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# ON A CLASS OF ARCHIMEDEAN INTEGRAL DOMAINS

RAYMOND A. BEAUREGARD AND DAVID E. DOBBS

**1. Introduction.** Our starting point is an observation in elementary number theory [**10**, Exercise 26, p. 17]: if $a$ and $b$ are positive integers such that each number in the sequence $a, b^2, a^3, b^4, \ldots$ divides the next, then $a = b$. Its proof depends only on $\mathbf{Z}$ being a unique factorization domain (*UFD*) whose units are $1, -1$. Accordingly, we abstract and say that a (commutative integral) domain $R$ *satisfies* (*) in case, whenever nonzero elements $a$ and $b$ in $R$ are such that each element in the sequence $a, b^2, a^3, b^4, \ldots$ divides the next, then $a$ and $b$ are associates in $R$ (that is, $a = bu$ for some unit $u$ of $R$). The main objective of this paper is the study of the class of domains satisfying (*).

This class is shown to be situated properly between the classes of completely integrally closed and of Archimedean domains (definitions recalled in Section 2) and, for integrally closed domains, it fits nicely into the hierarchy introduced by Ohm [**11**]. As may be expected from the *UFD* case above, any Krull domain satisfies (*). However, such is not the case for *GCD* (pseudo-Bézout) domains, for which the properties of being completely integrally closed, Archimedean, and (*) coincide (thus generalizing a result about Bézout domains [**5**, Proposition 3]). Finally, since (*) is reflected by polynomial extensions, consideration of nonmaximal orders in quadratic algebraic number fields shows, in contrast with the integrally closed case, that for each positive integer $n$, there is a Noetherian domain of Krull dimension $n$ which does not satisfy (*).

Throughout, $R$ denotes a domain, with $U(R)$ its group of units and $R^*$ its monoid of nonzero elements. It will be convenient to abbreviate "$a$ divides $b$" by "$a|b$" and "Krull dimension" by "dim." Any unexplained terminology is standard, as in [**4**] and [**6**].

**2. The property (*).** This section is concerned with the implications between condition (*) and some closely related conditions, as well as the stability of condition (*) under various domain-theoretic processes. Several related (counter) examples are presented in later sections.

We begin by recalling two definitions. The domain $R$ is said to be *completely integrally closed* (cic) if ($r \in R^*$, $rx^n \in R$ for each $n \geqq 1 \Rightarrow x \in R$). Following [**14**], $R$ is called *Archimedean* if $\bigcap Rr^n = 0$ for each nonunit $r$ of $R$. If $R$ is cic, then $R$ is integrally closed and, by [**5**, Corollary 5], also Archimedean. An Archimedean domain need not be cic, since a result of Ohm [**11**, Corollary 1.4]

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implies that any 1-dimensional domain is Archimedean. The relation of these properties to (*) is given next.

THEOREM 2.1. *Consider the following statements:*
(i) *$R$ is* cic;
(ii) *$R$ satisfies* (*);
(iii) *$R$ is Archimedean*;
(iv) *If $a$, $b \in R^*$ are such that each element in the sequence $a$, $b^2$, $a^3$, $b^4$, ... divides the next and if either $a|b$ or $b|a$, then $a$ and $b$ are associates in $R$*;
(v) *$R$ satisfies the ascending chain condition on principal ideals* (accp). *Then*: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftarrow$ (v).

*Proof.* (i) $\Rightarrow$ (ii): Let $R$ be cic, and let $a$, $b \in R^*$ such that $a|b^2|a^3|b^4|\ldots$. As $a[(ba^{-1})^2]^n \in R$ for each $n \geqq 1$, we have $(ba^{-1})^2 \in R$, and hence $ba^{-1} \in R$; i.e., $a|b$. A similar argument (using $ab^{-1}$ in place of $ba^{-1}$) shows that $b|a$, and so $a$ and $b$ are associates, as required.

(iv) $\Rightarrow$ (iii): Suppose (iv) holds, and let $0 \neq b \in \bigcap Rr^n$. We need to show that $r \in U(R)$. This follows since $b|(rb)^2|b^3|(rb)^4|\ldots$ and $b|rb$.

(ii) $\Rightarrow$ (iii): This is immediate from the implication just established, since (ii) $\Rightarrow$ (iv).

(iii) $\Rightarrow$ (iv): Let $R$ be Archimedean, and let $a$, $b \in R^*$ such that $a|b^2|a^3|b^4|\ldots$. Suppose, moreover, that $b|a$; write $a = rb$, with $r \in R$. As $a^{2n-1}|b^{2n}$ for each $n \geqq 1$ (and as $b \neq 0$), it follows readily that $r^{2n-1}|b$; thus, $b \in \bigcap Rr^n$. Since $R$ is Archimedean, $r \in U(R)$, so that $a$ and $b$ are associates, as desired. The case $a|b$ is handled similarly (using $b^{2n}|a^{2n+1}$ to show $(ba^{-1})^{2n}|a$).

(v) $\Rightarrow$ (iv): In the version (v) $\Rightarrow$ (iii), this is immediate, as (v) is well known to be equivalent to the condition that every strictly descending infinite chain of principal ideals has zero intersection. (This follows, for example, from [**1**, Proposition 1].) The proof is complete.

One consequence of the examples in Section 5 is the failure, even for integrally closed $R$, of all the possible implications amongst (i)–(v) which were not settled by Theorem 2.1.

*Remark* 2.2. Since any cic domain satisfies (*) by virtue of Theorem 2.1, our store of domains satisfying (*) includes all Krull domains; in particular, any Noetherian integrally closed domain and (to recapture the motivating exercise of Niven-Zuckerman) any *UFD* satisfy (*). Moreover, a valuation ring $R$ is cic if and only if dim $(R) \leqq 1$ (cf. [**4**, Theorem 14.5(3)]), and intersections of cic domains are again cic. Consequently, all domains that are locally cic – for example, almost Krull domains and 1-dimensional Prüfer domains – are cic and, hence, satisfy (*). As the proof of the following globalization result makes clear, the property of being preserved by intersections is one way in which (*) behaves like the property of being cic.

PROPOSITION 2.3. *If $R_M$ satisfies* (*) *for each maximal ideal $M$ of $R$, then $R$ satisfies* (*).

*Proof.* Let $a, b \in R^*$ such that $a|b^2|a^3|b^4| \ldots$. The successive quotients may be viewed in each $R_M$ and so, by hypothesis, $ab^{-1}, ba^{-1} \in \bigcap R_M = R$, as required.

We shall see later (Remark 4.1) that the converse of Proposition 2.3 is false, although some positive results along those lines are available (Theorem 4.3). The next result will be used in Section 6 to analyze the Noetherian case.

PROPOSITION 2.4. *The following are equivalent:*

(i) *$R$ satisfies* (*).

(ii) *$R[\{X_\alpha\}]$ satisfies* (*), *for each set $\{X_\alpha\}$ of algebraically independent indeterminates over $R$;*

(iii) (ii) *with "each" replaced by "some."*

*Proof.* Since a simple degree argument shows $U(R[\{X_\alpha\}]) = U(R)$, it follows easily that (iii) $\Rightarrow$ (i). As (ii) $\Rightarrow$ (iii) trivially, it remains only to show (i) $\Rightarrow$ (ii). Assume (i), and consider nonzero elements $f, g \in R[\{X_\alpha\}]$ such that $f|g^2|f^3|g^4| \ldots$. As $f, g$ involve only finitely many of the $X_\alpha$ nontrivially, we may suppose, by induction, that $f, g \in R[X]$. If $a$ and $b$ are the leading coefficients of $f$ and $g$, respectively, the divisibility conditions on the powers of $f$ and $g$ lead easily to $a|b^2|a^3|b^4| \ldots$. By (i), we have $a = bu$, for some $u \in U(R)$. However, if $K$ denotes the quotient field of $R$, we may view the successive quotients arising from $f|g^2|f^3|g^4| \ldots$ inside (the *UFD*) $K[X]$, so that $f = gv$, for some $v \in U(K[X])$. By comparing leading coefficients, $a = bv$, so that $u = v$. Hence $f = gu$, to complete the proof.

Recall that an example of Samuel [**12**, Theorem 9.1, p. 40] shows that $R$ a *UFD* $\not\Rightarrow R[[X]]$ a *UFD*. A similar result holds for (*); that is, $R$ satisfies (*) $\not\Rightarrow R[[X]]$ satisfies (*). To establish this invalidity of the analogue of Proposition 2.4 for power series, use the next proposition (for which we are indebted to R. Gilmer) and the fact that domains satisfying (*) need not be cic (c.f. Example 5.2 or Theorem 6.1 below).

PROPOSITION 2.5. *$R$ is* cic *if and only if $R[[X]]$ satisfies* (*).

*Proof.* The "only if" half follows immediately from [**4**, Theorem 12.9] and Theorem 2.1. For the converse, assume that $R$ is not cic; choose $a \in R^*$ and $d$ (in the quotient field of $R$ but) not in $R$ such that $ad^n \in R$ for each $n \geqq 1$. Setting $b = a(1 - dX)$, we have $a|b^2|a^3|b^4| \ldots$, with quotients in $R[[X]]$. As $ba^{-1} \notin R[[X]]$, it follows that $R[[X]]$ does not satisfy (*), to complete the proof.

We next note several additional methods (which are used later) for building domains satisfying (*) or related properties.

PROPOSITION 2.6. *Let $T$ be a domain containing $R$.*

(i) *If $R$ is integrally closed, $T$ is integral over $R$ and $T$ satisfies* (*), *then $R$ satisfies* (*).

(ii) *If* $U(R) = U(T)$ *and* $T$ *satisfies* accp (*respectively*, (*)), *then* $R$ *satisfies* accp (*respectively*, (*)).

*Proof.* (i) Let $a, b \in R^*$ such that $a|b^2|a^3|b^4| \ldots$. Viewing the successive quotients in $T$, we have $ab^{-1}, ba^{-1} \in T$, since $T$ satisfies (*). If $K$ is the quotient field of $R$, then $ab^{-1} \in U(T) \cap K = U(R)$, as required.

(ii) Let $Ra_1 \subset Ra_2 \subset \ldots$ be an ascending sequence of nonzero principal ideals of $R$. If $T$ satisfies accp, then for some $n$, $Ta_n = Ta_i$ for all $i > n$; then $a_n(a_i)^{-1} \in U(T) = U(R)$, so that $Ra_n = Ra_i$, and $R$ also satisfies accp. The proof of the final assertion may safely be omitted.

The following result is a partial converse of Proposition 2.6(i) and is motivated by a result of Krull (cf. [**4**, Theorem 12.8]) about cic domains.

PROPOSITION 2.7. *Let* $R$ *be integrally closed and satisfy* (*). *Let* $T$ *be the integral closure of* $R$ *in a purely inseparable field extension* $L$ *of* $K$, *the quotient field of* $R$. *Then* $T$ *satisfies* (*).

*Proof.* Let $a, b \in T^*$ such that $a|b^2|a^3|b^4| \ldots$. There is a positive integer $m$ such that $a^m, b^m \in K$, since $L/K$ is purely inseparable; select $r \in R^*$ such that $R$ contains $c = ra^m$ and $d = rb^m$. It is straightforward to verify that $c|d^2|c^3|d^4| \ldots$, with successive quotients in $T \cap K = R$. Since $R$ satisfies (*), $(ba^{-1})^m = dc^{-1} \in U(R)$. Thus $ba^{-1}$, $ab^{-1}$ are integral over $R$ and so lie in $T$; i.e., $a$ and $b$ are associates in $T$, completing the proof.

The section closes with an observation which will be of help in Sections 4 and 5. For background material on the $D + M$ construction, see [**4**, Appendix 2].

*Remark* 2.8. Let $V$ be a valuation ring of the form $K + M$, where $K$ is a field and $M(\neq 0)$ is the maximal ideal of $V$. If $D$ is a subring of $K$ such that $D + M$ satisfies (*), then $D = K$.

*Proof.* It suffices to show that any nonzero $\alpha \in K$ lies in $D$. Select $0 \neq m \in M$. Then $m|(\alpha m)^2|m^3|(\alpha m)^4| \ldots$, with successive quotients in $Km \subset M \subset D + M$. By (*), $\alpha = (\alpha m)m^{-1} \in U(D + M)$, so that $\alpha \in U(D + M) \cap K = U(D) \subset D$, as required.

Since 1-dimensional domains are Archimedean, the analogue of Remark 2.8 for the Archimedean property is invalid.

**3. The GCD case.** As we observed in the introduction, any *UFD* satisfies (*). Since $R$ is a *UFD* if and only if $R$ is both a Krull domain and a *GCD* domain, it is natural to ask whether any *GCD* domain satisfies (*). (Recall from Remark 2.2 that any Krull domain satisfies (*).) Since all valuation rings are *GCD* domains, a negative answer to this question is provided by any valuation ring of dimension $\geqq 2$. This fact follows from the earlier observation that a valuation ring is cic if and only if its dimension is $\leqq 1$, together with the next theorem, which characterizes the *GCD* domains which satisfy (*).

THEOREM 3.1. *For a GCD domain R, the following are equivalent:*
(a) *R is* cic*;*
(b) $R[[X]]$ *is integrally closed;*
(c) *R is Archimedean;*
(d) *R satisfies* (*).

*Proof.* By Theorem 2.1, (a) $\Rightarrow$ (d) $\Rightarrow$ (c); as in [**11**, Theorem 0.2], (a) $\Rightarrow$ (b) $\Rightarrow$ (c). Finally, to show (c) $\Rightarrow$ (a), assume (c), and suppose $rx^n \in R$ for each $n \geqq 1$, with $r \in R^*$. Write $x = ab^{-1}$ for relatively prime elements $a, b \in R$. For each $n$, $b^n$ is relatively prime to $a^n$, by [**6**, Theorem 49(c)]; as $b^n | ra^n$, [**6**, Exercise 7, p. 41] implies $b^n | r$. Hence, $r \in \bigcap Rb^n$ and, by (c), $b \in U(R)$, so that $x \in R$, as required.

Another *GCD* domain (this time, nonvaluation) which fails to satisfy (*) is given by $S = \mathbf{Z} + X\mathbf{Q}[X]$, the subring of $\mathbf{Q}[X]$ consisting of the polynomials with integral constant term. (It is straightforward to verify that $S$ is a *GCD* domain; to show that it does not satisfy (*), test any of the conditions in Theorem 3.1.) Note, moreover, that $S$ is a Bézout domain. For the special case of Bézout domains, Theorem 3.1 is a consequence of [**5**, Proposition 3] (together with Theorem 2.1 and [**11**, Theorem 0.2]), since any Bézout domain is *QR*, in the sense that each of its overrings is a localization. Indeed, Bézout domains may be characterized as *GCD* domains which are *QR*.

For a nonBézout illustration of Theorem 3.1, let $X$, $Y$ be algebraically independent indeterminates over a Bézout domain $B$. Then $T = B[X, Y]$ is a *GCD* nonBézout domain; moreover, by Proposition 2.4, $T$ satisfies (*) if and only if $B$ satisfies (*).

*Remark* 3.2. Instead of appealing to either the cic property or Theorem 3.1, we can easily give a direct proof that a valuation ring $V$ satisfies (*) if and only if $\dim(V) \leqq 1$. First, one checks easily that $V$ is Archimedean if and only if the value group $G$ of the associated valuation $v$ is Archimedean *qua* totally ordered group, i.e. if and only if $\dim(V) = \operatorname{rank}(G) \leqq 1$. On the other hand, $V$ is Archimedean if and only if $V$ satisfies (*), in view of Theorem 2.1 and the observation that, for $a, b \in V^*$, either $a|b$ or $b|a$.

It is of some interest to use $G$ in order to show directly, in case $\dim(V) \leqq 1$, that $a|b^2|a^3|b^4|\ldots$ implies $Va = Vb$. If $a = bc$ with $c \in V$, it follows from $a^{2n-1}|b^{2n}$ that $c^{2n-1}|b$, and so $v(b) \geqq (2n-1)v(c)$ for each $n \geqq 1$; as $G$ is Archimedean, $v(c) = 0$, so that $c \in U(V)$ and $a$ is associated to $b$. The case $a|b$ is handled similarly.

The above emphasis on the role of the value group is in the spirit of Krull [**7**, Satz 8, p. 170], who showed that a valuation ring is cic if and only if its associated value group is Archimedean.

It is convenient next to present an example which will be revisited in Example 5.1.

*Example* 3.3. Let $A$ be *the* ring of all algebraic integers, i.e. the integral closure of $\mathbf{Z}$ in some algebraic closure of $\mathbf{Q}$. Then dim $(A) = 1$, $A$ is a Bézout domain, $A$ satisfies (*) and $A$ is not a *UFD*. Indeed, the first assertion follows by integrality; the second is well known (cf. [**6**, Theorem 102]); the third then follows from Remark 2.2; and the final assertion follows, e.g., by observing the essentially different factorizations $2.3 = (1 + (-5)^{1/2})(1 - (-5)^{1/2})$.

*Remark* 3.4. Let $R$ be *a* ring of algebraic integers, i.e. the integral closure of $\mathbf{Z}$ in some finite (algebraic) field extension of $\mathbf{Q}$. Since $R$ is a Dedekind domain, Remark 2.2 implies that $R$ satisfies (*). Another proof may be obtained by applying Proposition 2.6(i) and Example 3.3.

The section closes with an attempt to get more mileage from the standard proof [**4**, Theorem 14.5] of the characterization of cic valuation rings. We assume the notion of a goingdown ring (cf. [**3**, Theorem 1]), familiar examples of which are valuation rings and 1-dimensional domains. By a result of McAdam [**9**, Corollary 11], a quasilocal integrally closed domain is a going-down ring if and only if each prime ideal is comparable to each principal ideal.

PROPOSITION 3.5. *Let $R$ be quasilocal and integrally closed. Then* dim $(R) \leqq 1$ *if and only if $R$ is an Archimedean going-down ring.*

*Proof*. The "only if" part is immediate. If the converse fails, select distinct nonzero prime ideals $P \subset M$ of $R$. Let $a \in M \backslash P$ and $0 \neq b \in P$. By McAdam's result, $P = Pa$, and so $b \in \bigcap Pa^n$, contradicting the assumption that $R$ is Archimedean.

**4. Localization.** We next consider the behavior of (*) under localization, and begin with the promised counterexample to the converse of Proposition 2.3.

*Remark* 4.1. If $R$ satisfies (*) and $M$ is a maximal ideal of $R$, then $R_M$ need not satisfy (*). For an example, let $R$ be constructed as in Example II of Sheldon [**15**]; then $R$ is a cic Bézout domain and dim $(R) = 2$. Since $R$ is cic, it does indeed satisfy (*). However, if $M$ is a height 2 prime of $R$, $R_M$ is a 2-dimensional valuation ring and, by Remark 3.2, cannot satisfy (*).

The following lemma paves the way for a positive statement about (*) and localization. As usual, we shall call a nonzero element in a domain *prime* in case it generates a prime ideal.

LEMMA 4.2. *Let $R$ satisfy* accp *and let $S$ be a* (*necessarily saturated*) *multiplicative subset of $R$ generated by $U(R)$ and a set of prime elements of $R$. Then:*

(1) *Each $a \in R^*$ may be expressed as $a = a_0 s_0$, where $s_0 \in S$ and $a_0 \in R$ is relatively prime to $S$* (*in the sense that $S$ contains no nonunit factor of $a_0$*).

(2) *If* $a_0 s_0 = a_1 s_1$ *with* $s_0$, $s_1 \in S$ *and* $0 \neq a_0$, $a_1$ *relatively prime to* $S$, *then* $Ra_0 = Ra_1$ (*and so* $Rs_0 = Rs_1$).

(3) *If* $a_0 \in R^*$ *is relatively prime to* $S$, *then* $R_S a_0 \cap R = Ra_0$.

(4) $S$ *satisfies* accp, *in the sense that there is no infinite strictly ascending chain* $\{Ss_n\}$ *with each* $s_n \in S$.

(5) $S$ *satisfies* (*), *in the sense that* $a|b^2|a^3|b^4| \ldots$ *for* $a, b \in S$ *implies* $Ra = Rb$.

*Proof.* (1). Deny. The accp provides $Rr$ maximal such that $r \in R^*$ is not expressible as desired. Consequently, $r$ is not relatively prime to $S$, and so $r = as$, with $a \in R$ and $s \in S \backslash U(R)$. Then $a$ is not expressible as desired, contradicting the maximality of $Rr$.

(2). Ape the standard proof for uniqueness of factorization in a principal ideal domain.

(3). It suffices to show that, if $r_1 s^{-1} a_0 = r \in R$ with $r_1$, $r \in R$ and $s \in S$, then $r \in Ra_0$. By (1), write $r_1 = b_0 s_0$, with $s_0 \in S$ and $b_0$ relatively prime to $S$; similarly, write $r = b_1 s_1$ with appropriate factors. As $a_0 b_0$ is relatively prime to $S$, (2) gives $u \in U(R)$ such that $b_1 = b_0 a_0 u$, so that $r = b_0 u s_1 a_0$, as required.

(4). It suffices to observe that, if $Ss_1 \subsetneqq Ss_2$ for $s_1$, $s_2 \in S$ and if $s_1$ is expressible (apart from unit factor) as a product of $n$ generating primes (counting multiplicities), then $s_2$ is associated to a product of fewer than $n$ generating primes.

(5). If $a|b^2|a^3|b^4| \ldots$ with $a$, $b$ nonunits of $S$, then $(ab)(ba^{-1})^n \in S$ for each $n \geqq 1$. Write $ba^{-1} = dc^{-1}$ where $d$, $c \in S$ have no prime factor in common. Aping the proof that (c) $\Rightarrow$ (a) in Theorem 3.1, we find $ab \in \bigcap Sc^n$. As $R$ is Archimedean, $c \in U(R)$, so that $a|b$. Finally, apply the implication (v) $\Rightarrow$ (iv) in Theorem 2.1 to show that $a$ and $b$ are associates.

THEOREM 4.3. *Let* $R$ *satisfy* accp *and let* $S$ *be a multiplicative subset of* $R$ *generated by a set of prime elements. Then* $R$ *satisfies* (*) *if and only if* $R_S$ *satisfies* (*).

*Proof.* It is harmless to augment $S$ by $U(R)$, so that Lemma 4.2 is applicable.

Assume $R$ satisfies (*). Consider nonzero elements $a$, $b$ of $R_S$ such that $a|b^2|a^3|b^4| \ldots$, with successive quotients in $R_S$. As our aim is to prove $R_S a = R_S b$, we may take $a$, $b \in R$; moreover, by part (1) of Lemma 4.2, $a$ and $b$ may be supposed relatively prime to $S$. Then (3) implies that $Ra \supset Rb^2 \supset Ra^3 \supset Rb^4 \supset \ldots$. As $R$ satisfies (*), $Ra = Rb$ and, *a fortiori*, $R_S a = R_S b$.

Conversely, assume that $R_S$ satisfies (*), and consider $a$, $b \in R^*$ such that $a|b^2|a^3|b^4| \ldots$. By (1), write $a = a_0 s_0$, $b = b_1 s_1$ with $s_0$, $s_1 \in S$ and $a_0$, $b_1$ relatively prime to $S$. An immediate consequence of (1)–(2) is that $a_0|(b_1)^2|(a_0)^3|(b_1)^4| \ldots$ and $s_0|(s_1)^2|(s_0)^3|(s_1)^4| \ldots$. By (5) and the hypothesis about $R_S$, we have $Rs_0 = Rs_1$ and $a_0(b_1)^{-1} \in U(R_S)$, respectively. The latter assertion yields $R_S a_0 = R_S b_1$, so that $Ra_0 = Rb_1$, by (3). Hence $Ra = Rb$, to complete the proof.

*Example* 4.4. The assumption that $R$ satisfies accp may not be deleted from Theorem 4.3. To illustrate this, consider the ring of restrained power series $R = \mathbf{Z}_{2\mathbf{Z}} + X\mathbf{Q}[[X]]$ and its multiplicative subset $S$ consisting of powers of 2. Now $R_S = \mathbf{Q}[[X]]$ satisfies (*) because it is a discrete (rank 1) valuation ring; however, Remark 2.8 implies that $R$ does not satisfy (*). (Of course, $R$ does not satisfy accp either: consider $\{RX2^{-n}\}$.)

**5. Integrally closed examples.** The examples presented below will settle the questions left open by Theorem 2.1, and will relate (*) to the properties studied by Ohm [**11**]. In particular, Examples 5.2 and 5.3 serve to eliminate generalizations of Theorem 3.1 which might, at first blush, seem possible.

*Example* 5.1. We present here two examples of 1-dimensional cic domains which do not satisfy accp. The first is the ring $A$ of all algebraic integers. This was shown to be 1-dimensional and cic in Example 3.3; to see that $A$ fails to satisfy accp, it suffices to consider the ascending chain $\{A2^{1/2^n}\}$.

The second example is a valuation ring $V$, whose corresponding valuation $v$ has (rank 1) value group $\mathbf{Q}$. (For instance, construct $V$ as in [**2**, Exemple 6, p. 107].) Select elements $a_n \in V$ such that $v(a_n) = 2^{-n+1}$ for each $n \geqq 1$. As $v(a_n(a_{n+1})^{-1}) = 2^{-n}$, each $a_n(a_{n+1})^{-1}$ is a nonunit of $V$, so that $\{Va_n\}$ is a strictly ascending chain, and $V$ fails to satisfy accp.

*Example* 5.2. This example is of an integrally closed domain which satisfies (*) and accp, but is not cic. Let $X$, $Y$ be algebraically independent indeterminates over a field $K$. We claim that $R = K \oplus K[X]Y \oplus K[X]Y^2 \oplus \ldots$, viewed as a subring of $K[X, Y]$, has the desired properties.

Indeed, since $K[X, Y]$ is a Noetherian *UFD*, the assertions in Proposition 2.6(ii) show that $R$ satisfies accp and (*). (The former was observed by Evans [**6**, Exercise 8, p. 114] in another connection.) Moreover, $R$ is not cic, since $YX^n \in R$ for each $n \geqq 1$ while $X \notin R$. Finally, to show that $R$ is integrally closed, observe first that $K[X, Y] = K[X] + R$ is integrally closed. Accordingly, it suffices to show that, if $\alpha \in K[X]$ is integral over $R$, then $\alpha \in K$. By focusing on the coefficient of $Y^0$ in an integrality equation of $\alpha$ over $R$, we see that $\alpha$ is integral over $K$, thus giving the desired result, since $K$ is algebraically closed in $K[X]$. (Indeed, appeals to [**11**, Theorem 3.2] and [**6**, Theorem 188] now yield more: the power series ring $R[[T]]$ is integrally closed.)

Note also that $R$ does not satisfy property ($e$) of [**11**], since [**6**, Exercise 8, p. 114] shows that $Y$ is a nonunit of $R$ contained in no height 1 prime ideal.

*Example* 5.3. The final example in this section is of a 1-dimensional domain $R$ which satisfies accp, does not satisfy (*), and is such that $R[[X]]$ is integrally closed. (By [**2**, Exercice 27, p. 76] or [**11**, Corollary 1.4], such $R$ must be Archimedean and, of course, integrally closed.)

The construction is due to Krull [**8**, pp. 670–671]. Let $r$, $s$ be indeterminates over a field $k$, and define a discrete (rank 1) valuation $v$ on $k(r, s)$ by setting

$v(r) = 1$ and $v(s) = 0$. Its valuation ring is $V = k(s) + M$, with maximal ideal $M$. We claim that $R = k + M$ has the desired properties. (Note that Krull observed $R$ is not cic.)

Indeed, [**4**, Theorem $A(c)$, $(d)$, $(f)$, p. 560] implies that dim $(R) = 1$ and $U(R) = R \backslash M$. To show that $R$ satisfies accp, deny, and select a strictly increasing chain $\{Ra_n\}$ of principal ideals of $R$. Since $V$ is Noetherian, $Va_n = Va_{n+1}$ for some $n$; then $a_n = ba_{n+1}$, with $b \in [R \cap U(V)] \backslash U(R) = \emptyset$, the desired contradiction. Hence, $R$ satisfies accp. Finally, Remark 2.8 shows that $R$ does not satisfy (*), and an argument of Ohm [**11**, p. 329] shows that $R[[X]]$ is integrally closed.

*Remark* 5.4. It is now straightforward to use the preceding examples in order to show that, for each of the possible implications amongst (i)–(v) which Theorem 2.1 did not resolve, there is an integrally closed counterexample. With the aid of Theorem 2.1, the above examples, and Ohm's summary of results [**11**, p. 322], it is also easy to settle all but two of the possible implications involving Ohm's properties $(a)$, $(b)$, $(c)$, $(d)$, $(e)$ (described in [**11**, p. 321]), together with $(f)$ $R$ is integrally closed and satisfies (*), and $(g)$ $R$ is integrally closed and satisfies accp. (Recall that $(b)$, $(c)$, $(d)$ are, respectively, the properties of $R$ that $R$ be cic, that $R[[X]]$ be integrally closed, and that $R$ be integrally closed Archimedean.) The two open questions concern $(f) \Rightarrow (c)$ and $(g) \Rightarrow (c)$; we suspect that both of these implications fail.

**6. Noetherian examples.** Recall that a Noetherian integrally closed domain, being completely integrally closed, satisfies (*). The next result and remark show that any attempt to classify the Noetherian domains satisfying (*) must contend with number-theoretic problems.

THEOREM 6.1. *Let $n$ be a positive integer. Then there exist Noetherian domains $R_n$ and $T_n$ such that* dim $(R_n) = n =$ dim $(T_n)$, *neither $R_n$ nor $T_n$ is integrally closed, $R_n$ satisfies* (*), *and $T_n$ does not satisfy* (*).

*Proof.* Recall (cf. [**2**, Corollaire 1, p. 19; **5**, Corollary 2]) that a domain $D$ is integrally closed if and only if the polynomial ring $D[X]$ is integrally closed. In view of the Hilbert basis theorem and Krull's result on the dimension of a polynomial ring over a Noetherian ring (cf. [**4**, Theorem 25.5]), Proposition 2.4 shows that it is enough to handle the case $n = 1$.

Let $d$ be a negative squarefree integer other than $-3$, such that $d \equiv 1 \pmod 4$; set $R = \mathbf{Z}[d^{1/2}]$. Note that dim $(R) = 1$ (by integrality) and $R$ is Noetherian (by Hilbert basis theorem); moreover, $R$ is not integrally closed, since the (unique) maximal order of $\mathbf{Q}(d^{1/2})$ is $S = \mathbf{Z}[(1 + d^{1/2})/2]$. Since $d \neq -3$, $U(S) = \{1, -1\}$ (cf. [**13**, Proposition 1, p. 76]); thus, $U(R) = U(S)$. As $S$ is Noetherian and integrally closed, $S$ satisfies (*), and so the second part of Proposition 2.6(ii) shows that $R$ also satisfies (*). Accordingly, $R$ is a suitable choice for $R_1$.

We claim that $T = \mathbf{Z}[(-3)^{1/2}]$ is a suitable $T_1$. Indeed, as above, dim $(T) = 1$, $T$ is Noetherian and $T$ is not integrally closed. To show that $T$ does not satisfy (*), let $a = 1 + (-3)^{1/2}$ and $b = 1 - (-3)^{1/2}$. After observing that $a^2 = -2b, b^2 = -2a$ and $a^3 = b^3 = -8$, one checks readily that $a|b^2|a^3|b^4| \ldots$, the successive quotients repeating in blocks of six. However, $a$ and $b$ are not associated in $T$, since $ab^{-1} = -b/2 \notin T$. This completes the proof.

*Remark* 6.2. (a) The rings $R_1$ and $T_1$ in Theorem 6.1 may also be constructed using real quadratic algebraic number fields. For instance, $\mathbf{Z}[(17)^{1/2}]$ is a satisfactory $R_1$; the verification proceeds as before, since an analysis of the Pell-Fermat equation shows that the fundamental unit of $\mathbf{Q}[(17)^{1/2}]$ is $4 + (17)^{1/2}$ (cf. [**13**, p. 77]), forcing $\mathbf{Z}[(17)^{1/2}]$ to contain every unit of the maximal order. To find a real $T_1$, use $\mathbf{Z}[d^{1/2}]$ with $d = 5$ (respectively, 13), and observe that the fundamental unit $u$ of $\mathbf{Q}(d^{1/2})$ is $(1 + 5^{1/2})/2$ (respectively, $(3 + (13)^{1/2})/2$); set $a = 1 - 5^{1/2}$ (respectively, $3 - (13)^{1/2}$) and $b = au = -2$. Further examples may be similarly constructed.

(b) The rings $T_n$ of Theorem 6.1 and (a) above are examples of Archimedean domains which do not satisfy (*) and are not integrally closed. A related integrally closed example was given in Example 5.3.

(c) The use of nonmaximal orders in algebraic number fields in constructing $T_1$ above, shows that (Proposition 2.6 (i) notwithstanding) a domain need not inherit (*) from its integral closure.

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*University of Rhode Island,*
*Kingston, Rhode Island;*
*Rutgers University,*
*New Brunswick, New Jersey*