

NEARLY INTEGRAL HOMOMORPHISMS OF COMMUTATIVE RINGS

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A unital homomorphism $f: R \rightarrow T$ of commutative rings is said to be *nearly integral* if the induced map $R/I \rightarrow T/IT$ is integral for each ideal I of R which properly contains $\ker(f)$. This concept leads to new characterisations of integral extensions and fields. For instance, if R is not a field, then an inclusion $R \rightarrow T$ is integral if and only if it is nearly integral and (R, T) is a lying-over pair. It is also proved that each overring extension of an integral domain R is nearly integral if and only if $\dim(R) \leq 1$ and the integral closure of R is a Prüfer domain. Related properties and examples are also studied.

1. INTRODUCTION

All rings considered in this paper are commutative with identity; and all ring-homomorphisms and subrings are unital. Our starting point is a result of fundamental importance: integrality is a universal property (see [2, Proposition 5, p.307], [3, Lemma, p.160]). This means that if $f: R \rightarrow T$ is an integral ring homomorphism and a ring-homomorphism $R \rightarrow S$ is viewed as a change of base (in the sense of [9]), then the induced ring-homomorphism $f_{(S)}: S \rightarrow S \otimes_R T$ is also integral. Our purpose here is to study the weakening of the "integral" concept obtained by restricting attention to algebras of the form $S = R/I$, where I is an ideal of R which properly contains $\ker(f)$. Specifically, we shall say that a ring-homomorphism $f: R \rightarrow T$ is *nearly integral* in case the induced map $R/I \rightarrow R/I \otimes_R T \cong T/IT$ is integral for each ideal I of the above type. (This work may be contrasted with [1], where we studied the question as to when certain factors R/I led to integrally closed rings.) Additional characterisations of "nearly integral" are given in Proposition 2.2; and Corollary 2.3 shows that $f: R \rightarrow T$ is nearly integral if and only if the inclusion $f(R) \rightarrow T$ is nearly integral.

Of course, any integral ring-homomorphism is nearly integral. As one would expect, "nearly integral" behaves much like "integral", and several results establish phenomena such as transitivity (Proposition 2.7(a)) and stability under localisation (Proposition 2.6(a)). However, such analogies are not perfect. Indeed, although "nearly integral" admits a globalisation result in the context of integral domains (Proposition 2.6(b)), this result fails in general (Remark 2.15(a)).

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In order to deepen our understanding of nearly integral ring-homomorphisms $R \rightarrow T$, we next mention some archetypical sufficient conditions: integrality; R a field; and $R = 0$ (in which case unitality forces $T = 0$). A main purpose of our work is to show how to recover these archetypes from hypotheses that involve “nearly integral”. For instance, it is shown in Theorem 2.13(c) that a ring R is either a field or 0 if (and only if) the polynomial ring $R[X]$ is nearly integral over R . Characterisations of integrality in this spirit appear in Theorem 2.13(b), Theorem 2.14, and Proposition 2.16. Indeed, Theorem 2.14 states that if R is not a field, then a ring extension $R \rightarrow T$ is integral if (and only if) it is nearly integral and (R, T) is a lying-over pair (in the sense of [5]). Perhaps our deepest result is Corollary 2.11: an integral domain R has the property that each of its overring extensions is nearly integral if and only if the (Krull) dimension of R is at most 1 and the integral closure of R is a Prüfer domain. This follows in part from the fact (Proposition 2.10(b)) that nearly integral overring extensions of integral domains satisfy the incomparability property, INC. As Remark 2.12(a) shows, “nearly integral” does not satisfy this integral-like behaviour in general.

Any unexplained terminology is standard, as in [2] and [10]. In Proposition 2.16, we assume some familiarity with the opening pages of [5].

2. RESULTS

We begin with an easy characterisation of integrality that motivates our interest in homomorphisms of the type $f_{(S)}$.

PROPOSITION 2.1. *For a ring-homomorphism $f: R \rightarrow T$, the following conditions are equivalent*

- (1) $f_{(S)}: S \rightarrow S \otimes_R T$ is integral for each ring-homomorphism $R \rightarrow S$;
- (2) $f_{(S)}$ is integral for each ring-homomorphism $R \rightarrow S$ which is not an isomorphism;
- (3) f is integral.

PROOF: (3) \Rightarrow (1) since, as we recalled in the introduction, integrality is a universal property. Of course, (1) \Rightarrow (2) trivially. Finally, assume (2), and consider the inclusion map of R into the polynomial ring $S = R[X]$. By (2), $f_{(S)}: R[X] \rightarrow R[X] \otimes_R T \cong T[X]$ is integral. Thus, for each t in T , we have

$$t^n + f_{(S)}(h_{n-1})t^{n-1} + \dots + f_{(S)}(h_0) = 0$$

for some $h_i \in R[X]$. Applying the T -algebra map $T[X] \rightarrow T$ that sends X to 0, we obtain an integrality equation for t over R . Hence, (2) \Rightarrow (3). \square

Notice that, in using condition (2) in the above proof, we chose $S = R[X]$. Another natural choice would be $S = R/I$, where I is a nonzero ideal of R . In this case, $f_{(S)}$

identifies with the canonical map $\bar{f}: R/I \rightarrow R/I \otimes_R T \cong T/IT$. This observation now leads us to our main object of study.

PROPOSITION 2.2. *For a ring-homomorphism, $f: R \rightarrow T$, the following four conditions are equivalent*

- (1) T is integral over $f(R) + IT$ for each ideal I of R such that $I \not\subseteq \ker(f)$;
- (2) $\bar{f}: R/I \rightarrow T/IT$ is integral for each ideal I of R such that $I \not\subseteq \ker(f)$;
- (3) T is integral over $f(R) + IT$ for each ideal I of R such that $\ker(f) \not\subseteq I$;
- (4) \bar{f} is integral for each ideal I of R such that $\ker(f) \not\subseteq I$.

If the above equivalent conditions hold, then we say that f is nearly integral (or that T is nearly integral over R).

PROOF: Let I be an ideal of R . Consider the commutative diagram

$$\begin{array}{ccc} f(R) + IT & \longrightarrow & T \\ \downarrow & & \downarrow \\ (f(R) + IT)/IT & \longrightarrow & T/IT \end{array}$$

whose horizontal maps are inclusions and whose vertical maps are the canonical surjections. Notice that the top horizontal map is integral if and only if the bottom horizontal map is integral. (This may be verified by direct calculation or by appeal to either [8, proof of Lemma 4.6] or [6, Corollary 1.5(5)].) Moreover, the bottom horizontal map is integral if and only if $\bar{f}: R/I \rightarrow T/IT$ is integral, since $(f(R) + IT)/IT = \text{im}(\bar{f})$. These observations yield the equivalences (1) \Leftrightarrow (2) and (3) \Leftrightarrow (4). Moreover, it is trivial that (1) \Rightarrow (3) (and, similarly, that (2) \Rightarrow (4)). To complete the proof, we shall show that (3) \Rightarrow (1).

Let I be an ideal of R such that $I \not\subseteq \ker(f)$. Put $J = I + \ker(f)$. Since $\ker(f) \not\subseteq J$, it follows from (3) that T is integral over $f(R) + JT$. But $JT = IT$ since $\ker(f) \cdot T = 0$. Therefore, T is integral over $f(R) + IT$. □

COROLLARY 2.3. *A ring-homomorphism $f: R \rightarrow T$ is nearly integral if and only if T is nearly integral over $f(R)$.*

PROOF: T is nearly integral over $f(R)$ if and only if T is integral over $f(R) + JT$ for each nonzero ideal J of $f(R)$. By a standard homomorphism theorem, such J are in one-to-one correspondence with ideals I of R such that $\ker(f) \not\subseteq I$, the bijection being given by $J \mapsto f^{-1}(J)$ and $I \mapsto f(I)$. Notice that if $I = f^{-1}(J)$, then $JT = f(I)T = IT$. Hence, T is nearly integral over $f(R)$ if and only if T is integral over $f(R) + IT$ for each ideal I of R such that $\ker(f) \not\subseteq I$; that is, by Proposition 2.2, if and only if f is nearly integral. □

Remark 2.4. (a) It is also of some interest to note the following alternate proof that (4) \Rightarrow (2) (instead of showing (3) \Rightarrow (1)) in Proposition 2.2. Given an ideal $I \not\subseteq \ker(f)$, put $J = I + \ker(f)$ and observe that $\bar{f}: R/I \rightarrow T/IT$ is the composite of the integral maps $R/I \rightarrow R/J$ and $R/J \rightarrow T/JT = T/IT$. Thus (see [2, Proposition 6, p.307]), \bar{f} is integral. (Notice that the key to both proofs is the observation that $JT = IT$.)

(b) Corollary 2.3 also admits a proof in the spirit of (a). With J and I as in the earlier proof of Corollary 2.3, recall that $JT = IT$, and note that $f(R)/J \cong R/I$ by a standard homomorphism theorem. Thus, $f(R)/J \rightarrow J/JT$ may be identified with $R/I \rightarrow T/IT$.

(c) The above technique can be used to explain why a related concept is less interesting than “nearly integral”. To wit: if $f: R \rightarrow T$ is a ring-homomorphism such that $\bar{f}: R/I \rightarrow T/IT$ is integral for all nonzero ideals I of R , then either f is an injection or f is integral. Indeed, if f is not an injection, take $I = \ker(f)$, and notice that f factors as the composite of integral maps $R \rightarrow R/I$ and $R/I \rightarrow T/IT = T/0 \cong T$, whence f is integral.

On the other hand, it does not follow that a ring-homomorphism $f: R \rightarrow T$ is integral given that f is nearly integral and $\ker(f) \neq 0$. To see this, let R be an integral domain of (Krull) dimension 1, let M be a maximal ideal of R , and let f be the composite of the canonical surjection $R \rightarrow F = R/M$ and the inclusion $F \rightarrow T = F[X]$. We have $\ker(f) = M \neq 0$ and, using condition (3) of Proposition 2.2, f is nearly integral. (The point is that $I = R$ leads to $f(R) + IT = T$.) Of course, (X in) T is not integral over R .

In view of Corollary 2.3, we lose no generality in studying *nearly integral extensions*, namely $R \subset T$ such that (R is a subring of T and) the inclusion map $R \rightarrow T$ is nearly integral. Accordingly, we shall focus henceforth on such inclusions $R \subset T$.

The next result records some useful sufficient conditions for “nearly integral”.

PROPOSITION 2.5. *Let $R \subset T$ be a ring extension. Then T is nearly integral over R in each of the following cases*

- (a) T is integral over R ;
- (b) R is a field;
- (c) R is an integral domain and T contains the quotient field of R .

PROOF: (a) follows from the fact that integrality is a universal property. As for (b), condition (3) of Proposition 2.2 is satisfied since $I = R$ leads to $R + IT = T$. (Another proof of (b) is available via condition (4) of Proposition 2.2 since $I = R$ leads to $\bar{f}: R/I \rightarrow T/IT$ being identified with $0 \rightarrow 0$, which is certainly integral.) Finally, (c) follows via condition (3) of Proposition 2.2. The point is that if I is a nonzero ideal of R and K is the quotient field of R , then $R + IT = T$ because

$$IT = I(KT) = (IK)T = KT = T. \quad \square$$

We next show that “nearly integral” is a local property, at least for integral domains.

PROPOSITION 2.6. *Let $R \subset T$ be a ring extension. Then*

- (a) *if T is nearly integral over R , then T_S is nearly integral over R_S for each multiplicatively closed subset S of R ;*
- (b) *if $T_{R \setminus M}$ is nearly integral over R_M for each maximal ideal M of R and R is an integral domain, then T is nearly integral over R .*

PROOF: (a) We shall show that T_S is integral over $R_S + JT_S$ for all nonzero ideals J of R_S . Any such J has the form $J = IR_S$, where I is the inverse image of J under $R \rightarrow R_S$. In particular, $I \neq 0$. Thus, by hypothesis, T is integral over $R + IT$. Applying the homomorphism $T \rightarrow T_S$, we see that T_S is integral over $(R + IT)_S = R_S + IR_S T_S = R_S + JT_S$.

(b) Let I be a nonzero ideal of R , and let A denote the integral closure of $R + IT$ in T . For each maximal ideal M of R , we claim that $A_{R \setminus M} = T_{R \setminus M}$. Since $T_{R \setminus M}$ is nearly integral over R_M , we have $T_{R \setminus M}$ integral over $R_M + IR_M T_{R \setminus M} = (R + IT)_{R \setminus M}$. (Notice that $IR_M \neq 0$ since R is an integral domain.) The claim now follows since taking integral closure commutes with localisation (see [2, Proposition 16, p.314]). Hence by globalisation [2, Theorem 1, p.88], $A = T$; that is, T is integral over $R + IT$, and so T is nearly integral over R . □

As Proposition 2.6 suggests, it is useful to study the behaviour of the “nearly integral” property. In this spirit, we record the following result:

PROPOSITION 2.7. *Let $R \subset S \subset T$ be a tower of rings. Then*

- (a) *if S is nearly integral over R and T is nearly integral over S , then T is nearly integral over R ;*
- (b) *let T be nearly integral over R , such that $IT \cap S = IS$ for each nonzero ideal I of R ; then S is nearly integral over R ;*
- (c) *let T be nearly integral over R , such that each nonzero ideal J of S satisfies $J \cap R \neq 0$; then T is nearly integral over S .*

PROOF: (a) If I is a nonzero ideal of R , then $R/I \rightarrow T/IT$ factors as the composite $R/I \rightarrow S/IS \rightarrow T/IT$ of integral homomorphisms, and so is integral.

(b) This follows from the factorisation noted in the proof of (a) by viewing S/IS as an R/I -subalgebra of T/IT .

(c) Let J be a nonzero ideal of S and put $I = J \cap R$. By hypothesis, $R/I \rightarrow T/IT$ is integral. By composing this with (the integral map) $T/IT \rightarrow T/JT$, we see that

$h: R/I \rightarrow T/JT$ is integral. Since h factors through S/J , it follows that $S/J \rightarrow T/JT$ is integral, and so T is nearly integral over S . \square

One sufficient condition for the second hypothesis in Proposition 2.7(c) is that R be an integral domain and S an overring of R . Much of our later work will involve overrings, and so we pause to record an example of a nearly integral overring extension that is not integral. By way of contrast, notice that the nonintegral nearly integral extension underlying the second paragraph of Remark 2.4(c) is $F \subseteq F[X]$, which is not an overring extension.

Example 2.8. Let R be a one-dimensional Prüfer domain and T a proper overring of R . Then T is nearly integral, but not integral, over R .

PROOF: The second assertion follows since R is integrally closed. As for the first, Proposition 2.6(b) allows us to suppose that R is a (one-dimensional) valuation domain. Although T , with the change of notation, need not remain proper, it is either R or the quotient field of R (see [10, Theorem 65]). Apply Proposition 2.5(a), (c) to conclude that T is nearly integral over R . \square

By a result of Richman [11, Theorem 4], an integral domain is a Prüfer domain if and only if each of its overrings is flat. In view of Example 2.8, one might ask whether each flat overring is nearly integral. The next result shows this is not the case even for localisations. Indeed, Proposition 2.9 implies that if R is a valuation domain of dimension at least 2 and P is a nonzero nonmaximal prime ideal of R , then R_P is not nearly integral over R .

PROPOSITION 2.9. Let R be a quasilocal going-down domain (in the sense of [4]) and let P be a nonzero prime ideal of R . Then R_P is nearly integral over R if and only if P is the maximal ideal of R .

PROOF: The “if” assertion is trivial, for R_P is then R , which is certainly nearly integral over itself. Conversely, suppose that R_P is nearly integral over R . Then, by taking $I = P$ in condition (3) of Proposition 2.2, we see that R_P is integral over $A = R + PR_P$. However, the hypothesis on R implies that A is integral over R (see [4, Proposition 2.1]). Therefore, R_P is an integral overring of R . As R_P is also R -flat, it follows that $R_P = R$ (see [11, Proposition 2]), whence P is the maximal ideal of R . \square

In view of the last two results, it seems reasonable to ask which integral domains R have the property that each overring is nearly integral. The answer will be given in Corollary 2.11. Before giving a preliminary result, we recall that INC and LO denote the incomparability and lying-over properties, respectively, as in [10, p.28].

PROPOSITION 2.10. *Let $R \subset T$ be a nearly integral ring extension. Then*

- (a) *let $Q_1 \subset Q_2$ be prime ideals of T such that $Q_1 \cap R = Q_2 \cap R = P$; if R_P is not a field, then $Q_1 = Q_2$;*
- (b) *if, in addition, R is an integral domain and T is an overring of R , then $R \subset T$ satisfies INC.*

PROOF: (a) Proposition 2.6(a) allows us to pass from $R \subset T$ to $R_P \subset T_{R \setminus P}$; that is, we may assume (R, P) is quasilocal. As $R = R_P$ is not a field, $P \neq 0$. Set $S = R + PT$. By condition (3) of Proposition 2.2, T is integral over S . Hence, $S \subset T$ satisfies INC (see [10, Theorem 44]). Since

$$P \subset PT \cap R \subset Q_i \cap R = P,$$

we have $PT \cap R = P$, and so a standard homomorphism theorem yields $S/PT \cong R/P$. In particular, PT is a maximal ideal of S . Since $PT \subset Q_i \cap S$, it follows that $Q_1 \cap S = PT = Q_2 \cap S$, whence $Q_1 = Q_2$ by INC.

(b) Deny the proposition. Then $Q_1 \cap R = Q_2 \cap R = P$ for some distinct primes $Q_1 \subset Q_2$ of T . As $Q_2 \neq 0$ and T is an overring of R , we have $P = Q_2 \cap R \neq 0$. Then R_P is not a field and so, by (a), $Q_1 = Q_2$, the desired contradiction. □

COROLLARY 2.11. *Let R be an integral domain with integral closure R' . Then the following conditions are equivalent*

- (1) *each overring of R is nearly integral over R ;*
- (2) *if $A \subset B$ are overrings of R , then B is nearly integral over A ;*
- (3) *$\dim(R) \leq 1$ and R' is a Prüfer domain.*

PROOF: (2) \Rightarrow (1): trivial.

(1) \Rightarrow (3): assume (1). If u is an element of the quotient field of R , then Proposition 2.10 (b) implies that $R \subset R[u]$ satisfies INC, and so $h(u) = 0$ for some polynomial $h \in R[X]$ with a unit coefficient (see [5, Theorem 2.3]). It follows from [7, Theorem 5] that R' is a Prüfer domain.

Since R' is integral over R , it follows readily from Proposition 2.7(c) and (1) that each overring of R' is nearly integral over R' . As $\dim(R) = \dim(R')$, we may assume $R = R'$; that is, R is a Prüfer domain. If M is a maximal ideal of R and T is an overring of R_M , we have T nearly integral over R , and so $T = T_{R \setminus M}$ is nearly integral over R_M , by Proposition 2.6(a). As $\dim(R) = \sup\{\dim(R_M)\}$, we may suppose R quasilocal. In particular, R is a valuation (hence, going-down) domain. Then Proposition 2.9, in conjunction with (1), yields $\dim(R) \leq 1$.

(3) \Rightarrow (2): given (3), R has valuative dimension at most 1, and so each overring of R inherits the hypotheses of (3) (see [10, Theorems 65 and 64]). Therefore it will

suffice to show that (3) \Rightarrow (1). Assume (3), and let S be an overring of R . Set $T = R'S$. By Example 2.8, T is nearly integral over R' . As R' is (nearly) integral over R , Proposition 2.7(a) yields that T is nearly integral over R . Hence, $R/I \rightarrow T/IT$ is integral for each nonzero ideal I of R . We shall show that $R/I \rightarrow S/IS$ is integral by using a “radical” variant of the condition in Proposition 2.7(b).

Let $s \in S$. As $s + IT$ is integral over R/I , we have

$$\xi = s^n + r_{n-1}s^{n-1} + \dots + r_0 \in IT \cap S$$

for some elements r_{n-1}, \dots, r_0 of R . We claim that $\xi \in \text{rad}_S(IS)$. If so, $\xi^N \in IS$ for some integer $N \geq 1$, thus producing

$$s^{nN} + a_{nN-1}s^{nN-1} + \dots + a_0 \in IS$$

for suitable $a_i \in R$. It would follow that $s + IS$ is integral over R/I , whence S/IS is integral over R/I , whence S is nearly integral over R .

It remains to verify the claim regarding ξ . We shall show, in fact, that $\xi \in P$ for each prime ideal P of S which contains IS . As T is integral over S , the extension $S \subset T$ satisfies LO (see [10, Theorem 44]). Choose $Q \in \text{Spec}(T)$ such that $Q \cap S = P$. Then

$$\xi \in IT = IST \subset PT \subset Q,$$

whence $\xi \in Q \cap S = P$, as desired. □

Remark 2.12. (a) The hypothesis “ R_P is not a field” cannot be deleted from Proposition 2.10(a). To see this, let R be a field F , and take $T = F[X]$, $Q_1 = 0$, $Q_2 = XT$ and $P = 0$. Then $R \subset T$ does not satisfy INC (since $Q_1 \not\subseteq Q_2$ and $Q_1 \cap R = Q_2 \cap R = P$), although Proposition 2.5(b) shows that T is nearly integral over R .

(b) The converse of Proposition 2.10(b) is false. Indeed, if P is a nonzero, non-maximal prime of a valuation domain R , then $R \subset R_P$ satisfies INC although, by Proposition 2.9, R_P is not nearly integral over R .

We next give two results showing that “nearly integral”, suitably enhanced, can be used to recover the archetypes from Proposition 2.5(a), (b). A rôle for polynomials is not surprising since a ring extension $R \subset T$ is integral if and only if $R[X] \subset T[X]$ satisfies GU (going-up): see [3, Lemma, p.160].

THEOREM 2.13. (a) *Let $R \subset T$ be a ring extension such that the conductor $(R : T)$ is nonzero. Then T is integral over R if (and only if) T is nearly integral over R .*

(b) *For a ring extension $R \subset T$ the following four conditions are equivalent*

- (1) $T[X]$ is nearly integral over $R[X]$;

- (2) $T[X]$ is nearly integral over $R[X] + XT[X]$;
- (3) $T[X]$ is integral over $R[X]$;
- (4) T is integral over R .

(c) For a ring R , the following four conditions are equivalent

- (i) $R[X]$ is nearly integral over R ;
- (ii) $R[X]$ is integral over $R + hR[X]$ for all nonunits $h \in R[X] \setminus R$;
- (iii) $R[X]$ is integral over each ring A such that $R \subsetneq A \subset R[X]$;
- (iv) either R is a field or $R = 0$.

PROOF: (a) Let $I = (R : T)$. By hypothesis, $I \neq 0$. Since $IT = I$ and T is nearly integral over R , it follows that $R/I \rightarrow T/I$ is integral. By analysing the pullback $R \cong T \times_{T/I} R/I$ via [8] or [6] as in the proof of Proposition 2.2, we conclude that T is integral over R .

(b) It follows easily from [2, Proposition 13, p.312] that (3) \Leftrightarrow (4). Of course, (3) \Rightarrow (1). We next show (1) \Rightarrow (4). Assume (1), and let $I = XR[X]$. It follows from (1) and condition (2) of Proposition 2.2 that $A \subset B$ is integral, where $A = R[X]/I \cong R$ and $B = T[X]/IT[X] = T[X]/XT[X] \cong T$. In other words, T is integral over R , giving (4).

Since (3) \Rightarrow (2) trivially, it suffices to prove (2) \Rightarrow (4). Assume (2). Since $J = XT[X]$ is a nonzero common ideal of $D = R[X] + XT[X]$ and $E = T[X]$, it follows from (a) that E is integral over D . Hence, $D/J \rightarrow E/J$ is integral; that is, T is integral over R , yielding (4).

(c) (iv) \Rightarrow (iii) follows from the proof of [5, Proposition 2.9]; and (iii) \Rightarrow (ii) trivially. We next show (ii) \Rightarrow (iv). Let $r \in R \setminus \{0\}$. By (ii), X is integral over $R + rXR[X]$; that is,

$$X^n + (a_{n-1} + rXf_{n-1})X^{n-1} + \dots + (a_0 + rXf_0) = 0$$

for some a_{n-1}, \dots, a_0 in R and f_{n-1}, \dots, f_0 in $R[X]$. Equating coefficients of X^n leads to $1 \in rR$. Hence, R is a field, giving (iv).

As Proposition 2.5(b) gives (iv) \Rightarrow (i), it now suffices to show (i) \Rightarrow (iv). Let $r \in R \setminus \{0\}$. By (i), $R + rR[X] \rightarrow R[X]$ is integral. Now, factoring out the common ideal $rR[X]$ preserves integrality. However,

$$(R + rR[X])/rR[X] \cong R/(R \cap rR[X]) = R/rR$$

and

$$R[X]/rR[X] \cong (R/rR)[X].$$

Hence, $\bar{R} = R/rR$ is such that the polynomial ring $\bar{R}[X]$ is integral over \bar{R} . We have that \bar{R} is the zero ring, whence $rR = R$, and $1 \in rR$. Therefore (iv) follows. \square

It follows from the proof of Theorem 2.13(b) that a ring extension $R \subset T$ is integral if and only if $T[\{X_\alpha\}]$ is nearly integral over $R[\{X_\alpha\}]$ for each (or some) nonempty subset $\{X_\alpha\}$ of algebraically independent indeterminates. Similarly, condition (i) in Theorem 2.13(c) has an equivalent variant in which X is replaced by $\{X_\alpha\}$.

Recall from [5] that (R, T) is called a *lying-over pair* (or *LO-pair*) in case $R \subset T$ is a ring extension such that $A \subset B$ satisfies LO for each tower $R \subset A \subset B \subset T$ of rings. Lying-over pairs with first coordinate a field have been studied extensively [5, Section 4]. We next show that if R is not a field, the concepts of “lying-over pair” and “nearly integral” jointly characterise the archetypical example of each, namely “integrality”.

THEOREM 2.14. *Let $R \subset T$ be a ring extension such that R is not a field. Then the following conditions are equivalent'*

- (1) T is nearly integral over R and (R, T) is an LO-pair;
- (2) T is integral over R .

PROOF: (2) \Rightarrow (1): apply Proposition 2.5(a) and [5, Corollary 3.3].

(1) \Rightarrow (2): Deny this. Then, by [5, Proposition 4.5], there exist a maximal ideal M of R , a nonmaximal prime ideal N of T such that $N \cap R = M$, and an element $X \in T/N$ such that X is transcendental over the field $F = R/M$ (and T/N is integral over $F[X]$). As $M \neq 0$ (since R is not a field) and T is nearly integral over R , it follows that T is integral over $R + MT$. Thus

$$A = (R + MT)/MT \rightarrow T/MT \rightarrow T/N$$

is a composite of integral maps, hence integral. As $MT \cap R = M$, we have $A \cong R/M = F$, and so T/N is integral (algebraic) over F . In particular, X is algebraic over F , the desired contradiction. \square

Remark 2.15. (a) Let R be a von Neumann regular (absolutely flat) ring which is neither a field nor 0. (For instance, $R = F \times F$ where F is a field.) Put $T = R[X]$. By Theorem 2.13(c), T is not nearly integral over R . However T is “locally nearly integral” over R , in the sense that $T_{R \setminus M}$ is nearly integral over R_M for each prime (maximal) ideal M of R . (This follows via Proposition 2.5(b) since each R_M is a field.) This example shows that the hypothesis “ R is an integral domain” cannot be deleted from Proposition 2.6(b).

(b) The hypothesis “ R is not a field” cannot be deleted from Theorem 2.14. To see this, let R be a field F and set $T = F[X]$. Then T is nearly integral over R by Proposition 2.5(b), and (R, T) is an LO-pair by [5, Proposition 2.9], but T is certainly not integral over R . A similar comment applies to Proposition 2.16 below.

(c) Theorem 2.13(a) can be used to prove the following companion to Proposition 2.9. Let R be an integral domain and let P be a nonzero prime ideal of R . Then R is

quasilocal with maximal ideal P if (and only if) P is divided in R (in the sense that $PR_P = P$) and R_P is nearly integral over R . We next indicate a proof of this result. By hypothesis, $P \subset (R : R_P)$, and so Theorem 2.13(a) implies that R_P is integral over R . Then, as in the proof of Proposition 2.9, an appeal to [11, Proposition 2] completes the proof.

(d) Theorem 2.13(b) can be used to show that “nearly integral” is not a universal property. For instance, let $f: R \rightarrow T$ be a nonintegral nearly integral inclusion map, and consider $S = R[X]$. Then $f_{(S)}: S \rightarrow S \otimes_R T \cong T[X]$ is not nearly integral since f is not integral.

In this regard, one has however the following easy positive result. If $R \rightarrow T$ is nearly integral and I is an ideal of R , then the induced map $R/I \rightarrow T/IT$ is also nearly integral.

It should be stressed that “nearly integral” often exhibits behaviour like that of “integral”. Another case in point is the following result, which is an easy consequence of facts in [9]. The direct limit of any directed family of nearly integral homomorphisms is itself nearly integral. Of course, mention of direct limits suggests finite-type subalgebras and possible uses of Zariski’s Main Theorem (ZMT). We shall close with one such result, which may be viewed as a companion to Theorem 2.14. It will be convenient to say that (R, T) is a *nearly integral-pair* in the case where $R \subset T$ is a ring extension such that B is nearly integral over A for each tower $R \subset A \subset B \subset T$ of rings. We refer the reader to [5, p.456] for the definition of “survival-pair”.

PROPOSITION 2.16. *Let R be an integral domain which is not a field and let T be an overring of R . Then the following conditions are equivalent*

- (1) (R, T) is a nearly integral-pair and a survival-pair;
- (2) T is integral over R .

PROOF: (2) \Rightarrow (1): apply Proposition 2.5(a) and [5, Theorem 2.1].

(1) \Rightarrow (2): in view of Proposition 2.10(b), this follows by applying [5, Remark 2.5]. We shall provide a self-contained proof here. By Proposition 2.7(c) and [2, Lemma 2, p.326], we may suppose R is integrally closed in T , and seek to show $R = T$. Without loss of generality, $T = R[u]$ for some element u . By “survival”, $MT \neq T$ for each maximal ideal M of R , and so some maximal ideal N of T lies over M . By Proposition 2.10(b), N is isolated in the fibre over M . Hence, by ZMT, we have $R_r = T_r$ for some $r \in R \setminus M$. Varying M , we find finitely many r_i such that $\Sigma Rr_i = R$ and $R_{r_i} = T_{r_i}$ for each i . Since $\prod R_{r_i}$ is R -faithfully flat [2, Proposition 3, p.109], it follows that $R = T$ (see [2, Proposition 1, p.27]). \square

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