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UNCOUNTABLE DISCRETE SETS IN EXTENSIONS AND METRIZABILITY

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ABSTRACT. If X is a topological space then $\exp X$ denotes the space of non-empty closed subsets of X with the Vietoris topology and λX denotes the superextension of X. Using Martin's axiom together with the negation of the continuum hypothesis the following is proved: If every discrete subset of $\exp X$ is countable the X is compact and metrizable. As a corollary, if λX contains no uncountable discrete subsets then X is compact and metrizable. A similar argument establishes the metrizability of any compact space X whose square $X \times X$ contains no uncountable discrete subsets.

1. **Introduction and preliminaries.** Our set-theoretic and topological terminology and notation are standard. For background material on set-theoretic topology the reader is referred to (4). All topological spaces discussed in this paper are assumed to be regular and Hausdorff and infinite.

If X is a space then $\exp X$ denotes the set of all non-empty closed subsets of X. If $G_1, G_2, \ldots, G_n$ are subsets of X, we define

$$\langle G_1, G_2, \ldots, G_n\rangle = \left\{F \in \exp X : F \subseteq \bigcup_{i=1}^{n} G_i \text{ and } F \cap G_i \neq \phi \text{ for } i = 1, 2, \ldots, n\right\}.$$

In particular, if $G \subseteq X$ then $\langle G\rangle = \{F \in \exp X : F \subseteq G\}$. The Vietoris topology on $\exp X$ is that topology having the sets of the form $\langle G_1, G_2, \ldots, G_n\rangle$ where $G_1, G_2, \ldots, G_n$ are open in X, as a basis. Endowed with this topology $\exp X$ is referred to as the *space of closed subsets* of X. A comprehensive discussion of the topological properties of $\exp X$ may be found in (7).

A family of sets $\mathcal{S}$ is said to be *linked* if $A \cap B \neq \phi$ for all $A, B \in \mathcal{S}$. For a topological space X, λX denotes the set of all maximal linked families of closed subsets of X. If G is an open subset of X, we let $G^+ = \{\alpha \in \lambda X :$ there exists $F \in \alpha$ such that $F \subseteq G\}$. λX is endowed with a topology by taking all sets of the form G^+, for G open in X, as a sub-base. With this topology λX is referred to as the *superextension* of X. Our basic reference for superextensions is (13).

In this paper we make use of the set theoretical statement known as Martin's

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axiom, denoted by MA. For an introduction to Martin's axiom and its applications in topology the reader is referred to (9). We will not need a precise statement of Martin's axiom in this paper, since we will only be using a theorem from (12) and one from (4) which are proved using Martin's axiom together with the negation of the continuum hypothesis, which we abbreviate by MA+ ~CH. For a discussion of the consistency of MA+ ~CH with the usual axioms of set theory the reader is referred to (11).

For the reader's convenience we now recall several topological notions which will be used in the sequel.

A space X is *Lindelof* if every open cover of X has a countable subcover. X is *hereditarily Lindelof* if every subspace of X is Lindelof.

X is *separable* if X has a countable dense subset. X is *hereditarily separable* if every subspace of X is separable.

As usual, a space S is called *discrete* if every point of S is open in S. Thus if S is a subspace of a space X, then S is discrete if for every point $s \in S$ there exists an open set G_s in X such that $G_s \cap S = \{s\}$.

We recall that a subset S of a space X is a G_δ*-set* in X if S is equal to the intersection of countably many open subsets of X. X is said to have a G_δ-*diagonal* if the diagonal $\Delta_X = \{(x, x) : x \in X\}$ is a G_δ- set in the square $X \times X$.

Finally we recall that a space X is said to be *first countable* if every point of X has a countable neighborhood base in X.

We will use the fact that, in a regular hereditarily Lindelof space, every closed set is a G_δ-set. Since a compact space, in which all points are G_δ-sets, is first countable, it follows that every hereditarily Lindelof compact space is first countable.

In this paper, our main interest is in the spaces $\exp X$, λX, and $X \times X$, with regard to the existence of uncountable discrete subsets. What does it mean for these extensions to contain no uncountable discrete subset? In the next section, using MA+ ~CH, we show that if all discrete subsets of $\exp X$ are countable, then X must be compact and metrizable. This answers a question posed by V. I. Malyhin in (6). As a corollary it follows that , if all discrete subsets of λX are countable then X must be compact and metrizable. A similar argument establishes the metrizability of any compact space X whose square $X \times X$ contains no uncountable discrete subset.

2. **Uncountable discrete subsets in exp** X, λX, **and** $X \times X$. In (6) it is shown that if $\exp X$ contains no uncountable discrete subsets then X must be compact, hereditarily Lindelof, and hereditarily separable. For the reader's convenience we sketch that argument here: Firstly one observes that the space of closed subsets of the discrete space N of natural numbers contains an uncountable discrete subset. In fact, if $\mathcal{A}$ is any antichain of subsets of N, then $\mathcal{A}$ is a discrete subset of $\exp N$, for clearly $\langle S \rangle \cap \mathcal{A} = \{S\}$ for all S in $\mathcal{A}$. Since

there exist uncountable antichains of subsets of N the statement follows. Therefore if $\exp X$ has no uncountable discrete subsets then X must be countably compact. For if X is not countably compact then X contains a closed subset homeomorphic to N, and therefore $\exp X$ would contain a copy of $\exp N$ and so would contain an uncountable discrete set. (We have used the fact that if F is a closed subset of a space X then the inclusion mapping is an embedding of $\exp F$ into $\exp X$.) Next one shows that, if $\exp X$ has no uncountable discrete subset, then X must be hereditarily Lindelof. If not, then X contains a sequence $\{G_\alpha : \alpha < \omega_1\}$ of non-empty open sets such that $\alpha < \beta \rightarrow G_\alpha \subseteq G_\beta$. For each α, choose $x_{\alpha+1} \in G_{\alpha+1} - G_\alpha$. Then the set $\mathcal{D} = \{X - G_{\alpha+1} : \alpha < \omega_1\}$ is an uncountable discrete subset of $\exp X$. In fact, $\langle X - \{x_{\alpha+1}\}, G_{\alpha+2} - \{x_{\alpha+1}\}\rangle$ is an open set in $\exp X$ whose intersection with $\mathcal{D}$ is precisely $\{X - G_{\alpha+1}\}$. Thus X must be hereditarily Lindelof. Therefore X is compact, being countably compact and Lindelof. Finally, X must also be hereditarily separable. Otherwise X would contain a sequence of points $\{x_\alpha : \alpha < \omega_1\}$ such that, for all $\alpha < \omega_1$, $x_\alpha \notin \mathrm{cl}\{x_\xi : \xi < \alpha\}$. (Here cl S denotes the closure of S in X.) Let $F_\alpha = \mathrm{cl}\{x_\xi : \xi \leq \alpha + 1\}$. Then $\mathcal{D} = \{F_\alpha : \alpha < \omega_1\}$ is an uncountable discrete subset of $\exp X$, since $\langle X - \{x_{\alpha+2}\}, X - \mathrm{cl}\{x_\xi : \xi \leq \alpha\}\rangle$ is an open set in $\exp X$ whose intersection with $\mathcal{D}$ is precisely $\{F_\alpha\}$.

Thus if $\exp X$ has no uncountable discrete subsets then X must be compact, hereditarily Lindelof and hereditarily separable. In (6) it is asked whether X must in fact be a compact metric space. In this section we will answer this question in the affirmative with the help of MA + ~CH. (Note that the converse statement, that $\exp X$ contains no uncountable discrete subset when X is compact metric, is trivial; for, if X is a compact metric space, then $\exp X$ is also a compact metrizable space and so has a countable basis; in fact, if X is a compact metric space with metric d, then the Vietoris topology on $\exp X$ is the same as that determined by the Hausdorff metric on $\exp X$ induced by d).

2.1 THEOREM (MA + ~CH). *Let X be a space and suppose exp X contains no uncountable discrete subsets. Then X is compact and metrizable.*

Proof. As pointed out at the beginning of this section, the assumption on $\exp X$ implies that X is compact, hereditarily Lindelof and hereditarily separable. These three properties imply that $\exp X$ is first countable. (see (3) and (7) for a discussion of the first countability and other cardinal invariants of $\exp X$) Since $\exp X$ is compact whenever X is, it follows that $\exp X$ is a first countable compact space containing no uncountable discrete subsets. But, by Theorem 5.6 of (4), MA + ~CH implies that any first countable compact space which contains no uncountable discrete subsets must be hereditarily separable. Therefore $\exp X$ is hereditarily separable. By Theorem 3 of (12), MA + ~CH implies that any hereditarily separable compact space is hereditarily Lindelof. Thus $\exp X$ is hereditarily Lindelof, and so every closed subset of $\exp X$ is a G_δ-set in

$\exp X$. In particular, the set $F_1 = \{\{x\} : x \in X\}$ is a G_δ-set in $\exp X$. Now, the function $f : X \times X \to \exp X$ defined by $f(x, y) = \{x, y\}$ is continuous, and so $f^{-1}(F_1)$ is a G_δ-set in $X \times X$. But $f^{-1}(F_1)$ is exactly the diagonal Δ_X. Therefore X is a compact space having a G_δ-diagonal. Any such compact space is metrizable, by the well-known result of (10).

The corresponding result for superextensions is a consequence of the following result, whose proof does not require MA + ~CH.

2.2 Lemma. *Let X be a topological space. If* $\exp X$ *contains an uncountable discrete subset then λX contains an uncountable discrete subset.*

Proof. Of course this would follow immediately if we knew that $\exp X$ were embedded in λX. This does not seem to be known, however. It is known that for any *proper* closed subset F of X, $\exp F$ is embedded in λX (see the proposition on page 94 of (13).) Thus it is sufficient to show that, if $\exp X$ contains an uncountable discrete subset, then so does $\exp F$ for some proper closed subset F of X. To this end, let Σ be a discrete subset of $\exp X$ with $|\Sigma| = \omega_1$. For the sake of contradiction, assume that, for every proper closed subset F of X, $\exp F$ does not contain any uncountable discrete subset. Then, by the remarks at the beginning of this section, every proper closed subset F is compact, hereditarily Lindelof, and hereditarily separable. Since clearly X can be expressed as the union of two proper closed sets, it follows that X is separable. Let D be a countable dense subset of X. Now every point of X has a proper closed neighborhood F. Since F is compact and hereditarily Lindelof, and therefore first countable, it follows that X is first countable. We now show that there exists a point $d \in D$ such that $\{F \in \Sigma : d \notin F\}$ is uncountable. For D is dense, so each of the sets $X - F$, for F in Σ, contains some element d of D. Since $|\Sigma| = \omega_1$ and D is countable, the pigeon-hole principle implies the existence of such an element d, say d_0. Let $\Sigma_0 = \{F \in \Sigma : d_0 \notin F\}$. Thus $|\Sigma_0| = \omega_1$. Let $\{G_n : n \in \omega\}$ be a countable neighborhood base for d_0 in X. For each F in Σ_0, $X - F$ is an open set containing d_0, and so there exists an $n \in \omega$ such that $G_n \subseteq X - F$. Since $|\Sigma_0| = \omega_1$, it follows that there exists an integer k such that $\{F \in \Sigma_0 : F \subseteq X - G_k\}$ is uncountable. Let $\Sigma_1 = \{F \in \Sigma_0 : F \subseteq X - G_k\}$, and let $R = X - G_k$. Then R is a proper closed subset of X and $F \in \Sigma_1 \to F \subseteq R$. That is, $\Sigma_1 \subseteq \exp R$. Since $\Sigma_1 \subseteq \Sigma$, it follows that $\exp R$ contains an uncountable discrete subset, which is a contradiction.

2.3 Corollary (MA + ~CH). *Let X be a space whose superextension λX contains no uncountable discrete subsets. Then X is compact and metrizable.*

Finally we look at the square $X \times X$.

2.4 Theorem (MA + ~CH). *Let X be a compact space whose square $X \times X$ contains no uncountable discrete subsets. Then X is metrizable.*

Proof. By a theorem due to Zenor in (14), if $X \times Y$ contains no uncountable discrete subsets then either X is hereditarily Lindelof, or Y is hereditarily separable. In the present case this implies that either X is hereditarily Lindelof or X is hereditarily separable. Now, by Theorem 3 of (12), $MA + \sim$CH implies that any compact, hereditarily separable space is hereditarily Lindelof. Combining this with Zenor's theorem we conclude that X is hereditarily Lindelof. Therefore X is first countable, and so $X \times X$ is first countable. Therefore $X \times X$ is a first countable compact space containing no uncountable discrete subsets. By Theorem 5.6 of (4), cited above, this implies that $X \times X$ is hereditarily separable. Applying Theorem 3 of (12) once again, we conclude that $X \times X$ must be hereditarily Lindelof. Therefore every closed subset of $X \times X$ is a G_δ-set in $X \times X$. In particular X has a G_δ-diagonal and so X is metrizable.

Remarks. Using the continuum hypothesis CH, examples have been constructed of non-metrizable compact spaces whose squares contain no uncountable discrete subsets; the reader is referred to (5) and (8). Thus the existence of a non-metrizable compact space whose square contains no uncountable discrete subsets is independent of the usual axioms of set theory. The authors have been unable to establish the independence of Theorem 2.1.

We note that, although the compactness of X is part of the conclusions of 2.1 and 2.3, it is part of the hypothesis of 2.4. To see that the assumption of compactness in 2.4 is not entirely specious, we observe that the square of any countable space contains no uncountable discrete subset.

Finally, it is worth noting that the metrizability of a compact space X does not follow simply from the assumption that X itself contains no uncountable discrete subsets. The well-known "double-arrow" space of Alexandroff and Urysohn (1), (see also 9C of (2)—this space is the same as the top and bottom of the lexicographically ordered unit square), serves as a counterexample.

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