$ for $1 \leq i \leq k$ and by Lemma 1 (i), we have $A_iI = 0$ so that $A_i \subseteq A_\ell(I)$ for each i. $A_\ell(I) \neq 0$ by the semi-primeness of R.

Assume that J satisfies ACC on quadratic ideals. Then there is some $j = 1, \ldots, k$ such that $A_\ell(I) \cap J \subseteq B_j \cap J$. Set $Q = A_j \cap J \subseteq B_j \cap J$ and obtain the contradiction $JU_Q = 0$.

Finally, assume that J satisfies DCC. Since $A_iI = 0$ for each $i = 1, \ldots, k$, $I \subseteq A_r(\sum_i^k A_i)$. By the minimality of the $A_i \cap J$'s there is some $1 \leq j \leq k$ such that $A_r(I) \cap J \supseteq A_j \cap J$. Set $Q = A_j \cap J$ and obtain $JU_Q = 0$.

Proof of Theorem 1. By Lemma 2, R/B_i is an involution prime ring for each $B_i \in \beta$ where the involution on R/B_i is determined by applying * to the representative of the coset. The homomorphic image of J in R/B_i is closed ample quadratic Jordan subring of $H(R/B_i)$ which satisfies either ACC or DCC

accordingly as J does. That is, R/B_i satisfies the hypothesis of [6, Theorem 2] and hence R/B_i is *-prime Goldie. Thus R is semiprime Goldie. The last sentence of Theorem 1 follows from Theorem 2 after arguing, as Montgomery did in the proof of [5, Corollary 2], that regular elements of J are regular in R.

Before closing we point out that the open question (stated in the closing remarks of [5]) as to the necessity of the common multiple property of a Jordan ring J to have a ring of quotients may be answered negatively by considering the Jordan plus structure of the left Ore domain that is not a right Ore domain. However, the sufficiency of the common multiple property for J to have a ring of quotients remains open.

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Department of Mathematics
University of Windsor,
Windsor, Ontario,
Canada