BULL. AUSTRAL. MATH. SOC. MOS 18E40, 16A64, 16A21
VOL. 4 (1971), 355-359.

Rings whose modules form few torsion classes

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Characterizations are obtained of rings R such that the only torsion classes (respectively, hereditary torsion classes) of left unital R-modules are $\{0\}$ and the class of all modules.

Throughout this note all rings are associative, with identity, and except where otherwise specified, all modules are left, unital modules. A *torsion class* of modules is a non-void class closed under homomorphic images, extensions and direct sums. A torsion class which is also closed under submodules is called *hereditary*. Every module M is contained in a smallest torsion class, namely

$$T(M) = \{N \mid \mathrm{Hom}_R(M, K) = 0 \Rightarrow \mathrm{Hom}_R(N, K) = 0\} .$$

For a proof of this, as well as for further details concerning torsion classes see [3].

We shall be investigating the following conditions for a ring R :

(*) the only torsion classes of R-modules are $\{0\}$ and $\mathrm{mod}(R)$, the class of all R-modules;

(**) the only hereditary torsion classes of R-modules are $\{0\}$ and $\mathrm{mod}(R)$.

We shall obtain characterizations of rings satisfying these conditions and show that they are not in general equivalent, though for commutative rings and for rings with zero Jacobson radical they are.

PROPOSITION 1. *A ring* R *satisfies* (*) *if and only if:*

Received 2 December 1970.

(i) all simple R-modules are isomorphic,

(ii) all non-zero R-modules have simple submodules, and

(iii) all non-zero R-modules have maximal submodules.

Proof. Assume R satisfies (*). If S_1, S_2 are simple modules, then $T(S_1) = T(S_2) = \text{mod}(R)$, so $\text{Hom}_R(S_1, S_2) \neq 0$ and thus $S_1 \cong S_2$. If a module M has no simple submodules, then for any simple module S, we have $\text{Hom}_R(S, M) = 0$. But $M \in T(S) = \text{mod}(R)$, so $M = 0$. Any non-zero module M satisfies $T(M) = \text{mod}(R)$, so in particular for any simple module S there is a non-zero homomorphism $M \to S$, whose kernel is a maximal submodule.

Conversely if R satisfies *(i)*, *(ii)* and *(iii)* and if $T \neq \text{mod}(R)$ is a torsion class, then for any $M \in T$ there exists a non-zero module N such that $\text{Hom}_R(M, N) = 0$. (Otherwise $\text{mod}(R) = T(M) \subseteq T$.) Hence $\text{Hom}_R(M, S) = 0$ for any simple submodule S of N. *(i)* and *(ii)* then imply that M has no maximal submodules, so $M = 0$. //

There is a similar description for the rings which satisfy (**):

PROPOSITION 2. *A ring R satisfies (**) if and only if*

(i) all simple R-modules are isomorphic, and

(ii) all non-zero R-modules have simple submodules.

Proof. If S is simple, then $T(S)$ is hereditary (see [3], Proposition 4.2). Thus if R satisfies (**), $T(S) = \text{mod}(R)$, and *(i)*, *(ii)* follow as in the previous proof. For the converse, let S be any simple module; then $\text{Hom}_R(S, M) \neq 0$ for every non-zero module M, and so $T(S) = \text{mod}(R)$. But by *(ii)*, any hereditary torsion class H with non-zero members must contain S, whence $T(S) \subseteq H$. //

Andrunakievič and Rjabuhin [1] have characterized the rings which satisfy (**) in terms of hereditary radical properties of R-algebras. Also, it is easy to see that a ring R satisfies condition *(ii)* of Proposition 2 if and only if for every left ideal $L \neq R$, the set of left ideals K with $L \subsetneq K$ has a minimal element, so Proposition 2 is essentially the same as Theorem 3 of [*1*].

Before proceeding, we recall some terminology from [*2*]. An ideal I of a ring R is called *left T-nilpotent* if for any sequence (a_i) of elements of I, there exists an integer n such that $a_1a_2 \ldots a_n = 0$. *Right T-nilpotence* is similarly defined. In what follows, $J(R)$ denotes the Jacobson radical of a ring R.

THEOREM 3. *A ring R satisfies* (*) *if and only if $J(R)$ is left and right T-nilpotent and $R/J(R)$ satisfies* (*). *In this case $R/J(R)$ is (left) primitive.*

Proof. For any ring R, the simple R-modules can be given simple $R/J(R)$-module structures in an obvious way, and conversely. Thus all simple R-modules are isomorphic if and only if all simple $R/J(R)$-modules are.

Every non-zero R-module has a simple submodule if and only if $J(R)$ is right T-nilpotent and every non-zero $R/J(R)$-module has a simple submodule ([*6*], Proposition 3.2), while every non-zero R-module has a maximal submodule if and only if $J(R)$ is left T-nilpotent and every non-zero $R/J(R)$-module has a maximal submodule ([7], Théorème 1.1). Finally, if all simple R-modules are isomorphic, then $J(R)$ is equal to the annihilator of any one of them, and every simple $R/J(R)$-module is therefore faithful. //

Similarly we obtain

THEOREM 4. *A ring R satisfies* (**) *if and only if $J(R)$ is right T-nilpotent and $R/J(R)$ satisfies* (**). *$R/J(R)$ is then (left) primitive.*

COROLLARY 5. *For commutative rings,* (*) *and* (**) *are equivalent.*

THEOREM 6. *The following conditions are equivalent for a left primitive ring R:*

(i) R satisfies (*);

(ii) R satisfies (**);

(iii) R is a full ring of matrices over a division ring.

Proof. Obviously *(i)* $\Rightarrow$ *(ii)*.

(ii) ⇒ *(iii)*: Since $J(R) = 0$, R has no non-zero nilpotent one-sided ideals, so the (left) socle I of R is a non-zero ideal, and, since all simple modules are isomorphic, a simple ring (see [4], p. 65). Thus I is a non-zero idempotent ideal. The class

$$T_I = \{M \in \text{mod}(R) \mid IM = 0\}$$

is therefore a hereditary torsion class ([5], Corollary 2.3) which does not contain all R-modules. Hence $T_I = \{0\}$. But T_I contains the R-module R/I , so $R = I$ is a simple ring, and each simple R-module is projective. For an arbitrary R-module M , let M_0 denote the socle and let M_1 be defined by the exact sequence

$$0 \to M_0 \to M_1 \to (M/M_0)_0 \to 0 .$$

Since the simple modules are projective, the sequence splits, and M_1 is a direct sum of simple modules. But this means that M/M_0 has zero socle, whence $M = M_0$ is projective. R is therefore semi-simple artinian as well as simple, that is a full matrix ring over a division ring.

(iii) ⇒ *(i)*: Let S be a simple R-module. Then any non-zero R-module is a direct sum of copies of S so it is clear from the defining closure properties of torsion classes that any torsion class containing non-zero modules must contain S and therefore all modules. //

(*) and (**) are not equivalent for rings in general: Bass ([2], p. 476) gives an example of a ring R such that $R/J(R)$ is a field and $J(R)$ is left T-nilpotent but not right T-nilpotent. By Theorems 3 and 4, the opposite ring of R satisfies (**) but not (*). This example also shows that (**) is not equivalent to the corresponding statement for right R-modules. In contrast, it is clear from Theorems 3 and 6 that a ring satisfies (*) if and only if there are no non-trivial torsion classes of right R-modules.

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