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Sequential Order under PFA

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Abstract. It is shown that it follows from PFA that there is no compact scattered space of height greater than ω in which the sequential order and the scattering heights coincide.

1 Introduction

In 1974, Bashkirov [2] proved that it follows from CH that there are compact spaces of any sequential order up to and including ω_1. We are interested in the question of weakening the CH assumption to either of Martin's Axiom or even ZFC. It is also interesting to ask for a compact scattered space in which the scattering levels of points coincides with their sequential order with respect to the first level. This is the natural way to construct such spaces and is the method employed by Bashkirov.

The reader is referred to the excellent surveys [5] and [8] for general background information. The survey by Shakhmatov [5] will provide detailed information about the relations among convergence properties such as sequentiality and the Frechet-Urysohn property. Vaughan's article [8] is a good source for information about the combinatorial consequences of Martin's Axiom (and PFA) such as the values (and meanings) of the cardinals such as $\mathfrak{p}$ and $\mathfrak{b}$.

The sequential order of a point p with respect to a set A (see [5, 3.9]) is the minimum ordinal α such that there is a sequence of points with sequential order less than α that converges to p (the points of A have sequential order 0 with respect to A and $A^{(\alpha)}$ denotes the set of points that have sequential order with respect to A at most α). A space X is *scattered* [4] if every subset has a relatively isolated point, and this gives rise to the Cantor–Bendixson (scattering) levels of a scattered space. Namely, the set X_0 of isolated points are the level 0 of the space, and for each ordinal α, the points at level α, denoted X_α, are the points that are isolated in the subspace remaining when all points of level less than α are removed. The height of a scattered space is the minimum ordinal α for which $X_{\alpha+1}$ is empty. If we just refer to the sequential order of a point in a scattered space, we will understand it to mean with respect to the set of points at level 0. It is easily seen that every compact scattered space with countable scattering height is sequential. We define a space X to be *CB-sequential* if it is sequential, compact, scattered, and points in X_α have sequential order α with respect to X_0. We introduce the term CB-sequential because of the required connection between the Cantor–Bendixson levels and the sequential order.

There is a compelling motivation for determining the maximum possible sequential order in the presence of the Proper Forcing Axiom (PFA) that comes from the

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Moore–Mrowka problem (see [1]). First of all, it is quite remarkable that the best known lower bounds are 2 in ZFC and 4 under PFA [3]. Secondly, Balogh proved that each compact space of countable tightness is sequential if PFA is assumed (PFA is known to imply Martin's Axiom and $\mathfrak{c} = \omega_2$). If there is some finite bound on the sequential order of compact sequential spaces in models of PFA, it would mean that compact spaces of countable tightness are literally only a few steps away from being Fréchet–Urysohn.

To illustrate the two concepts above, let us recall the natural topological space associated with an almost disjoint family $\mathcal{A}$ of subsets of the integers, $\mathbb{N}$. On the point set $\mathbb{N} \cup \mathcal{A}$, let $\mathbb{N}$ be open and discrete, and a typical neighborhood of $a \in \mathcal{A}$ is any set of the form $\{a\} \cup a \setminus n$ (we treat the integers as ordinals and recall that an ordinal α is equal to its set of predecessors). This topology on $\mathbb{N} \cup \mathcal{A}$ is locally compact, scattered, and certainly sequential, in fact first-countable. Let X be the one-point compactification and notice that there is a subset of $\mathbb{N}$ converging to the point at infinity if and only if $\mathcal{A}$ is not a maximal almost disjoint family. Thus the point at infinity will be at scattering level 2 and will have sequential order at most 2; it will attain sequential order 2 exactly when the family $\mathcal{A}$ is a maximal almost disjoint family.

2 A Proper Poset

In this section we use PFA [6] and a technique introduced by Todorcevic [7] for constructing proper posets. The technique is known as using elementary submodels as side-conditions. We refer the reader to [6, 7] for any undefined notions from this section. The Proper Forcing Axiom (analogous to $\mathrm{MA}(\omega_1)$) is the statement that, for each family of ω_1 many dense subsets of a proper poset P, there is a P-filter that meets each of these dense sets.

Definition 2.1 A condition p in a poset P is said to be (P, M)-generic if for each dense $A \subset P$ $(A \in M)$ and each $q < p$, there is an $r \in A \cap M$ such that r is compatible with q. A poset P is proper if there is a cardinal λ such that for every regular $\theta \geq \lambda$ and every countable $M \prec H(\theta)$ with $P \in M$, there is, for each $p \in P \cap M$, a $q < p$ that is (P, M)-generic.

Definition 2.2 For families $\mathcal{A}, \mathcal{U} \subset [\omega]^\omega$, we say that $\mathcal{A}$ is $\mathcal{U}$-*large* if for each countable $\mathcal{W} \subset \mathcal{U}$, there is an $a \in \mathcal{A}$ such that $a \cap W$ is infinite for each $W \in \mathcal{W}$. If $a \cap W$ is infinite for each $W \in \mathcal{W}$, we can denote this as $a \in \mathcal{W}^+$. A family of sets $\mathcal{M}$ *separates* a set A if for each pair $\{a, a'\} \subset A$, there is an $M \in \mathcal{M}$ such that $|M \cap \{a, a'\}| = 1$. Say that a family $\mathcal{A} \subset [\omega]^\omega$ is *inseparable* if there do not exist disjoint subsets of ω, each of which contains (mod finite) uncountably many members of $\mathcal{A}$.

We define a poset for the purpose of selecting an uncountable inseparable subcollection of a collection $\mathcal{A}$.

Definition 2.3 Given a family $\mathcal{A} \subset [\omega]^\omega$ and an ultrafilter $\mathcal{U}$ on ω, define the poset $\mathbb{P}(\mathcal{A}, \mathcal{U})$ to be those $p = (\mathcal{A}_p, n_p, \mathcal{M}_p)$ such that $\mathcal{A}_p$ is a finite subset of $\mathcal{A}$, $\mathcal{M}_p$ is a

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finite elementary $\in$-chain of countable elementary submodels of $\langle H((2^{\mathfrak{c}})^+), \in, \mathcal{A}, \mathcal{U}\rangle$, $n_p \in \omega$, $\mathcal{M}_p$ separates $\mathcal{A}_p$, and if $M \in \mathcal{M}_p$ and $a \in A_p \setminus M$, then $a \in (\mathcal{U} \cap M)^+$. We define $p < q$ if usual set extension holds in each coordinate, and for each $M \in \mathcal{M}_q$, $a \in \mathcal{A}_q \setminus M$, and $a' \in M \cap (\mathcal{A}_p \setminus \mathcal{A}_q)$, we have $a \cap a' \not\subset n_q$.

Lemma 2.4 *If G is a $\mathbb{P}(\mathcal{A}, \mathcal{U})$-generic filter, then the family $\mathcal{A}_G = \bigcup_{p\in G} \mathcal{A}_p$ is inseparable.*

Proof If the family $\mathcal{A}_G$ is countable, then it would be vacuously inseparable (although it will follow from later results that it is uncountable). In any event, the family $\mathcal{A}_G$ satisfies what can be called the Hausdorff–Luzin condition. That is, there is a well-ordering $\prec$ on $\mathcal{A}_G$ such that for each $a' \in \mathcal{A}_G$ and $n \in \omega$, there are only finitely many $a \in \mathcal{A}_G$ such that $a \prec a'$ and $a \cap a' \subset n$. The well-ordering is naturally defined by $a \prec a'$ if there is a $p \in G$ and $M \in \mathcal{M}_p$ such that $M \cap \{a, a'\} = \{a\}$. It then follows easily from the definition of extension in $\mathbb{P}(\mathcal{A}, \mathcal{U})$ that this Hausdorff–Luzin condition holds and that the order type of $(\mathcal{A}_G, \prec)$ is (at most) ω_1. Now we give the standard argument that the family $\mathcal{A}_G$ is inseparable. Assume otherwise and suppose that there is a $Y \subset \omega$ and $n \in \omega$ such that both $\mathcal{A}_Y = \{a \in \mathcal{A}_G : a \setminus Y \subset n\}$ and $\mathcal{A}_{\neg Y} = \{a \in \mathcal{A}_G : a \cap Y \subset n\}$ are uncountable. Choose any $a' \in \mathcal{A}_Y$ such that $\{a \in \mathcal{A}_{\neg Y} : a \prec a'\}$ is infinite and observe that this contradicts the Hausdorff–Luzin requirement on a'. ■

For each $m \in \omega$ and family $\mathcal{A}$, let $\mathcal{A}_m = \{a \in \mathcal{A} : m \in a\}$, and for $t \subset \omega$, $\mathcal{A}_t = \{a \in \mathcal{A} : t \subset a\}$.

Lemma 2.5 *If $\mathcal{U}$ is an ultrafilter on ω and $\mathcal{A}^*$ is any $\mathcal{U}$-large family, then $\{m \in \omega : \mathcal{A}^*_m$ is $\mathcal{U}$-large$\} \in \mathcal{U}$.*

Proof Let $U \in \mathcal{U}$ and assume that for each $m \in U$, $\mathcal{A}^*_m$ is not $\mathcal{U}$-large. For each $m \in U$, there is some countable $\mathcal{W}_m \subset \mathcal{U}$ such that no member of $\mathcal{A}^*_m$ hits infinitely each member of $\mathcal{W}_m$. Consider the collection $\mathcal{W} = \{U\} \cup \bigcup_m \mathcal{W}_m$ and, using that $\mathcal{A}^*$ is $\mathcal{U}$-large, select $a \in \mathcal{A}^*$ such that $a \cap W$ is infinite for each $W \in \mathcal{W}$. Fix any $m \in a \cap U$. It follows that $a \cap W$ is infinite for each $W \in \mathcal{W}_m$, contradicting the assumption on $\mathcal{W}_m$. ■

Lemma 2.6 *For each $\delta \in \omega_1$, the set D_δ is a dense subset of $\mathbb{P}(\mathcal{A}, \mathcal{U})$, where*

$$D_\delta = \{p \in \mathbb{P}(\mathcal{A}, \mathcal{U}) : (\exists M \in \mathcal{M}_p) M \cap \omega_1 \not\subset \delta \text{ and } \mathcal{A}_p \setminus M \neq \varnothing\}.$$

Proof Fix any $p \in \mathbb{P}(\mathcal{A}, \mathcal{U})$ and any countable elementary submodel $M \prec H((2^{\mathfrak{c}})^+)$ such that $\delta, \mathcal{A}, \mathcal{U}, p \in M$. Set $\bar{p} = (\mathcal{A}_p, n_p, \mathcal{M}_p \cup \{M\})$. We need to add an a to $\mathcal{A}_p$ with $a \notin M$. If $\mathcal{A}_p$ is empty, then we may select any $a \in \mathcal{A}$ such that $a \in (\mathcal{U} \cap M)^+$ using that $\mathcal{A}$ is $\mathcal{U}$-large. Otherwise, enumerate $\mathcal{A}_p$ in *increasing* order $\{a_0, \dots, a_\ell\}$ (*i.e.*, so that for each $i < \ell$, there is an $M_i \in \mathcal{M}_p$ so that $M_i \cap \mathcal{A}_p = \{a_0, \dots, a_i\}$) and inductively choose $t = \{m_0, \dots, m_\ell\}$ so that $\mathcal{A}_t$ is $\mathcal{U}$-large and $m_i \in a_i \setminus n_p$ for each i. To see how to choose m_{i+1} (having chosen $t_i = \{m_0, \dots, m_i\}$), we note that by the definition of $\mathbb{P}(\mathcal{A}, \mathcal{U})$ and Lemma 2.5, a_{i+1} must meet the set $\{m \in \omega : \mathcal{A}_{t_i \cup \{m\}}$ is $\mathcal{U}$-large$\}$.

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Now that we have selected t and we know that $\mathcal{A}_t$ is $\mathcal{U}$-large, we may select $a \in \mathcal{A}_t$ such that a meets every member of $M \cap \mathcal{U}$. The condition $q = (\mathcal{A}_p \cup \{a\}, n_p, \mathcal{M}_{\bar{p}})$ is easily seen to be below p and in D_p. ■

Lemma 2.7 *If $\mathcal{A}$ is $\mathcal{U}$-large, then $\mathbb{P}(\mathcal{A}, \mathcal{U})$ is proper.*

Proof Let $\mathbb{P}(\mathcal{A}, \mathcal{U}) \in H(\theta)$ for some regular cardinal θ. Assume that p and $\mathbb{P}(\mathcal{A}, \mathcal{U})$ are members of some countable $M \prec H(\theta)$. Let $n = n_p$. We show that $(\mathcal{A}_p, n, \mathcal{M}_p \cup \{M \cap H((2^{\mathfrak{c}})^+)\})$ is an $(\mathbb{P}(\mathcal{A}, \mathcal{U}), M)$-generic condition.

Let $r < (\mathcal{A}_p, n, \mathcal{M}_p \cup \{M \cap H((2^{\mathfrak{c}})^+)\})$ and $r \in D \in M$ with D a dense subset of $\mathbb{P}(\mathcal{A}, \mathcal{U})$. Let $\mathcal{A}' = \mathcal{A}_r \cap M$ and $\vec{A}_r = \mathcal{A}_r \setminus M = \{a_0, \dots, a_{\ell-1}\}$ listed in increasing order (*i.e.*, for each $i < \ell$, there is an $M_i \in \mathcal{M}_r$ such that $M_i \cap \mathcal{A}_p = \mathcal{A}' \cup \{a_j : j < i\}$). Set

$$E_0 = \{\vec{A} \in \mathcal{A}^\ell : (\exists q \in D)(\exists \bar{M} \in \mathcal{M}_q)(\bar{M} \cap \mathcal{A}_q = \mathcal{A}' \text{ and } \vec{A} = \mathcal{A}_q \setminus \bar{M})\},$$

$$E_1 = \{\vec{A} \in \mathcal{A}^{\ell-1} : \{b : \vec{A} \cup \{b\} \in E_0\} \text{ is } \mathcal{U}\text{-large}\},$$

and more generally,

$$E_k = \{\vec{A} \in \mathcal{A}^{\ell-k} : \{b : \vec{A} \cup \{b\} \in E_{k-1}\} \text{ is } \mathcal{U}\text{-large}\} \text{ for } 0 < k \le \ell.$$

By induction on k, we check that $\{a_i : i < \ell - k\} \in E_k$. We assume that $\{a_0, \dots, a_i\} \in E_{k-1}$, and we, of course, have that $M_i \cap \{a_0, \dots, a_i\} = \{a_0, \dots, a_{i-1}\}$. If $\mathcal{Y} = \{b : \{a_0, \dots, a_{i-1}\} \cup \{b\} \in E_{k-1}\}$ were not $\mathcal{U}$-large, then, by elementarity, there would be a subcollection $\mathcal{W} \subset \mathcal{U} \cap M_i$ for which no member of $\mathcal{Y}$ hits each element in an infinite set. However, a_i is a member of $\mathcal{Y}$ and is required to hit each member of $\mathcal{W}$ in an infinite set.

Now that we know that $\varnothing \in E_\ell$, let $\mathcal{Y}^0 = \{b : \{b\} \in E_{\ell-1}\} \in M_0$. There is an $m_0 \in a_0 \setminus n$ such that $\mathcal{Y}^0_{m_0}$ is $\mathcal{U}$-large. Similarly we may choose $m_1 \in a_1 \setminus n$ so that $\mathcal{Y}^0_{m_0,m_1}$ is $\mathcal{U}$-large, and, recursively, there is a finite set t_0 such that $\mathcal{Y}^0_{t_0}$ is $\mathcal{U}$-large and $t_0 \cap a_i \setminus n$ is not empty for each $i < \ell$.

Fix any $t_0 \subset b_0 \in \mathcal{Y}^0_{t_0} \cap M \subset E_{\ell-1}$, and define $\mathcal{Y}^1 = \{b : \{b_0, b\} \in E_{\ell-2}\} \in M$. Repeat the process to find a finite t_1 so that $\mathcal{Y}^1_{t_1}$ is $\mathcal{U}$-large and $t_1 \cap a_i \setminus n$ is not empty for each $i < \ell$. Select any $t_1 \subset b_1 \in \mathcal{Y}^1_{t_1}$ and continue the induction. In this way we find a sequence $\langle b_i : i < \ell\rangle$ so that $t_i \subset b_i$ and $t_i \cap a_j \setminus n$ is not empty for each $i, j < \ell$ and $\langle b_i : i < \ell\rangle \in E_0 \cap M$.

It follows, by definition of E_0 and elementarity, that there is a $q \in D \cap M$ with $\mathcal{A}_q = \mathcal{A}' \cup \{b_0, \dots, b_{\ell-1}\}$. The fact that q is compatible with r follows from the choices of the t_i and the fact that $\mathcal{M}_q \subset M \cap H((2^{\mathfrak{c}})^+)$, and so each $\tilde{M} \in \mathcal{M}_q$ is an elementary submodel of M_0. ■

Following Lemma 2.4, for a filter $H \subset \mathbb{P}(\mathcal{A}, \mathcal{U})$, let $\mathcal{A}_H$ denote the set $\bigcup_{p \in H} \mathcal{A}_p$.

Lemma 2.8 *Let X be a compact scattered sequential space containing a discrete copy of ω. Suppose that $\mathcal{A}$ is any maximal almost disjoint family of infinite subsets of ω consisting of sequences converging in X. For each $a \in \mathcal{A}$, let x_a denote the point in X to which a converges. If $\mathcal{U}$ extends the neighborhood trace of some point $z \in \overline{\omega} \setminus (\omega)^{(1)}$, then*

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there is a family of ω_1-many dense subsets of $\mathbb{P}(\mathcal{A},\mathcal{U})$ such that for each $\mathbb{P}(\mathcal{A},\mathcal{U})$-filter H that meets each of these dense sets, z is a complete accumulation point of $X_H = \{x_a : a \in \mathcal{A}_H\}$.

Proof Fix any $q \in \mathbb{P}(\mathcal{A},\mathcal{U})$ and $M_0 \in \mathcal{M}_q$. Let $p < q$ be arbitrary such that there is some $M \in \mathcal{M}_p$ with $q \in M$. We may assume that M is a countable union of countable elementary submodels $\{M_n : n \in \omega\} \subset M$. It suffices to show that there is a countable family $\mathcal{D}$ of sets that are predense below p such that for any $\mathcal{D}$-generic filter $H \ni p$, z is in the closure of $X_H \setminus M_0$. Let $T \subset [\omega]^{<\omega}$ be the set of all t such that $\mathcal{A}_t$ is $\mathcal{U}$-large. For each $t \in T$, let $T_t = \{t' \in T : t \subset t'\}$. For $t \in T$, let $X_t^M = \{x_a : a \in \mathcal{A}_t \cap M\}$.

Define $K_M = \{y : (\exists t_y \in T) \ \ y \in \overline{X_t^M \setminus M_n}$ for all $t \in T_{t_y}$ and $n \in \omega\}$. We first check that $z \in K_M$ with witness $t_z = \varnothing$. Fix any $n \in \omega$. Since $\mathcal{A}_t$ is $\mathcal{U}$-large for each $t \in T$ and $\mathcal{A}_t \cap M_n$ is countable, it follows that $\mathcal{A}_t \setminus M_n$ is again $\mathcal{U}$-large. Furthermore, $z \in M$ and X has countable tightness, hence z is a limit of $X_t^M \cap M \setminus M_n$. Therefore $z \in K_M$. Now, using that K_M is scattered, we have that there is a countable set of relatively isolated points of K_M, the closure of which contains z. Our proof is complete once we show that for any isolated point y of K_M, there is a countable family $\mathcal{D}$ of predense below p subsets of $\mathbb{P}(\mathcal{A},\mathcal{U})$ such that y is in the closure of X_H for each $\mathcal{D}$-generic filter H. Fix any clopen set W_y that satisfies $W_y \cap K_M = \{y\}$. For each $x \in W_y \setminus \{y\}$, fix a clopen set $W_x \subset W_y \setminus \{y\}$ such that W_x contains no other points at the same scattering level as x.

Claim 1. For each $t \in T_{t_y}$ and each $n \in \omega$, the set of $r \in \mathbb{P}(\mathcal{A},\mathcal{U})$ such that $x_a \in W_y$ for some $a \in \mathcal{A}_t \cap \mathcal{A}_r \cap M \setminus M_n$ is predense below p.

Let $r < p$ and $t \in T_{t_y}$ be arbitrary and assume (by possibly increasing n) that n is large enough so that $\mathcal{M}_r \cap M \subset M_n$. As in the proof of Lemma 2.7, there is $t' \in T_t$ such that $t' \cap a \setminus n_r$ is not empty for each $a \in \mathcal{M}_r \setminus M$. Since $W_y \cap X_{t'} \cap M \setminus M_n$ is not empty, we may fix some $a \in \mathcal{A}_{t'} \cap M \setminus M_n$ such that $x_a \in W_y$. The condition $\langle \mathcal{A}_r \cup \{a\}, n_r, \mathcal{M}_r \cup \{M_n\}\rangle$ is below r and is a member of the prescribed predense below p set. This completes the proof of the claim.

Claim 2. For each $x \in W_y \setminus \{y\}$ and $t \in T_{t_y}$, there is a $t' \in T_t$ and $n \in \omega$ such that $W_x \cap \mathcal{A}_{t'} \cap M \setminus M_n$ is empty.

We prove this claim by induction on the scattering level of x. Since $x \notin K_M$, there is a $t_0 \in T_t$ and an $n_0 \in \omega$ such that x is not in the closure of $X_{t_0} \cap M \setminus M_{n_0}$. Since X is scattered, there is a finite subset $\{x_1, \dots, x_\ell\}$ of W_x such that $W_x \setminus (W_{x_1} \cup \dots \cup W_{x_\ell})$ contains only finitely many points of $X_{t_0} \cap M \setminus M_{n_0}$. Recursively applying the inductive hypothesis, there is a $t_\ell \in T_{t_0}$ and an $n_\ell \in \omega$ such that $(W_{x_1} \cup \dots \cup W_{x_\ell})$ is disjoint from $\mathcal{A}_{t_\ell} \cap M \setminus M_{n_\ell}$. We then have that $W_x \cap X_{t_\ell} \cap M \setminus M_{n_\ell}$ is finite, and so for some $n \geq n_\ell$, W_x is disjoint from $X_{t_\ell} \cap M \setminus M_n$.

Claim 3. There is a countable collection $\mathcal{D}$ of predense below p sets such that for each $\mathcal{D}$-generic filter H, y is in the closure of $X_H \cap M$.

Let $\mathcal{D}$ include the set described in Claim 1 for each $t \in T_{t_y}$ and $n \in \omega$. It follows then that $W_y \cap X_H \cap X_t \cap M \setminus M_n$ is infinite for each $t \in T_{t_y}$. A typical neighborhood W of y has the form $W_y \setminus (W_{x_1} \cup \dots \cup W_{x_\ell})$ with $\{x_1, \dots, x_\ell\} \subset W_y \setminus \{y\}$. By

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Claim 2, we can again, recursively, select $t_\ell \in T_{t_y}$ and n_ℓ such that $W_{x_i} \cap X_{t_\ell} \cap M \setminus M_{n_\ell}$ is empty for each $1 \le i \le \ell$. It follows then that $W \cap X_H \cap X_{t_\ell} \cap M$ is infinite. ∎

Theorem 2.9 (PFA) *If X is compact, scattered, and sequential and z is a limit point of a set A but no sequence from A converges to z, then there is an uncountable set $\{x_\alpha : \alpha \in \omega_1\}$ of points in $A^{(1)}$ such that z is the unique complete accumulation point.*

Proof Without loss of generality, A is countable and so we may identify the set A with ω. Let $\mathcal{U}$ be any ultrafilter on ω that extends the neighborhood trace of z. Set $\mathcal{A}$ to be any maximal almost disjoint family of converging subsequences of ω (in the sequentially compact space X). Before applying Lemmas 2.7 and 2.8, we have to show that $\mathcal{A}$ is $\mathcal{U}$-large. Let $\mathcal{W}$ be any countable subset of $\mathcal{U}$. Of course we can assume that $\mathcal{W}$ is descending. There is a sequence $b \subset \omega$ that is (mod finite) contained in each $W \in W$ and that converges in X. It follows then that there is an $a \in \mathcal{A}$ that meets b in an infinite set.

Now by Lemma 2.7, $\mathbb{P}(\mathcal{A}, \mathcal{U})$ is proper. Then by Lemma 2.8 and PFA, we may assume that we have a filter $H \subset \mathbb{P}(\mathcal{A}, \mathcal{U})$ that ensures that z is a complete accumulation point of X_H and that $\mathcal{A}_H$ is inseparable. The uniqueness of z follows from the fact that X is Hausdorff and that each neighborhood of a complete accumulation point will (mod finite) contain uncountably many members of $\mathcal{A}_H$. ∎

Corollary 2.10 (PFA) *Each madf on ω contains an inseparable (Hausdorff–Luzin) subfamily of cardinality ω_1.*

3 Obstructions to Large Sequential Order

Theorem 3.1 (PFA) *There is no CB-sequential space with sequential order greater than ω.*

Proof Assume that X is a CB-sequential space and that $\{w_n : n \in \omega\}$ is a subset of the set of points that have scattering rank ω. Let $\{W_n : n \in \omega\}$ be a family of pairwise disjoint clopen sets such that $W_n \cap X_\omega$ is $\{w_n\}$ for each n. For each n, fix any sequence $\{w(n,m) : m \in \omega\}$ of points in W_n such that $w(n,m) \in X_m$. Since the scattering levels diverge to ω, it follows that $\{w(n,m) : m \in \omega\}$ converges to w_n. Now we apply Theorem 2.9. For each $n < m$, select $\{x(n,m,\alpha) : \alpha \in \omega_1\} \subset X_n \cap W_n$ so that $w(n,m)$ is the unique complete accumulation point.

We start an ω_1 length induction. Set $\xi_0 = 0$. Choose an infinite set $I_0 \subset \omega$ so that for each $n \in \omega$, the sequence $\{x(n,m,0) : m \in I_0\}$ is a converging sequence with limit $y(n,0) \in X_{n+1} \cap W_n$. Next choose an infinite $J_0 \subset \omega$ so that $\{y(n,0) : n \in J_0\}$ is a converging sequence with limit $v_0 \in X_\omega$. Fix a clopen neighborhood V_0 of v_0 such that v_0 is the only point of X_ω in V_0. We observe that $v_0 \notin W_n$ for each n, and therefore that $V_0 \cap \{w(n,m) : m \in \omega\}$ is finite for each n. Furthermore, for each n, m such that $w(n,m) \notin V_0$, $V_0 \cap \{x(n,m,\alpha) : \alpha \in \omega_1\}$ must be countable. Therefore, for sufficiently large $\xi_1 \in \omega_1$, we have that for each n, $V_0 \cap \{x(n,m,\xi_1) : m \in \omega\}$ is finite.

For the next step of the induction, we select an infinite $I_1 \subset I_0$, so that, for each n, $\{x(n,m,\xi_1) : m \in I_1\}$ converges to a point $y(n,1) \in X_{n+1} \cap W_n \setminus V_0$. Then

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select $J_1 \subset J_0$ so that $\{y(n,1) : n \in J_1\}$ converges to some point $v_1 \in X_\omega \setminus V_0$, and select V_1 so that $V_1 \cap X_\omega = \{v_1\}$. We continue to choose the sequence of sets $\{I_\alpha : \alpha \in \omega_1\}$ (descending mod finite), $\{J_\alpha : \alpha \in \omega_1\}$ (descending mod finite), ordinals $\{\xi_\alpha : \alpha \in \omega_1\} \subset \omega_1$, points $y(n,\alpha) \in X_{n+1} \cap W_n \setminus \bigcup_{\beta<\alpha} V_\beta$ so that $\{x(n,m,\xi_\alpha) : m \in I_\alpha\}$ converges to $y(n,\alpha)$ for all $n \in J_\alpha$, and $\{y(n,\alpha) : n \in J_\alpha\}$ converges to $v_\alpha \in X_\omega \setminus \bigcup_{\beta<\alpha} V_\beta$.

After these ω_1 many steps, we have a look at the sequence $\{V_\alpha : \alpha \in \omega_1\}$, the canonical clopen sets for w_α. Since PFA implies that $\mathfrak{p} > \omega_1$, there are infinite sets I, J that are mod finite contained in every member of $\{I_\alpha : \alpha \in \omega_1\}$ and $\{J_\alpha : \alpha \in \omega_1\}$ respectively.

For each $\alpha \in \omega_1$, there is a function f_α such that

$$\{x(n,m,\alpha) : m \in I \setminus f_\alpha(n)\} \subset V_\alpha, \text{ and } \{w(n,m) : m > f_\alpha(n)\} \cap V_\alpha = \varnothing$$

for all but finitely many n. Again by PFA, $\mathfrak{b} > \omega_1$, and we may choose some $f^* > f_\alpha$ for all $\alpha \in \omega_1$, and consider any limit $w \in X_\omega$ of the sequence $S = \{w(n, f(n)) : n \in J\}$. It follows that $w \neq v_\alpha$, since $V_\alpha \cap S$ is finite. We now assume that W is a clopen neighborhood of w satisfying that $W \cap X_\omega = \{w\}$. Since $w(n, f(n)) \in W$ for infinitely many $n \in J$, there is an α such that $x(n, f(n), \xi_\alpha) \in W$ for infinitely many $n \in J$. But this means that $W \cap V_\alpha$ contains an infinite subset of $\{x(n, f(m), \xi_\alpha) : n \in J\}$ contradicting that $W \cap V_\alpha \cap X_\omega$ is supposed to be empty. ■

4 Questions

Question 1. Does Theorem 2.9 hold for sequential spaces that are not necessarily scattered?

Question 2. Does Theorem 3.1 follow from Martin's Axiom and $\mathfrak{c} > \omega_1$?

Question 3. Does Corollary 2.10 follow from Martin's Axiom and $\mathfrak{c} > \omega_1$?

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