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ANALYTIC LOG PICARD VARIETIES

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Dedicated to Professor Luc Illusie

Abstract. We introduce a log Picard variety over the complex number field by the method of log geometry in the sense of Fontaine-Illusie, and study its basic properties, especially, its relationship with the group of log version of $\mathbb{G}_m$-torsors.

Introduction

In [8], we introduce the notions log complex torus and log abelian variety over $\mathbb{C}$, which are new formulations of degenerations of complex torus and abelian variety over $\mathbb{C}$, and compare them with the theory of log Hodge structures. Classical theories of semi-stable degenerations of abelian varieties over $\mathbb{C}$ can be regarded in our theory as theories of proper models of log abelian varieties.

In this paper, we introduce the notion of log Picard variety over $\mathbb{C}$. Log Picard varieties are some kind of degenerations of Picard varieties, which live in the world of log geometry in the sense of Fontaine-Illusie. We define an analytic log Picard variety as a log complex torus by the method of log Hodge theory via [8], and study its relationship with the group of $\mathbb{G}_{m,\log}$-torsors.

If we take proper models, our construction is similar to Namikawa's one ([20]). See [6] and [7] for some arithmetic studies of log Picard varieties in the framework of log geometry by using the group of $\mathbb{G}_{m,\log}$-torsors.

We also define log Albanese variety and discuss several open problems.

This paper is logically a continuation of [8] and [9]. Though [9] mainly concerns the algebraic theory of log abelian varieties, it contains some analytic computations, which we will use in Section 6 of this paper.

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§1. Review of the classical theory

1.1. Let $\mathcal{A}$ be the category of complex tori and let $\mathcal{H}$ be the category of Hodge structures H of weight -1 satisfying $F^{-1}H_{\mathbb{C}} = H_{\mathbb{C}}$ and $F^1H_{\mathbb{C}} = 0$. Then we have an equivalence

$$\mathcal{H} \simeq \mathcal{A}$$

which sends an object H of $\mathcal{H}$ to the complex torus $H_{\mathbb{Z}}\backslash H_{\mathbb{C}}/F^0H_{\mathbb{C}}$.

Let $\mathcal{A}^+$ be the category of abelian varieties over $\mathbb{C}$ and let $\mathcal{H}^+$ be the full subcategory of $\mathcal{H}$ consisting of polarizable objects. Then the above equivalence induces an equivalence

$$\mathcal{H}^+ \simeq \mathcal{A}^+.$$

1.2. For a complex torus A corresponding to an object H of $\mathcal{H}$, the complex torus A^* corresponding to the object $\mathrm{Hom}(H, \mathbb{Z})(1)$ is called the dual complex torus of A. We have

$$A^* \simeq \mathrm{Ext}^1(A, \mathbb{G}_m).$$

If A is an abelian variety, A^* is also an abelian variety.

1.3. Let P be a compact Kähler manifold. Then for each integer m, $H_{\mathbb{Z}} = H^m(P, \mathbb{Z})/(\text{torsion})$ with the Hodge filtration on $H_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}$ is a Hodge structure of weight m, which we denote by $H^m(P)$. The dual of $H^1(P)$ is an object of $\mathcal{H}$. The complex torus A_P corresponding to the dual of $H^1(P)$ under the equivalence in 1.1 is called the Albanese variety of P. The twist $H^1(P)(1)$ is also an object of $\mathcal{H}$. The complex torus A_P^* corresponding to $H^1(P)(1)$, i.e., the dual complex torus of A_P, is called the Picard variety of P.

If P is a projective manifold, A_P and A_P^* are abelian varieties.

1.4. For the Picard variety, we have a canonical embedding

$$A_P^* \subset H^1(P, \mathbb{G}_m).$$

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1.5. Assume that P is connected. Then the Albanese variety A_P of P has the following universality. Fix $e \in P$. Then there exists a unique morphism $\psi_e : P \to A_P$ called the Albanese map of P with respect to e satisfying the following (i) and (ii).

(i) $\psi_e(e) = 0$.

(ii) For any complex torus B, the map

$$\mathrm{Hom}(A_P, B) \longrightarrow \{\text{morphism } f : P \to B \mid f(e) = 0\}; \quad h \longmapsto h \circ \psi_e$$

is bijective.

§2. Log versions (summary)

In this paper, the above 1.1–1.4 are generalized to the log versions 2.1–2.4, respectively. In this section, we give rough descriptions of the generalizations. See later sections for details.

Let S be an fs log analytic space.

2.1. This part was done in [8].

Let $\mathcal{A}_S$ be the category of log complex tori over S and let $\mathcal{H}_S$ be the category of log Hodge structures H on S of weight -1 satisfying $F^{-1}H_{\mathcal{O}} = H_{\mathcal{O}}$ and $F^1 H_{\mathcal{O}} = 0$. Then we have an equivalence

$$\mathcal{H}_S \simeq \mathcal{A}_S.$$

See 3.1–3.3 for details.

Let $\mathcal{A}_S^+$ be the category of log abelian varieties over S and let $\mathcal{H}_S^+$ be the full subcategory of $\mathcal{H}_S$ consisting of all objects whose pull backs to all points of S are polarizable. Then the above equivalence induces an equivalence

$$\mathcal{H}_S^+ \simeq \mathcal{A}_S^+.$$

2.2. For a log complex torus A over S corresponding to an object H of $\mathcal{H}_S$, the log complex torus A^* corresponding to the object $\mathcal{H}om(H, \mathbb{Z})(1)$ is called the dual log complex torus of A.

We have

$$\mathcal{E}xt^1(A, \mathbb{G}_m) \subset A^* \subset \mathcal{E}xt^1(A, \mathbb{G}_{m,\log}).$$

See 6.1 for details.

If A is a log abelian variety, A^* is also a log abelian variety.

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2.3. Let $f\colon P \to S$ be a proper, separated and log smooth morphism of fs log analytic spaces. To discuss the log Albanese and the log Picard varieties of P/S, we assume some conditions on P/S which roughly say that higher direct images $R^m f_*^{\log}\mathbb{Z}$ carry natural log Hodge structures $\mathcal{H}^m(P)$ on S for some m. See Section 7 for the precise conditions. For example, if S is log smooth over $\mathbb{C}$, and if f is projective locally over S, vertical, and for any $p \in P$, the cokernel of $M^{\mathrm{gp}}_{S,f(p)}/\mathcal{O}^\times_{S,f(p)} \to M^{\mathrm{gp}}_{P,p}/\mathcal{O}^\times_{P,p}$ is torsion free, then the above condition on $R^m f_*^{\log}\mathbb{Z}$ is satisfied for any m. Here f is said to be *vertical* if for any $p \in P$, any element of $M_{P,p}$ divides the image of some element of $M_{S,f(p)}$. See Section 9 for further discussions of when the conditions are satisfied. In the rest of this paragraph, we assume the above condition on $R^m f_*^{\log}\mathbb{Z}$ for $m = 1$. Then the dual of $\mathcal{H}^1(P)$ and $\mathcal{H}^1(P)(1)$ are objects of $\mathcal{H}_S$. We define the log Albanese variety $A_{P/S}$ as the log complex torus corresponding to the dual log Hodge structure of $\mathcal{H}^1(P)$, and the log Picard variety $A^*_{P/S}$ as the log complex torus corresponding to $\mathcal{H}^1(P)(1)$.

2.4. For the log Picard variety, we have a canonical embedding

$$A^*_{P/S} \subset \mathcal{H}^1(P, \mathbb{G}_{m,\log})$$

under some conditions. See 8.2 for more details.

2.5. As in the classical case 1.5, we expect that the log Albanese varieties have the universal property. We discuss this, a partial result, and related problems in Section 10.

2.6. Plan of this paper. In Section 3, we review some concerned parts of [8]. In Section 4, we calculate the log Betti cohomologies of a log complex torus. In Section 5, we introduce several variants of the extension group of the unit log Hodge structure $\mathbb{Z}$ by the log Hodge structure corresponding to the log complex torus, which should be related to the log Picard variety. In Section 6, we relate these groups with some geometric extension groups for the use in Section 8. In Section 7, we introduce the conditions mentioned in 2.3. See Section 9 for the situation when they are satisfied. Under this condition, we prove our main theorem on log Picard varieties in Section 8. Section 10 discusses some problems including these on log Albanese varieties.

§3. Review of the paper [8]

Here we recall an equivalence of the category of log complex tori and that of log Hodge structures, and models of log complex tori.

Let S be an fs log analytic space.

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3.1. We review the functors which give the equivalence of the categories stated in 2.1. We define a functor from $\mathcal{H}_S$ to $\mathcal{A}_S$ in this paragraph, and its inverse functor from $\mathcal{A}_S$ to $\mathcal{H}_S$ in 3.3 after a preliminary in 3.2. For an object H of $\mathcal{H}_S$, we define the sheaf of abelian groups $\mathcal{E}xt^1(\mathbb{Z}, H)$ on (fs/S) by

$$\mathcal{E}xt^1(\mathbb{Z}, H)(T) = \mathrm{Ext}^1(\mathbb{Z}, H_T)$$

for fs log analytic spaces T over S, where H_T denotes the pull back of H to T, and Ext^1 is taken for the category of log mixed Hodge structures over T. Note that the category of log mixed Hodge structures has the evident definitions of "exact sequence" and "extension (short exact sequence)". We consider Ext^1 as the set of isomorphism classes of extensions, with the group structure given by Baer sums.

We proved in [8] that the above $\mathcal{E}xt^1(\mathbb{Z}, H)$ is a log complex torus over S. This gives the functor from $\mathcal{H}_S$ to $\mathcal{A}_S$. When H belongs to $\mathcal{H}_S^+$, the sheaf $\mathcal{E}xt^1(\mathbb{Z}, H)$ is a log abelian variety so that the functor $\mathcal{H}_S \to \mathcal{A}_S$ induces $\mathcal{H}_S^+ \to \mathcal{A}_S^+$.

3.2. The site $(\mathrm{fs}/S)^{\log}$.

To define the inverse functor, we review the site $(\mathrm{fs}/S)^{\log}$.

Let $(\mathrm{fs}/S)^{\log}$ be the following site. An object of $(\mathrm{fs}/S)^{\log}$ is a pair (U, T), where T is an fs log analytic space over S and U is an open set of $T^{\log}$. The morphisms are defined in the evident way. A covering is a family of morphisms $((U_\lambda, T_\lambda) \to (U, T))_\lambda$, where each $T_\lambda \to T$ is an open immersion and the log structure of T_λ is the inverse image of that of T, and $(U_\lambda)_\lambda$ is an open covering of U.

We have a morphism of topoi $\{\text{sheaf on } (\mathrm{fs}/S)^{\log}\} \xrightarrow{\tau} \{\text{sheaf on } (\mathrm{fs}/S)\}$. This is defined as follows. For a sheaf F on $(\mathrm{fs}/S)^{\log}$, the image $\tau_*(F)$ on (fs/S) is defined by $\tau_*(F)(T) = F(T^{\log}, T)$. For a sheaf F on (fs/S), the inverse image $\tau^{-1}(F)$ on $(\mathrm{fs}/S)^{\log}$ is defined as follows. For an object (U, T) of $(\mathrm{fs}/S)^{\log}$, the restriction of $\tau^{-1}(F)$ to the usual site of open sets of U (i.e., the restriction to (U', T) for open sets U' of U) coincides with the inverse image of the restriction of F to the site of open sets of T under the composite map $U \to T^{\log} \to T$. The functor $\tau_*\tau^{-1}$ is naturally equivalent to the identity functor.

We will denote the sheaf $(U, T) \mapsto \mathcal{O}_T^{\log}(U)$ on $(\mathrm{fs}/S)^{\log}$ simply by $\mathcal{O}_S^{\log}$.

3.3. Now we describe the inverse functor $\mathcal{A}_S \to \mathcal{H}_S; A \mapsto H$. For a log complex torus A over S, the Ext^1 sheaf $\mathcal{E}xt^1(\tau^{-1}(A), \mathbb{Z})$ on $(\mathrm{fs}/S)^{\log}$ for the inverse image $\tau^{-1}(A)$ of A on $(\mathrm{fs}/S)^{\log}$ is a locally constant sheaf of finitely generated free abelian groups of $\mathbb{Z}$-rank $2\dim(A)$. Here $\dim(A)$ is understood as a locally constant function on S ([8] 3.7.4). We define

$$H_{\mathbb{Z}} = \mathcal{H}om_{\mathbb{Z}}(\mathcal{E}xt^1(\tau^{-1}(A), \mathbb{Z}), \mathbb{Z}).$$

Next we define

$$H_{\mathcal{O}} = \tau_*(\mathcal{O}_S^{\log} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}).$$

The canonical homomorphism $\mathcal{O}_S^{\log} \otimes_{\tau^{-1}(\mathcal{O}_S)} \tau^{-1}(H_{\mathcal{O}}) \to \mathcal{O}_S^{\log} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}$ is an isomorphism. Furthermore, there is a canonical surjective homomorphism $H_{\mathcal{O}} \to \mathrm{Lie}(A)$ of $\mathcal{O}_S$-modules. We define $F^p H_{\mathcal{O}}$ to be $H_{\mathcal{O}}$ if $p \le -1$, $\mathrm{Ker}(H_{\mathcal{O}} \to \mathrm{Lie}(A))$ if $p = 0$, and 0 if $p \ge 1$. Then this gives an object H of $\mathcal{H}_S$.

3.4. Models. We review models of log complex tori. Let A be a log complex torus over S. Let $\mathbb{G}_{m,\log} = \mathbb{G}_{m,\log,S}$ be the sheaf on (fs/S) defined by $\mathbb{G}_{m,\log}(T) = \Gamma(T, M_T^{\mathrm{gp}})$ for $T \in (\mathrm{fs}/S)$. Let the situation be as in [8] 5.1.1, that is, there exist finitely generated free $\mathbb{Z}$-modules X and Y, and a non-degenerate pairing $\langle\ ,\ \rangle\colon X \times Y \to \mathbb{G}_{m,\log}$ over S such that A is its associated quotient $Y \backslash \mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)}$, and there exist an fs monoid $\mathcal{S}$, $\mathcal{S}$-admissible pairing $X \times Y \to \mathcal{S}^{\mathrm{gp}}$, and a homomorphism $\mathcal{S} \to M_S/\mathcal{O}_S^\times$ of fs monoids such that the induced map $X \times Y \to M_S^{\mathrm{gp}}/\mathcal{O}_S^\times$ coincides with $\langle\ ,\ \rangle$ modulo $\mathbb{G}_m$. Here $\mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)} \subset \mathcal{H}om(X, \mathbb{G}_{m,\log})$ is the (Y)-part of $\mathcal{H}om(X, \mathbb{G}_{m,\log})$ defined in [8] 1.3.1, that is, for an object T of (fs/S),

$$\begin{aligned}&\mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)}(T)\\ &\quad := \big\{\varphi \in \mathrm{Hom}(X, M_T^{\mathrm{gp}}) \mid \text{for each } x \in X, \text{ locally on } T,\\ &\qquad\quad \text{there exist } y, y' \in Y \text{ such that } \langle x, y\rangle | \varphi(x) | \langle x, y'\rangle \text{ in } M_T^{\mathrm{gp}}\big\},\end{aligned}$$

where for $f, g \in M_T^{\mathrm{gp}}$, $f \mid g$ means $f^{-1}g \in M_T$. Note that such data always exist locally on S.

Now we consider the cone

$$C := \big\{(N, l) \in \mathrm{Hom}(\mathcal{S}, \mathbb{N}) \times \mathrm{Hom}(X, \mathbb{Z}) \mid l(X_{\mathrm{Ker}(N)}) = \{0\}\big\}$$

([8] 3.4.2). A cone decomposition Σ is by definition a fan in $\mathrm{Hom}(\mathcal{S}^{\mathrm{gp}} \times X, \mathbb{Q})$ whose support is contained in the cone $C_{\mathbb{Q}}$ of the non-negative rational linear combinations of elements of C. Assume that Σ is stable under the action of Y, where $y \in Y$ acts on C by $(N, l) \mapsto (N, l + N(\langle\ , y\rangle))$. Then we define the subsheaf $A^{(\Sigma)}$ of A as $Y \backslash \mathcal{H}om(X, \mathbb{G}_{m,\log})^{(\Sigma)}$, where $\mathcal{H}om(X, \mathbb{G}_{m,\log})^{(\Sigma)} =$

$\bigcup_{\Delta\in\Sigma} V(\Delta) \subset \mathcal{H}om(X,\mathbb{G}_{m,\log})^{(Y)}$. Here $V(\Delta) \subset \mathcal{H}om(X,\mathbb{G}_{m,\log})$ is the (Δ)-part of $\mathcal{H}om(X,\mathbb{G}_{m,\log})$ defined in [8] 3.5.2, that is, for an object T of (fs/S),

$$V(\Delta)(T) := \{\varphi \in \mathrm{Hom}(X, M_T^{\mathrm{gp}}) \mid \mu\cdot(\varphi(x) \bmod \mathcal{O}_T^\times) \in M_T/\mathcal{O}_T^\times \text{ for every } (\mu,x)\in\Delta^\vee\}.$$

This $A^{(\Sigma)}$ is a subsheaf of A and is always representable in the category of fs log analytic spaces and the representing object, which is also denoted by $A^{(\Sigma)}$, is called the model of A associated to Σ. We say that a model is a proper model if it is proper over S. There always exists a fan Σ such that $A^{(\Sigma)}$ is proper.

§4. Log Betti cohomology

In the classical theory, if A is a complex torus, we have $H^1(A(\mathbb{C}),\mathbb{Z}) \simeq \mathbb{Z}^{2g}$, where $g = \dim(A)$, and the cup product induces an isomorphism $\bigwedge^m H^1(A(\mathbb{C}),\mathbb{Z}) \xrightarrow{\simeq} H^m(A(\mathbb{C}),\mathbb{Z})$ for any m. In this section, we prove the log version of these.

4.1. For a sheaf F on $(\mathrm{fs}/S)^{\log}$, $\mathcal{H}^m(F,\ \ )$ denotes the right derived functor of the direct image functor {abelian sheaf on $(\mathrm{fs}/S)^{\log}/F$} $\to$ {abelian sheaf on $(\mathrm{fs}/S)^{\log}$}.

For a sheaf F on (fs/S), $\mathcal{H}^m(F,\ \ )$ denotes the right derived functor of the direct image functor {abelian sheaf on $(\mathrm{fs}/S)/F$} $\to$ {abelian sheaf on (fs/S)}.

THEOREM 4.2. *Let A be a log complex torus over S.*

(1) *There is a natural isomorphism*

$$\mathcal{E}xt^1(\tau^{-1}(A),\mathbb{Z}) = \mathcal{H}^1(\tau^{-1}(A),\mathbb{Z})$$

of locally constant sheaves on $(\mathrm{fs}/S)^{\log}$.

(2) *The cup product induces an isomorphism*

$$\bigwedge^m \mathcal{H}^1(\tau^{-1}(A),\mathbb{Z}) \simeq \mathcal{H}^m(\tau^{-1}(A),\mathbb{Z}).$$

(3) *We have*

$$\mathcal{E}xt^m(\tau^{-1}(A),\mathbb{Z}) = 0 \quad \textit{for all } m \neq 1.$$

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(4) *Let the situation be as in* 3.4. *Let P be a proper model of A. Then we have*

$$\mathcal{H}^m(\tau^{-1}(A),\mathbb{Z}) \simeq \mathcal{H}^m(\tau^{-1}(P),\mathbb{Z}) \quad \textit{for all } m.$$

(5) *Let H be the object of $\mathcal{H}_S$ corresponding to the dual of A. Then*

$$\tau_* H_{\mathbb{Z}} = \mathcal{E}xt^1(A,\mathbb{Z}).$$

4.3. We first prove (4). This is reduced to $Rf_*^{\log}\mathbb{Z} = \mathbb{Z}$, where f is the canonical morphism $P \to A$ and $f^{\log}$ is the induced morphism $\tau^{-1}(P) \to \tau^{-1}(A)$. We use

LEMMA 4.3.1. *Let $g\colon X \to Y$ be a morphism which is locally a base change of a birational proper equivariant morphism of toric varieties. Then $Rg_*^{\log}\mathbb{Z} = \mathbb{Z}$.*

Proof. By [10] Propositions 5.3 and 5.3.2. ∎

To reduce the above $Rf_*^{\log}\mathbb{Z} = \mathbb{Z}$ to this lemma, it is enough to show that for any $T \in (\mathrm{fs}/A)$, the morphism $T \times_A P \to T$ is represented by a morphism which is locally a base change of a birational proper equivariant morphism of toric varieties. We will prove this. By $\mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)} = \bigcup_\Delta V(\Delta)$ ([8] 3.5.4; see 3.4 for the definition of $V(\Delta)$), the canonical map $\coprod_\Delta V(\Delta) \to A$ is surjective as a map of sheaves, where Δ ranges over all finitely generated subcones of C. Here C is the cone in 3.4. Hence we may assume that $T = V(\Delta)$ for some Δ. On the other hand, let Σ be the complete cone decomposition in C which defines the model P. Let $\Sigma \sqcap \Delta$ be the fan $\{\sigma \cap \tau \mid \sigma \in \Sigma, \tau \text{ is a face of } \Delta\}$. Then it is easy to see that $T \times_A P$ is represented by $V(\Sigma \sqcap \Delta)$, which is an fs log analytic space over $T = V(\Delta)$ whose structure morphism is a base change of a birational proper equivariant morphism of toric varieties. Thus the desired representability is proved, and hence (4) is proved.

4.4. Next we prove (2). We may assume that the situation is as in 3.4. By (4), it is sufficient to prove that $\bigotimes^m \mathcal{H}^1(\tau^{-1}(P),\mathbb{Z}) \to \mathcal{H}^m(\tau^{-1}(P),\mathbb{Z})$ induces the isomorphism $\bigwedge^m \mathcal{H}^1(\tau^{-1}(P),\mathbb{Z}) \simeq \mathcal{H}^m(\tau^{-1}(P),\mathbb{Z})$. Since any log complex torus, locally on the base, comes from a log smooth base ([8] Proposition 3.10.3), and since models are, locally on the base, constructed already on the log smooth base, we are reduced to the log smooth base case. Then we are reduced to the case where the log structure of the base is

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trivial because $\mathcal{H}^m(\tau^{-1}(P),\mathbb{Z})$ is locally constant by [10] Theorem 0.1 and by the proper base change theorem. (We note that if $P \to S$ is proper and separated, $P^{\log} \to S^{\log}$ is also proper and separated ([10] Lemma 3.2.1).) Hence (2) is proved.

4.5. We prove (1). First recall that $\mathcal{E}xt^1(\tau^{-1}(A),\mathbb{Z})$ is a locally constant sheaf on $(\mathrm{fs}/S)^{\log}$ (3.3). Next, $\mathcal{H}^1(\tau^{-1}(A),\mathbb{Z})$ is also locally constant by (4) and [10]. Hence (1) is reduced to the classical case, for, as in the previous paragraph, any log complex torus comes locally on the base from a log complex torus over a log smooth base. In the classical theory, it is known (or follows from the argument in the next paragraph using the exact sequence $0 \to H_{\mathbb{Z}} \to V \to A \to 0$ ($V = \mathrm{Lie}(A)$) and the spectral sequence which converges to $\mathcal{E}xt$).

4.6. We prove (3). By [19], [4], we have a resolution $M_*(\tau^{-1}(A)) \to \tau^{-1}(A)$ of $\tau^{-1}(A)$ as in [1] §3. It gives a spectral sequence which converges to $\mathcal{E}xt^m(\tau^{-1}(A),\mathbb{Z})$ and which has $\mathcal{H}^j(\mathbb{Z}^s \times \tau^{-1}(A^t),\mathbb{Z})$ for various s and t as $E_1^{*,j}$-terms ($j > 0$) and has the kernel of the canonical map $\mathcal{H}^0(\mathbb{Z}^s \times \tau^{-1}(A^t),\mathbb{Z}) \to \mathbb{Z}$ as $E_1^{*,0}$-terms. By this spectral sequence, by [10] Theorem 0.1 and by the fact that $X^{\log}$ is locally connected for any fs log analytic space X ([10] Lemma 3.6), we see that $\mathcal{E}xt^m(\tau^{-1}(A),\mathbb{Z})$ is a successive extension of locally constant sheaves. Hence we are reduced to the classical case. In that case, by the exact sequence $0 \to H_{\mathbb{Z}} \to V \to A \to 0$ ($V = \mathrm{Lie}(A)$), we see that it is enough to show $\mathcal{E}xt^m(V,\mathbb{Z}) = 0$ for any m, which is seen by the same kind of spectral sequence for V.

4.7. Before we prove (5), we give a preliminary on spectral sequences which relate $\mathcal{E}xt^m$ for sheaves on $(\mathrm{fs}/S)^{\log}$ with $\mathcal{E}xt^m$ for sheaves on (fs/S). Let F be a sheaf of abelian groups on (fs/S). For a sheaf of abelian groups G on $(\mathrm{fs}/S)^{\log}$, let

$$\theta_F(G) := \tau_*\,\mathcal{H}om(\tau^{-1}(F),G) = \mathcal{H}om(F,\tau_*(G)).$$

Let $R^m\theta_F$ be the m-th right derived functor of θ_F. We have spectral sequences

$$E_2^{p,q} = R^p\tau_*\,\mathcal{E}xt^q(\tau^{-1}(F),G) \Rightarrow E_\infty^m = R^m\theta_F(G), \tag{1}$$

$$E_2^{p,q} = \mathcal{E}xt^p(F,R^q\tau_*G) \Rightarrow E_\infty^m = R^m\theta_F(G). \tag{2}$$

4.8. We prove (5). Let $F = A$ and $G = \mathbb{Z}$ in 4.7. By the spectral sequence (1) in 4.7 and by $\mathcal{H}om(\tau^{-1}(A),\mathbb{Z}) = 0$ ([8] 3.7.5), we have

$$R^1\theta_A(\mathbb{Z}) = \tau_*\,\mathcal{E}xt^1(\tau^{-1}(A),\mathbb{Z}) = \tau_*(H_{\mathbb{Z}}).$$

By the spectral sequence (2) of 4.7 and the fact that any homomorphism

$$A \longrightarrow R^1\tau_*\mathbb{Z} = \mathbb{G}_{m,\log}/\mathbb{G}_m$$

is the zero map (which is seen by the fact that any morphism from $V(\Delta)$ to $\mathbb{G}_{m,\log}/\mathbb{G}_m$ is locally constant on S), we have

$$R^1\theta_A(\mathbb{Z}) = \mathcal{E}xt^1(A, \mathbb{Z}).$$

§5. Subgroups of $\mathcal{E}xt^1_{\mathrm{naive}}(\mathbb{Z}, H)$

Let S be an fs log analytic space and let H be an object of $\mathcal{H}_S$.

In this section, we consider sheaves of abelian groups on (fs/S) related to $\mathcal{E}xt^1(\mathbb{Z}, H)$, having the following relations:

$$G_H \subset \mathcal{E}xt^1(\mathbb{Z}, H) \subset \mathcal{E}xt^1_{\mathrm{naive}}(\mathbb{Z}, H)_0 \subset \mathcal{E}xt^1_{\mathrm{naive}}(\mathbb{Z}, H).$$

5.1. As in [8] 3.6, let

$$\mathcal{V}_H = (\mathcal{O}_S^{\log} \otimes H_{\mathbb{Z}})/(\mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} F^0 H_{\mathcal{O}}),$$

and define

$$\mathcal{E}xt^1_{\mathrm{naive}}(\mathbb{Z}, H) := \tau_*(H_{\mathbb{Z}} \backslash \mathcal{V}_H).$$

The exact sequence

$$0 \longrightarrow H_{\mathbb{Z}} \longrightarrow \mathcal{V}_H \longrightarrow H_{\mathbb{Z}} \backslash \mathcal{V}_H \longrightarrow 0$$

of sheaves on $(\mathrm{fs}/S)^{\log}$ induces (take $R\tau_*$ and use $R\tau_*\mathcal{O}^{\log} = \mathcal{O}$ ([5] (3.7))) an exact sequence

$$0 \longrightarrow \tau_*(H_{\mathbb{Z}}) \longrightarrow H_{\mathcal{O}}/F^0 H_{\mathcal{O}} \longrightarrow \mathcal{E}xt^1_{\mathrm{naive}}(\mathbb{Z}, H) \longrightarrow R^1\tau_*(H_{\mathbb{Z}}) \longrightarrow 0.$$

As in [8], $\mathcal{E}xt^1(\mathbb{Z}, H)$ is embedded in $\mathcal{E}xt^1_{\mathrm{naive}}(\mathbb{Z}, H)$. In the following, we show that $\mathcal{E}xt^1(\mathbb{Z}, H)$ is the inverse image of a certain subgroup sheaf of $R^1\tau_*(H_{\mathbb{Z}})$ under the above connecting homomorphism $\mathcal{E}xt^1_{\mathrm{naive}}(\mathbb{Z}, H) \to R^1\tau_*(H_{\mathbb{Z}})$. We also consider the other subgroup sheaves G_H and $\mathcal{E}xt^1_{\mathrm{naive}}(\mathbb{Z}, H)_0$ of $\mathcal{E}xt^1_{\mathrm{naive}}(\mathbb{Z}, H)$, which are also the inverse images of certain subgroup sheaves of $R^1\tau_*(H_{\mathbb{Z}})$.

5.2. Let

$$\begin{aligned} G_H &= \tau_*(H_{\mathbb{Z}}) \backslash (H_{\mathcal{O}}/F^0 H_{\mathcal{O}}) \\ &= \mathrm{Ker}(\mathcal{E}xt^1_{\mathrm{naive}}(\mathbb{Z}, H) \to R^1\tau_*(H_{\mathbb{Z}})). \end{aligned}$$

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5.3. We consider $R^1\tau_*(H_{\mathbb{Z}})$. Let

$$Y_H = H_{\mathbb{Z}}/\tau^{-1}\tau_* H_{\mathbb{Z}}.$$

Then $\tau^{-1}\tau_* Y_H = Y_H$. We will often denote $\tau_* Y_H$ simply by Y_H.

The evident exact sequence

$$0 \longrightarrow \tau^{-1}\tau_*(H_{\mathbb{Z}}) \longrightarrow H_{\mathbb{Z}} \longrightarrow Y_H \longrightarrow 0$$

induces (by taking $R\tau_*$ and using $R^1\tau_*\mathbb{Z} = \mathbb{G}_{m,\log}/\mathbb{G}_m$ ([13] (1.5))) an exact sequence

$$0 \longrightarrow Y_H \xrightarrow{\partial} \tau_*(H_{\mathbb{Z}}) \otimes \mathbb{G}_{m,\log}/\mathbb{G}_m \longrightarrow R^1\tau_*(H_{\mathbb{Z}}) \longrightarrow Y_H \otimes \mathbb{G}_{m,\log}/\mathbb{G}_m.$$

5.4. Let $\mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H)_0 \subset \mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H)$ be the inverse image of $(\tau_*(H_{\mathbb{Z}}) \otimes \mathbb{G}_{m,\log}/\mathbb{G}_m)/Y_H \subset R^1\tau_*(H_{\mathbb{Z}})$ by the connecting homomorphism $\mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H) \to R^1\tau_*(H_{\mathbb{Z}})$.

5.5. We define the subgroup sheaf $(\tau_*(H_{\mathbb{Z}}) \otimes \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y_H)}$ of $\tau_*(H_{\mathbb{Z}}) \otimes \mathbb{G}_{m,\log}/\mathbb{G}_m$ as follows:

For an fs log analytic space T over S and for $\varphi \in (\tau_*(H_{\mathbb{Z}}) \otimes \mathbb{G}_{m,\log}/\mathbb{G}_m)(T)$, φ belongs to $(\tau_*(H_{\mathbb{Z}}) \otimes \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y_H)}(T)$ if and only if for each $t \in T$ and each x in $\mathrm{Hom}((\tau_*(H_{\mathbb{Z}}))_t, \mathbb{Z})$, there exist $y_1, y_2 \in Y_H$ such that $x(y_1)|x(\varphi)|x(y_2)$ in $M^{\text{gp}}_{T,t}/\mathcal{O}^\times_{T,t}$.

Here for $f, g \in M^{\text{gp}}_{T,t}/\mathcal{O}^\times_{T,t}$, $f|g$ means $f^{-1}g \in M_{T,t}/\mathcal{O}^\times_{T,t}$, and $x(y_j)$ is defined by the above boundary map ∂.

By the embedding $Y_H \subset \tau_*(H_{\mathbb{Z}}) \otimes \mathbb{G}_{m,\log}/\mathbb{G}_m$, we have $Y_H \subset (\tau_*(H_{\mathbb{Z}}) \otimes \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y_H)}$.

Proposition 5.6. *$\mathcal{E}xt^1(\mathbb{Z}, H)$ coincides with the inverse image of $(\tau_*(H_{\mathbb{Z}}) \otimes \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y_H)}/Y_H$ in $\mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H)$.*

Proof. We may assume that H comes from a non-degenerate pairing $(X, Y, \langle\ ,\ \rangle)$ into $\mathbb{G}_{m,\log}$ over S.

Taking τ_* of the morphism $\mathcal{H}om(X, \mathbb{Z})\backslash\mathcal{V}_H \to H_{\mathbb{Z}}\backslash\mathcal{V}_H$ and using $\tau_*(\mathcal{O}^{\log}/\mathbb{Z}(1)) = \mathbb{G}_{m,\log}$ ([8] Lemma 3.2.5), we have the morphism $f\colon \mathcal{H}om(X, \mathbb{G}_{m,\log}) \to \mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H)$. By [8] Proposition 3.6.4, $\mathcal{E}xt^1(\mathbb{Z}, H)$ is the image of $\mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)}$ by f. Since $\mathcal{H}om(X, \mathbb{G}_{m,\log}) \xrightarrow{f} \mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H) \xrightarrow{\partial} R^1\tau_*(H_{\mathbb{Z}})$ is compatible with $g\colon \mathcal{H}om(X, \mathbb{G}_{m,\log}) = \tau_*(\mathcal{H}om(X, \mathbb{Z})\backslash\mathcal{V}_H) \xrightarrow{\partial} R^1\tau_*\mathcal{H}om(X, \mathbb{Z}) = \mathcal{H}om(X, \mathbb{Z}) \otimes \mathbb{G}_{m,\log}/\mathbb{G}_m \to$

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$\tau_*(H_{\mathbb{Z}}) \otimes \mathbb{G}_{m,\log}/\mathbb{G}_m$, the desired statement is reduced to that the map $h\colon \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m) \hookrightarrow \tau_*(H_{\mathbb{Z}}) \otimes \mathbb{G}_{m,\log}/\mathbb{G}_m$, induced by g, sends $\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}$ onto $(\tau_*(H_{\mathbb{Z}}) \otimes \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y_H)}$. To see it, we will work at stalks. Let $T \in (\mathrm{fs}/S)$ and $t \in T$. Let $x \in \mathrm{Hom}((\tau_*(H_{\mathbb{Z}}))_t, \mathbb{Z})$ and $y \in Y$, denote by $\bar{x}$ the image of x in X and denote by $\bar{y}$ the image of y in Y_H. Then we have $x(\bar{y}) = \langle \bar{x}, y \rangle$ (cf. [8] Proposition 3.3.5). This implies that h sends the (Y)-part into the (Y_H)-part. Conversely, let $\varphi \in (\tau_*(H_{\mathbb{Z}}) \otimes \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y_H)}$. Then for any $x \in \mathrm{Hom}((\tau_*(H_{\mathbb{Z}}))_t, \mathbb{Z})$, there exist $y_1, y_2 \in Y_H$ such that $x(y_1) | x(\varphi) | x(y_2)$ in $M^{\mathrm{gp}}_{T,t}/\mathcal{O}^{\times}_{T,t}$. If $\bar{x} = 0$, then $x(\bar{y}_j) = \langle \bar{x}, y_j \rangle = 1$ $(j = 1, 2)$. Hence $x(\varphi) = 1$ in $M^{\mathrm{gp}}_{T,t}/\mathcal{O}^{\times}_{T,t}$. This implies that φ belongs to $\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)$, and belongs to the (Y)-part. Thus we have the desired surjectivity. □

PROPOSITION 5.7. *G_H is a commutative Lie group over S.*

Proof. We may assume that H comes from a non-degenerate pairing $(X, Y, \langle\ ,\ \rangle)$ into $\mathbb{G}_{m,\log}$ over S. We may further assume that we are given an $\mathcal{S}$-admissible pairing $X \times Y \to \mathcal{S}^{\mathrm{gp}}$ with an fs monoid $\mathcal{S}$ and a homomorphism $\mathcal{S} \to M_S/\mathcal{O}_S^{\times}$ such that the induced map $X \times Y \to M_S^{\mathrm{gp}}/\mathcal{O}_S^{\times}$ coincides with $\langle\ ,\ \rangle$ modulo $\mathbb{G}_m$.

Let Σ be the fan determined by all faces of the monoid $\mathrm{Hom}(\mathcal{S}, \mathbb{N})$. Then $G_H = A(\Sigma)$, where A is the log complex torus corresponding to H. Thus we conclude that G_H is a smooth analytic space over S by [8] Theorem 5.3.2 (1). □

§6. Dual log complex tori and $\mathcal{E}xt^1(A, \mathbb{G}_{m,\log})$

Let S be an fs log analytic space.

In this section we prove the following.

THEOREM 6.1. *Let A be a log complex torus over S and let H be the log Hodge structure corresponding to the dual A^* of A. Then there are canonical isomorphisms*

$$\mathcal{E}xt^1(A, \mathbb{G}_m) \simeq G_H, \quad \mathcal{E}xt^1(A, \mathbb{G}_{m,\log}) \simeq \mathcal{E}xt^1_{\mathrm{naive}}(\mathbb{Z}, H)_0$$

for which the following diagram is commutative.

$$\begin{array}{ccc} \mathcal{E}xt^1(A, \mathbb{G}_m) & \subset & \mathcal{E}xt^1(A, \mathbb{G}_{m,\log}) \\ \wr\| & & \wr\| \\ G_H & \subset & \mathcal{E}xt^1_{\mathrm{naive}}(\mathbb{Z}, H)_0. \end{array}$$

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In particular, we have canonical embeddings

$$\mathcal{E}xt^1(A, \mathbb{G}_m) \subset A^* \subset \mathcal{E}xt^1(A, \mathbb{G}_{m,\log}).$$

More precisely, we have an exact sequence

$$0 \longrightarrow \mathcal{E}xt^1(A, \mathbb{G}_m) \longrightarrow \mathcal{E}xt^1(A, \mathbb{G}_{m,\log}) \longrightarrow (\tau_*(H_{\mathbb{Z}}) \otimes \mathbb{G}_{m,\log}/\mathbb{G}_m)/Y_H \longrightarrow 0,$$

and $A^* \subset \mathcal{E}xt^1(A, \mathbb{G}_{m,\log})$ *coincides with the inverse image of* $(\tau_*(H_{\mathbb{Z}}) \otimes \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y_H)}/Y_H$.

6.2. By taking Lie of $\mathcal{E}xt^1(A, \mathbb{G}_{m,\log}) \simeq \mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H)_0$ in 6.1, we have

$$\mathcal{E}xt^1(A, \mathbb{G}_a) \cong H_{\mathcal{O}}/F^0 H_{\mathcal{O}}.$$

We will prove this isomorphism directly first.

Proof. We may assume that A comes from a non-degenerate pairing $X \times Y \to \mathbb{G}_{m,\log}$. Then, by the exact sequence $0 \to Y \to T^{(Y)}_{\log} \to A \to 0$ together with $\mathcal{H}om(T^{(Y)}_{\log}, \mathbb{G}_a) = \mathcal{E}xt^1(T^{(Y)}_{\log}, \mathbb{G}_a) = 0$ ([9] 10.3), we have the above isomorphism (cf. [8] 3.2.8). □

Remark. This gives an alternative description of the functor $\mathcal{A}_S \xrightarrow{\sim} \mathcal{H}_S$, which gives the equivalence in [8] Theorem 3.1.5 (2). In fact, let $A \in \mathcal{A}_S$ and let $H \in \mathcal{H}_S$ be the corresponding object. Then the 0-th Hodge filtration $F^0 H_{\mathcal{O}}$ of $H_{\mathcal{O}}$ in [8] is nothing but the kernel of the natural homomorphism $H_{\mathcal{O}} \twoheadrightarrow \mathcal{E}xt^1(A^*, \mathbb{G}_a)$. Comparing this with the definition of $F^0 H_{\mathcal{O}}$ in [8], we also see the equality $\mathcal{E}xt^1(A, \mathbb{G}_a) = \operatorname{Lie} A^*$.

6.3. We prove

$$\mathcal{E}xt^1(A, \mathbb{G}_m) = G_H.$$

By the exponential sequence and 6.2, this is equivalent to the statement that $\mathcal{E}xt^1(A, \mathbb{G}_m) \to \mathcal{E}xt^2(A, \mathbb{Z})$ is the zero map. We may assume that A comes from a non-degenerate pairing $X \times Y \to \mathbb{G}_{m,\log}$. Then, by using the exact sequence $0 \to Y \to T^{(Y)}_{\log} \to A \to 0$, we are reduced to showing that the map $\mathcal{E}xt^1(T^{(Y)}_{\log}, \mathbb{G}_m) \to \mathcal{E}xt^2(T^{(Y)}_{\log}, \mathbb{Z})$ is the zero map. But $\mathcal{E}xt^1(T^{(Y)}_{\log}, \mathbb{G}_m) = 0$ by [9] 10.3.

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6.4. Next, we define a canonical homomorphism

$$\mathcal{E}xt^1(A, \mathbb{G}_{m,\log}) \longrightarrow \mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H)_0.$$

Consider an exact sequence

$$0 \longrightarrow \mathbb{G}_{m,\log} \longrightarrow A' \longrightarrow A \longrightarrow 0$$

on $(\text{fs}/S)^{\log}$ which gives a section of $\mathcal{E}xt^1(A, \mathbb{G}_{m,\log})$. We give the corresponding extension of $\mathbb{Z}$ by H. Taking $\mathcal{E}xt^1(\tau^{-1}(-), \mathbb{Z})$ of the above exact sequence, we have an exact sequence of the type

$$0 \longrightarrow H_{\mathbb{Z}} \longrightarrow\ ? \longrightarrow \mathcal{E}xt^1(\tau^{-1}(\mathbb{G}_{m,\log}), \mathbb{Z}) \longrightarrow \mathcal{E}xt^2(\tau^{-1}(A), \mathbb{Z}).$$

(Note that $\mathcal{H}om(\tau^{-1}(\mathbb{G}_{m,\log}), \mathbb{Z}) = 0$ because $\mathbb{Z}$ is not divisible, though any section of $\tau^{-1}(\mathbb{G}_{m,\log})$ is divisible ket locally in the sense of [5].) But $\mathcal{E}xt^2(\tau^{-1}(A), \mathbb{Z}) = 0$ (4.2 (3)). Hence by the canonical map $\mathbb{Z} \to \mathcal{E}xt^1(\tau^{-1}(\mathbb{G}_{m,\log}), \mathbb{Z}(1))$ (which comes from the exact sequence $0 \to \mathbb{Z}(1) \to \mathcal{L} \to \tau^{-1}(\mathbb{G}_{m,\log}) \to 0$), we obtain an exact sequence of the type

$$0 \longrightarrow H_{\mathbb{Z}} \longrightarrow H'_{\mathbb{Z}} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Let $H'_{\mathcal{O}} = \tau_*(H'_{\mathbb{Z}} \otimes \mathcal{O}_S^{\log})$. Let $F^0 H'_{\mathcal{O}} = \text{Ker}(H'_{\mathcal{O}} \to \mathcal{E}xt^1(A', \mathbb{G}_a))$. Then, 6.2 and the snake lemma give the exact sequence $0 \to F^0 H_{\mathcal{O}} \to F^0 H'_{\mathcal{O}} \to \mathcal{O}_S \to 0$. Thus we have an extension of pre-log mixed Hodge structures (see [8] 2.3.3 or [14] Definition 2.6.1 for the definition of pre-log mixed Hodge structure).

This gives

$$\mathcal{E}xt^1(A, \mathbb{G}_{m,\log}) \longrightarrow \mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H).$$

We show that this map sends $\mathcal{E}xt^1(A, \mathbb{G}_{m,\log})$ into $\mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H)_0$. It suffices to show that $\mathcal{E}xt^1(A, \mathbb{G}_{m,\log})$ goes to zero in $Y_H \otimes \mathbb{G}_{m,\log}/\mathbb{G}_m$ because

$$\text{Ker}(\mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H) \to Y_H \otimes \mathbb{G}_{m,\log}/\mathbb{G}_m) = \mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H)_0.$$

For this, we may assume that S is an fs log point. In this case $Y_H = H_{\mathbb{Z}}/\tau^{-1}\tau_* H_{\mathbb{Z}}$ is a constant sheaf. Define a torus T over S by $T = \mathcal{H}om(Y_H, \mathbb{G}_m)$. Then we have an embedding $T \subset A$.

We consider the exact sequence

$$0 \longrightarrow \mathcal{E}xt^1(A, \mathbb{G}_{m,\log}) \longrightarrow R^1\theta_A(\mathbb{G}_{m,\log}) \longrightarrow \mathcal{H}om(A, R^1\tau_*(\mathbb{G}_{m,\log})),$$

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(see 4.7 for θ_A) the isomorphism

$$R^1\tau_*(\mathbb{G}_{m,\log}) \simeq \mathbb{G}_{m,\log} \otimes (\mathbb{G}_{m,\log}/\mathbb{G}_m),$$

and the fact that the composite

$$\begin{aligned}\mathcal{E}xt^1(A, \mathbb{G}_{m,\log}) &\longrightarrow \mathcal{H}om(A, \mathbb{G}_{m,\log} \otimes (\mathbb{G}_{m,\log}/\mathbb{G}_m))\\ &\longrightarrow \mathcal{H}om(T, \mathbb{G}_{m,\log} \otimes (\mathbb{G}_{m,\log}/\mathbb{G}_m))\end{aligned}$$

coincides with the composite

$$\begin{aligned}\mathcal{E}xt^1(A, \mathbb{G}_{m,\log}) &\longrightarrow \mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H) \longrightarrow Y_H \otimes \mathbb{G}_{m,\log}/\mathbb{G}_m\\ &\longrightarrow \mathcal{H}om(T, \mathbb{G}_{m,\log} \otimes (\mathbb{G}_{m,\log}/\mathbb{G}_m)).\end{aligned}$$

Since the last arrow is injective, we see that $\mathcal{E}xt^1(A, \mathbb{G}_{m,\log})$ goes to zero in $Y_H \otimes \mathbb{G}_{m,\log}/\mathbb{G}_m$. Thus the desired homomorphism is defined.

6.5. We define the inverse map

$$\mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H)_0 \longrightarrow \mathcal{E}xt^1(A, \mathbb{G}_{m,\log}).$$

Assume that we are given an extension $0 \to H \to H' \to \mathbb{Z} \to 0$. Then we have exact sequences $0 \to \mathbb{Z}(1) \to H'^*(1) \to H^*(1) \to 0$ and $0 \to \mathbb{Z}(1)\backslash\mathcal{O}^{\log} \to H'^*_{\mathbb{Z}}(1)\backslash\mathcal{O}^{\log} \otimes H'^*_{\mathcal{O}}(1)/F^0(H'^*_{\mathcal{O}}(1)) \to H^*_{\mathbb{Z}}(1)\backslash\mathcal{O}^{\log} \otimes H^*_{\mathcal{O}}(1)/F^0(H^*_{\mathcal{O}}(1)) \to 0$. Since $\tau_*(\mathbb{Z}(1)\backslash\mathcal{O}^{\log}) = \mathbb{G}_{m,\log}$, we have an exact sequence of the form

$$0 \longrightarrow \mathbb{G}_{m,\log} \longrightarrow\ ?\ \longrightarrow \mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H^*(1)) \longrightarrow R^2\tau_*\mathbb{Z},$$

where $? = \tau_*(H'^*_{\mathbb{Z}}(1)\backslash\mathcal{O}^{\log} \otimes H'^*_{\mathcal{O}}(1)/F^0(H'^*_{\mathcal{O}}(1)))$.

We show that if the class of the original extension of $\mathbb{Z}$ by H belongs to $\mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H)_0$, then the composite

$$A \longrightarrow \mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H^*(1)) \longrightarrow R^2\tau_*\mathbb{Z} = \bigwedge^2(\mathbb{G}_{m,\log}/\mathbb{G}_m)$$

is the zero map and hence we obtain an extension of A by $\mathbb{G}_{m,\log}$. To prove this, we may assume that S is an fs log point. In this case Y_H is a constant sheaf. Let $T'_{\log}$ be the (Y_{H^*})-part of $\mathcal{H}om(Y_H, \mathbb{G}_{m,\log})$. The composite

$$\mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H) \longrightarrow Y_H \otimes (\mathbb{G}_{m,\log}/\mathbb{G}_m) \longrightarrow \mathcal{H}om(T'_{\log}, \bigwedge^2(\mathbb{G}_{m,\log}/\mathbb{G}_m))$$

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coincides with the composite

$$\mathcal{E}xt^1_{\rm naive}(\mathbb{Z}, H) \longrightarrow \mathcal{H}om(A, \bigwedge^2(\mathbb{G}_{m,\log}/\mathbb{G}_m)) \longrightarrow \mathcal{H}om(T'_{\log}, \bigwedge^2(\mathbb{G}_{m,\log}/\mathbb{G}_m)).$$

Since the last arrow is injective (in fact, the kernel of this map is $\mathcal{H}om(A/T'_{\log}, \bigwedge^2(\mathbb{G}_{m,\log}/\mathbb{G}_m))$, but there is a surjective homomorphism from a (usual non-log) complex torus to $A/T'_{\log}$ and any homomorphism from a complex torus to $\bigwedge^2(\mathbb{G}_{m,\log}/\mathbb{G}_m)$ is the zero map), the induced map $\mathcal{E}xt^1_{\rm naive}(\mathbb{Z}, H)_0 \to \mathcal{H}om(A, \bigwedge^2(\mathbb{G}_{m,\log}/\mathbb{G}_m))$ is the zero map.

Thus we have a homomorphism $\mathcal{E}xt^1_{\rm naive}(\mathbb{Z}, H)_0 \to \mathcal{E}xt^1(A, \mathbb{G}_{m,\log})$. This is the inverse.

This completes the proof of $\mathcal{E}xt^1(A, \mathbb{G}_{m,\log}) \cong \mathcal{E}xt^1_{\rm naive}(\mathbb{Z}, H)_0$, and the proof of 6.1.

Remark. There seems to be an alternative, simple proof of $\mathcal{E}xt^1(A, \mathbb{G}_{m,\log}) \cong \mathcal{E}xt^1_{\rm naive}(\mathbb{Z}, H)_0$. In fact, we can prove $\mathcal{E}xt^1_{\rm naive}(\mathbb{Z}, H) = R^1\theta_A(\mathcal{O}^{\log}/\mathbb{Z}(1))$ by 6.2. On the other hand, since $\mathbb{G}_{m,\log} = \tau_*(\mathcal{O}^{\log}/\mathbb{Z}(1))$, we have an exact sequence $0 \to \mathcal{E}xt^1(A, \mathbb{G}_{m,\log}) \to R^1\theta_A(\mathcal{O}^{\log}/\mathbb{Z}) \to \mathcal{H}om(A, R^2\tau_*(\mathbb{Z}))$. It seems possible to prove directly that $\mathcal{E}xt^1_{\rm naive}(\mathbb{Z}, H)_0$ coincides with the kernel of the induced homomorphism $\mathcal{E}xt^1_{\rm naive}(\mathbb{Z}, H) \to \mathcal{H}om(A, R^2\tau_*(\mathbb{Z}))$, which implies $\mathcal{E}xt^1(A, \mathbb{G}_{m,\log}) = \mathcal{E}xt^1_{\rm naive}(\mathbb{Z}, H)_0$.

§7. $\mathcal{H}^m(P)$

We define log Albanese varieties and log Picard varieties.

7.1. Let P be a proper, separated and log smooth fs log analytic space over an fs log analytic space S.

Let $m \geq 0$. We say that P/S is *good* for $\mathcal{H}^m$ if the following (i)–(iii) are satisfied.

(i) The two homomorphisms

$$\mathcal{O}_S^{\log} \otimes_{\mathbb{Z}} \mathcal{H}^m(\tau^{-1}(P), \mathbb{Z}) \longleftarrow \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} \tau_*(\mathcal{O}_S^{\log} \otimes_{\mathbb{Z}} \mathcal{H}^m(\tau^{-1}(P), \mathbb{Z})) \longrightarrow \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} \mathcal{H}^m(P, \omega^\bullet_{P/S})$$

are isomorphisms, where the latter homomorphism is induced from

$$\tau_*\mathcal{H}^m(\tau^{-1}(P), \mathcal{O}_S^{\log}) \longrightarrow \tau_*\mathcal{H}^m(\tau^{-1}(P), \omega^{\bullet,\log}_{P/S}) \longrightarrow \mathcal{H}^m(P, R\tau_*\omega^{\bullet,\log}_{P/S}) \xleftarrow{\simeq} \mathcal{H}^m(P, \omega^\bullet_{P/S}).$$

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Here $\omega^\bullet_{P/S}$ is the relative analytic log de Rham complex.

(ii) $\mathcal{H}^m(P, \omega^{\bullet \geq r}_{P/S}) \to \mathcal{H}^m(P, \omega^\bullet_{P/S})$ are injective, and $\mathcal{H}^m(P, \omega^{\bullet \geq r}_{P/S}) \to \mathcal{H}^m(P, \omega^r_{P/S})$ are surjective for all r.

(iii) Let

$$H_{\mathbb{Z}} = \mathcal{H}^m(\tau^{-1}(P), \mathbb{Z}), \quad H_{\mathcal{O}} = \mathcal{H}^m(P, \omega^\bullet_{P/S}).$$

Let

$$\iota : \mathcal{O}^{\log}_S \otimes_{\mathbb{Z}} H_{\mathbb{Z}} \xrightarrow{\simeq} \mathcal{O}^{\log}_S \otimes_{\mathcal{O}_S} H_{\mathcal{O}}$$

be the isomorphism given by (i). Then $H_{\mathbb{Z}}$ is the pullback of a sheaf of abelian groups on the small site of $S^{\log}$, which is also denoted by $H_{\mathbb{Z}}$, and, with the Hodge filtration on $H_{\mathcal{O}}$, the triple $(H_{\mathbb{Z}}, H_{\mathcal{O}}, \iota)$ is a log Hodge structure of weight m.

7.2. We remark that in case where $P \to S$ is exact, instead of the above first condition (i), we can consider the following simpler one.

(i) The two homomorphisms

$$\mathcal{O}^{\log}_S \otimes_{\mathbb{Z}} \mathcal{H}^m(\tau^{-1}(P), \mathbb{Z}) \to \mathcal{H}^m(\tau^{-1}(P), \omega^{\bullet,\log}_{P/S}) \leftarrow \mathcal{O}^{\log}_S \otimes_{\mathcal{O}_S} \mathcal{H}^m(P, \omega^\bullet_{P/S})$$

are isomorphisms.

This definition may be equivalent to the former, but we have not yet proved it. The results in Section 8 and Section 10 are valid with this modified definition of goodness under exactness assumption.

In Section 9, we will discuss when P/S is good.

If P/S is good for $\mathcal{H}^m$, we will denote by $\mathcal{H}^m(P)$ the log Hodge structure of weight m defined above.

7.3. Assume that P/S is good for $\mathcal{H}^1$. We define the log Albanese variety $A_{P/S}$ (resp. log Picard variety $A^*_{P/S}$) of P/S as the log complex torus associated to $\mathcal{H}^1(P)^*$ (resp. $\mathcal{H}^1(P)(1)$). Here $(\ )^*$ denotes the dual.

§8. Log Picard varieties

We show that the log Picard variety is a subgroup sheaf of $\mathcal{H}^1(\ , \mathbb{G}_{m,\log})$.

In this section, we always assume that $f : P \to S$ is proper, separated, log smooth, and exact. Assume also that for any $p \in P$, the cokernel of $M^{\rm gp}_{S,f(p)}/\mathcal{O}^\times_{S,f(p)} \to M^{\rm gp}_{P,p}/\mathcal{O}^\times_{P,p}$ is torsion free.

In this section, we always assume further that the following statement, which is the local invariant cycle theorem of degree 1, is valid.

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Assumption 8.1. *The natural homomorphism*

$$\mathcal{H}^1(P,\mathbb{Z}) \longrightarrow \tau_*\mathcal{H}^1(\tau^{-1}(P),\mathbb{Z})$$

is an isomorphism.

Remark. This assumption is satisfied, for example, when $f\colon P \to S$ is projective and vertical (see 2.3 for the definition of the verticality). A proof for this fact is outlined as follows: First, it is known for a log deformation in the sense of [22] whose all irreducible components of the underlying analytic space are smooth ([22] (5.6), [21]). The general case is reduced to this case as follows. Let $s \in S$ and we explain how one can prove that $R^1f_*\mathbb{Z} \to \tau_*R^1f_*^{\log}\mathbb{Z}$ is an isomorphism at s. Using the semistable reduction theorem ([18]), I. Vidal showed that there are a morphism from the standard log point $S' \to S$ whose image is $\{s\}$ and a log blowing up $P' \to P \times_S S'$ such that $f'\colon P' \to S'$ is a log deformation whose all irreducible components of the underlying analytic space are smooth ([24] Proposition 2.4.2.1; we remark that, in that proof, we can always take as the concerned proper subdivisions those corresponding to log blowing ups.) By the admissibility and 4.3.1, the problem is reduced to the case for f'.

Theorem 8.2. *Let P/S be as above. Assume that it is good for $\mathcal{H}^0$ and $\mathcal{H}^1$.*

(1) *Let*

$$\mathcal{H}^1(P,\mathbb{G}_m)_0 = \mathrm{Ker}(\mathcal{H}^1(P,\mathbb{G}_m) \to \mathcal{H}^2(P,\mathbb{Z})),$$
$$\mathcal{H}^1(P,\mathbb{G}_{m,\log})_0 = \mathrm{Ker}(\mathcal{H}^1(P,\mathbb{G}_{m,\log}) \to \tau_*\mathcal{H}^2(\tau^{-1}(P),\mathbb{Z})).$$

Let $H = \mathcal{H}^1(P)(1)$. Then we have canonical isomorphisms

$$\mathcal{H}^1(P,\mathbb{G}_m)_0 \simeq G_H, \quad \mathcal{H}^1(P,\mathbb{G}_{m,\log})_0 \simeq \mathcal{E}xt^1_{\text{naive}}(\mathbb{Z},H)_0,$$

for which the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{H}^1(P,\mathbb{G}_m)_0 & \subset & \mathcal{H}^1(P,\mathbb{G}_{m,\log})_0 \\ \wr\| & & \wr\| \\ G_H & \subset & \mathcal{E}xt^1_{\text{naive}}(\mathbb{Z},H)_0. \end{array}$$

In particular, we have canonical embeddings

$$\mathcal{H}^1(P,\mathbb{G}_m)_0 \subset A^*_{P/S} \subset \mathcal{H}^1(P,\mathbb{G}_{m,\log})_0.$$

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More precisely, we have an exact sequence

$$0 \longrightarrow \mathcal{H}^1(P, \mathbb{G}_m)_0 \longrightarrow \mathcal{H}^1(P, \mathbb{G}_{m,\log})_0 \longrightarrow (\tau_*(H_{\mathbb{Z}}) \otimes \mathbb{G}_{m,\log})/Y_H \longrightarrow 0$$

and $A^*_{P/S} \subset \mathcal{H}^1(P, \mathbb{G}_{m,\log})_0$ *coincides with the inverse image of* $(\tau_*(H_{\mathbb{Z}}) \otimes \mathbb{G}_{m,\log})^{(Y_H)}/Y_H$.

(2) *If* P/S *is furthermore good for* $\mathcal{H}^2$, *we have a canonical isomorphism*

$$\mathcal{H}^1(P, \mathbb{G}_{m,\log})/\mathcal{H}^1(P, \mathbb{G}_{m,\log})_0 \simeq \mathcal{H}om(\mathbb{Z}, \mathcal{H}^2(P)(1)).$$

Here the right-hand side is the $\mathcal{H}om$ *for log Hodge structures.*

(1) understands the part $\mathcal{H}^1(P, \mathbb{G}_{m,\log})_0$ of $\mathcal{H}^1(P, \mathbb{G}_{m,\log})$ by the log Hodge structure $\mathcal{H}^1(P)$, and (2) understands the quotient $\mathcal{H}^1(P, \mathbb{G}_{m,\log})/\mathcal{H}^1(P, \mathbb{G}_{m,\log})_0$ by the log Hodge structure $\mathcal{H}^2(P)$.

COROLLARY 8.3. *Let* P/S *be as in the beginning of this section. Assume that it is good for* $\mathcal{H}^m$ *with* $m = 0, 1, 2$. *Then we have an exact sequence*

$$0 \longrightarrow \mathcal{E}xt^1(A_{P/S}, \mathbb{G}_{m,\log}) \longrightarrow \mathcal{H}^1(P, \mathbb{G}_{m,\log}) \longrightarrow \tau_*\mathcal{H}^2(\tau^{-1}(P), \mathbb{Z}) \longrightarrow \mathcal{H}^2(P, \mathcal{O}_P).$$

Remark 8.4. (2) of Theorem seems to be a part of some unknown log Hodge conjecture. The non-log case of this part of the Hodge conjecture (i.e., the Hodge conjecture for divisors) is just a consequence of the exponential sequence.

The rest of this section is devoted to the proof of the above theorem and corollary.

8.5. Let P/S be as in the beginning of this section. In the rest of this section, we always assume that it is good for $\mathcal{H}^0$ and $\mathcal{H}^1$. Let $H = \mathcal{H}^1(P)(1)$.

Since we consider $\mathbb{G}_m$ and $\mathbb{G}_{m,\log}$ of (fs/P) and also $\mathbb{G}_m$ and $\mathbb{G}_{m,\log}$ of (fs/S), we denote them by $\mathbb{G}_{m,P}$, $\mathbb{G}_{m,\log,P}$, $\mathbb{G}_{m,S}$, and $\mathbb{G}_{m,\log,S}$, respectively, to avoid confusions. Furthermore, let $\mathbb{G}_{m,\log,S,P}$ be the subgroup of $\mathbb{G}_{m,\log,P}$ generated by $\mathbb{G}_{m,P}$ and $\mathbb{G}_{m,\log,S}$.

In the following, we will sometimes use the same notation for a sheaf and for the inverse image of it.

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PROPOSITION 8.6. *We have an exact sequence*

$$0 \longrightarrow \mathcal{H}^1(P, \mathbb{Z}) \longrightarrow \mathcal{H}^1(\tau^{-1}(P), \mathbb{Z}) \longrightarrow \mathcal{H}^0(P, \mathbb{G}_{m,\log,P}/\mathbb{G}_{m,\log,S,P}) \longrightarrow \mathcal{H}^2(P, \mathbb{Z}) \longrightarrow \mathcal{H}^2(\tau^{-1}(P), \mathbb{Z})$$

on $(\mathrm{fs}/S)^{\log}$.

Here sheaves on (fs/S) are identified with their inverse images on $(\mathrm{fs}/S)^{\log}$.

Proof. We denote $P \to S$ by f. Let a topological space Q be the fiber product of P and $S^{\log}$ over S, and let $g\colon P^{\log} \to Q$ and $f'\colon Q \to S^{\log}$ be the canonical maps. The spectral sequence for the composition $f^{\log} = f' \circ g$ gives an exact sequence

$$0 \longrightarrow R^1 f'_* g_* \mathbb{Z} \longrightarrow R^1 f^{\log}_* \mathbb{Z} \longrightarrow f'_* R^1 g_* \mathbb{Z} \longrightarrow R^2 f' g_* \mathbb{Z} \longrightarrow R^2 f^{\log}_* \mathbb{Z}.$$

We have

$$g_* \mathbb{Z} = \mathbb{Z}, \quad R^1 g_* \mathbb{Z} = \mathbb{G}_{m,\log,P}/\mathbb{G}_{m,\log,S,P}.$$

(These are seen at stalks by using the fact that for any $p \in P$, $M^{\mathrm{gp}}_{S,f(p)}/\mathcal{O}^\times_{S,f(p)}$ is a direct summand of $M^{\mathrm{gp}}_{P,p}/\mathcal{O}^\times_{P,p}$, which is by the exactness and torsion freeness assumptions in the beginning of this section.)

Hence the proper base change theorem for (f, f') gives the result. ∎

8.7. We prove

$$\mathcal{H}^1(P, \mathbb{G}_{m,P})_0 \simeq G_H.$$

Via the exponential sequence, we see

$$\mathcal{H}^1(P, \mathbb{G}_{m,P})_0 = \mathrm{Coker}(\mathcal{H}^1(P, \mathbb{Z}) \to \mathcal{H}^1(P, \mathcal{O}_P)).$$

By 8.1, $\mathcal{H}^1(P, \mathbb{Z}) = \tau_* H_{\mathbb{Z}}$. On the other hand, $\mathcal{H}^1(P, \mathcal{O}_P) = H_{\mathcal{O}}/F^0 H_{\mathcal{O}}$. Thus $\mathcal{H}^1(P, \mathbb{G}_{m,P})_0 \simeq G_H$.

8.8. Consider the exact sequence

$$0 \longrightarrow \mathbb{G}_{m,S} \longrightarrow \mathbb{G}_{m,P} \oplus \mathbb{G}_{m,\log,S} \longrightarrow \mathbb{G}_{m,\log,S,P} \longrightarrow 0.$$

By applying $\mathcal{H}^m(P, \quad)$, and by using

$$\mathcal{H}^m(P, \mathbb{G}_{m,S}) = \mathcal{H}^m(P, \mathbb{Z}) \otimes \mathbb{G}_{m,S},$$
$$\mathcal{H}^m(P, \mathbb{G}_{m,\log,S}) = \mathcal{H}^m(P, \mathbb{Z}) \otimes \mathbb{G}_{m,\log,S},$$

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we obtain an exact sequence

$$0 \longrightarrow \mathcal{H}^1(P,\mathbb{Z}) \otimes \mathbb{G}_{m,S} \longrightarrow \mathcal{H}^1(P,\mathbb{G}_{m,P}) \oplus \mathcal{H}^1(P,\mathbb{Z}) \otimes \mathbb{G}_{m,\log,S} \\ \longrightarrow \mathcal{H}^1(P,\mathbb{G}_{m,\log,S,P}) \longrightarrow 0.$$

8.9. Let $\mathcal{H}^1(P,\mathbb{G}_{m,\log,S,P})_0$ be the subgroup sheaf of $\mathcal{H}^1(P,\mathbb{G}_{m,\log,S,P})$ generated by the images of $\mathcal{H}^1(P,\mathbb{G}_{m,P})_0$ and $\mathcal{H}^1(P,\mathbb{Z}) \otimes \mathbb{G}_{m,\log,S}$.

By 8.8, there is a unique homomorphism $\mathcal{H}^1(P,\mathbb{G}_{m,\log,S,P}) \to \mathcal{H}^2(P,\mathbb{Z})$ which extends the homomorphism $\mathcal{H}^1(P,\mathbb{G}_{m,P}) \to \mathcal{H}^2(P,\mathbb{Z})$ and which annihilates the image of $\mathcal{H}^1(P,\mathbb{G}_{m,\log,S})$.

We have

$$\mathcal{H}^1(P,\mathbb{G}_{m,\log,S,P})_0 = \operatorname{Ker}(\mathcal{H}^1(P,\mathbb{G}_{m,\log,S,P}) \to \mathcal{H}^2(P,\mathbb{Z})).$$

PROPOSITION 8.10. $\mathcal{H}^1(P,\mathbb{G}_{m,\log,S,P})_0 \simeq \tilde{E}(H) := \tau_*(\tau^{-1}\tau_*(H_{\mathbb{Z}}\backslash\mathcal{V}_H))$.

Proof. By 8.1, 8.7, and 8.8,

$$\begin{aligned}
&\mathcal{H}^1(P,\mathbb{G}_{m,\log,S,P})_0 \\
&\quad \simeq \operatorname{Coker}(\mathcal{H}^1(P,\mathbb{Z}) \otimes \mathbb{G}_{m,S} \to \mathcal{H}^1(P,\mathbb{G}_{m,P})_0 \oplus \mathcal{H}^1(P,\mathbb{Z}) \otimes \mathbb{G}_{m,\log,S}) \\
&\quad \simeq \operatorname{Coker}(\tau_* H_{\mathbb{Z}} \otimes \mathbb{G}_{m,S} \to G \oplus \tau_* H_{\mathbb{Z}} \otimes \mathbb{G}_{m,\log,S}) \simeq \tilde{E}(H).
\end{aligned}$$

(For the last isomorphism, note that $(\tau_* H_{\mathbb{Z}} \otimes \mathbb{G}_{m,\log,S})/F^0 H_{\mathcal{O}} \simeq \tilde{E}(H)$.) ∎

8.11. Let $(\mathrm{fil}^r \subset \mathcal{H}^2(\tau^{-1}(P),\mathbb{Z}))_r$ be the filtration induced by the spectral sequence for the composition $f^{\log} = f' \circ g$. Then

$$\begin{gathered}
\mathrm{fil}^0 = \mathcal{H}^2(\tau^{-1}(P),\mathbb{Z}), \;\; \mathrm{fil}^2 = \operatorname{Image}(\mathcal{H}^2(P,\mathbb{Z}) \to \mathcal{H}^2(\tau^{-1}(P),\mathbb{Z})), \;\; \mathrm{fil}^3 = 0, \\
\mathrm{gr}^0 \subset \mathcal{H}^0(P, R^2 g_*\mathbb{Z}) = \mathcal{H}^0(P, \textstyle\bigwedge^2(\mathbb{G}_{m,\log,P}/\mathbb{G}_{m,\log,S,P})), \\
\mathrm{gr}^1 \subset \mathcal{H}^1(P, R^1 g_*\mathbb{Z}) = \mathcal{H}^1(P, \mathbb{G}_{m,\log,P}/\mathbb{G}_{m,\log,S,P}).
\end{gathered}$$

Furthermore, the image of the connecting map

$$\mathcal{H}^1(P,\mathbb{G}_{m,\log,P}) \longrightarrow \mathcal{H}^2(\tau^{-1}(P),\mathbb{Z})$$

is contained in fil^1.

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8.12. We will prove

$$\mathcal{H}^1(P, \mathbb{G}_{m,\log,P})_0 \simeq \mathcal{E}xt^1_{\mathrm{naive}}(\mathbb{Z}, H)_0.$$

LEMMA 8.12.1. *Assume that P/S is good for $\mathcal{H}^0$ and $\mathcal{H}^1$ and $\mathcal{H}^0(P) = \mathbb{Z}$. Then $\mathcal{H}^0(P, \mathbb{G}_{m,\log,P}) = \mathbb{G}_{m,\log,S}$.*

Proof. Let $f : P \to S$, $u = f \circ \tau_P = \tau_S \circ f^{\log} : P^{\log} \to S$. From the exact sequence

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathcal{O}_P^{\log} \longrightarrow \mathcal{O}_P^{\log}/\mathbb{Z}(1) \longrightarrow 0$$

of sheaves on $P^{\log}$, we have an exact sequence

$$\begin{aligned} 0 \longrightarrow u_*\mathbb{Z}(1) \longrightarrow u_*(\mathcal{O}_P^{\log}) \longrightarrow u_*(\mathcal{O}_P^{\log}/\mathbb{Z}(1)) \longrightarrow R^1u_*\mathbb{Z}(1) \\ \longrightarrow R^1u_*(\mathcal{O}_P^{\log}) \end{aligned}$$

of sheaves on S. Since P/S is good for $\mathcal{H}^0$ and $\mathcal{H}^0(P) = \mathbb{Z}$, we have $u_*\mathbb{Z}(1) = \mathbb{Z}(1)$ and $u_*(\mathcal{O}_P^{\log}) = \mathcal{O}_S$. Since

$$\tau_{P*}(\mathcal{O}_P^{\log}/\mathbb{Z}(1)) = \mathbb{G}_{m,\log,P},$$

we have

$$u_*(\mathcal{O}_P^{\log}/\mathbb{Z}(1)) = f_*(\mathbb{G}_{m,\log,P}).$$

Hence the above exact sequence is rewritten as

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathcal{O}_S \longrightarrow f_*(\mathbb{G}_{m,\log,P}) \longrightarrow R^1u_*(\mathbb{Z}(1)) \longrightarrow R^1f_*(\mathcal{O}_P).$$

We consider the last arrow of this exact sequence. We have an exact sequence

$$0 \longrightarrow R^1\tau_{S*}(f_*^{\log}(\mathbb{Z}(1))) \longrightarrow R^1u_*(\mathbb{Z}(1)) \longrightarrow \tau_{S*}(R^1f_*^{\log}(\mathbb{Z}(1))),$$

and isomorphisms

$$R^1\tau_{S*}(f_*^{\log}(\mathbb{Z}(1))) \simeq R^1\tau_{S*}(\mathbb{Z}(1)) \simeq \mathbb{G}_{m,\log,S}/\mathbb{G}_{m,S}.$$

Furthermore, since P/S is good for $\mathcal{H}^1$,

$$\tau_{S*}R^1f_*^{\log}(\mathbb{Z}(1)) \longrightarrow R^1f_*(\mathcal{O}_P)$$

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is injective. (This is because if we denote the Hodge structure $\mathcal{H}^1(P)(1)$ by H, then $\tau_{S*}R^1f_*^{\log}(\mathbb{Z}(1)) = \tau_{S*}(H_{\mathbb{Z}}(1))$ and $R^1f_*(\mathcal{O}_P) = H_{\mathcal{O}}/F^0H_{\mathcal{O}}$, and H is of weight -1.) In conclusion, we have an exact sequence

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathcal{O}_S \longrightarrow f_*(\mathbb{G}_{m,\log,P}) \longrightarrow \mathbb{G}_{m,\log,S}/\mathbb{G}_{m,S} \longrightarrow 0.$$

Hence $\mathbb{G}_{m,\log,S} = f_*(\mathbb{G}_{m,\log,P})$. □

By 8.6 and the above lemma, we have a commutative diagram of exact sequences

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathcal{H}^0(\mathbb{G}_{m,\log,P}/\mathbb{G}_{m,\log,S,P}) & \longrightarrow & Y_H\backslash\mathcal{H}^0(\mathbb{G}_{m,\log,P}/\mathbb{G}_{m,\log,S,P}) \\
\downarrow & & \downarrow \\
\mathcal{H}^1(\mathbb{G}_{m,\log,S,P}) & \longrightarrow & \mathcal{H}^2(\mathbb{Z}) \\
\downarrow & & \downarrow \\
\mathcal{H}^1(\mathbb{G}_{m,\log,P}) & \longrightarrow & \mathrm{fil}^1\,\mathcal{H}^2_{\log}(\mathbb{Z}) \\
\downarrow & & \downarrow \\
\mathcal{H}^1(\mathbb{G}_{m,\log,P}/\mathbb{G}_{m,\log,S,P}) & = & \mathcal{H}^1(\mathbb{G}_{m,\log,P}/\mathbb{G}_{m,\log,S,P}).
\end{array}$$

Here $\mathcal{H}^m(\ \ ) = \mathcal{H}^m(P,\ \ )$ and $\mathcal{H}^m_{\log}(\ \ ) = \mathcal{H}^m(\tau^{-1}(P),\ \ )$. This diagram gives an exact sequence

$$0 \longrightarrow Y_H \longrightarrow \mathcal{H}^1(P, \mathbb{G}_{m,\log,S,P})_0 \longrightarrow \mathcal{H}^1(P, \mathbb{G}_{m,\log,P})_0 \longrightarrow 0.$$

Hence by 8.10, we have

$$\mathcal{H}^1(P, \mathbb{G}_{m,\log,P})_0 \simeq Y_H\backslash\tilde{E}(H) \simeq \mathcal{E}xt^1_{\mathrm{naive}}(\mathbb{Z}, H)_0,$$

which completes the proof of 8.2.

8.13. By 8.2, we have a canonical map

$$\mathcal{H}^1(P, \mathbb{G}_{m,\log,P})_0 \longrightarrow \mathcal{E}xt^1_{\mathrm{naive}}(\mathbb{Z}, H).$$

Here we note that this homomorphism can be obtained more simply as follows (but from this definition, it is not clearly seen that the image is in $\mathcal{E}xt^1_{\mathrm{naive}}(\mathbb{Z}, H)_0$).

Since $\mathbb{G}_{m,\log,P} = \tau_*(\mathcal{O}_P^{\log}/\mathbb{Z})$, there is a homomorphism $\mathcal{H}^1(P, \mathbb{G}_{m,\log,P}) \to \tau_*\mathcal{H}^1(\tau^{-1}(P), \mathcal{O}_P^{\log}/\mathbb{Z})$, which induces a homomorphism $\mathcal{H}^1(P, \mathbb{G}_{m,\log,P})_0 \to \tau_* \operatorname{Coker}(\mathcal{H}^1(\tau^{-1}(P), \mathbb{Z}) \to \mathcal{H}^1(\tau^{-1}(P), \mathcal{O}_P^{\log}))$. On the other hand, as explained below, we have

$$(*) \qquad \mathcal{H}^1(\tau^{-1}(P), \mathcal{O}_P^{\log}) \cong \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} \mathcal{H}^1(P, \mathcal{O}_P),$$

which suffices for the definition because, then, the above τ_* of the cokernel is isomorphic to $\mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H)$. As for $(*)$, it reduces to the following general fact. For any proper separated exact morphism $f\colon X \to Y$ of fs log analytic spaces, if for any $x \in X$, the cokernel $M^{\text{gp}}_{Y,f(x)}/\mathcal{O}^\times_{Y,f(x)} \to M^{\text{gp}}_{X,x}/\mathcal{O}^\times_{X,x}$ is torsion free, then the natural homomorphism $\mathcal{O}_Y^{\log} \otimes_{\mathcal{O}_Y} Rf_*\mathcal{O}_X \to Rf_*^{\log}\mathcal{O}_X^{\log}$ is an isomorphism. This fact is the non-ket version of [5] (8.6.6) and proved similarly: reduce to the case where the underlying morphism of f is an isomorphism, and use the case where $n = 1$ of [5] Lemma (8.6.3.1).

8.14. We now prove 8.3. Assume that P/S is good also for $\mathcal{H}^2$. Then we have a sequence

$$\mathcal{H}^1(P, \mathbb{G}_{m,\log,P}) \longrightarrow \tau_*\mathcal{H}^2(\tau^{-1}(P), \mathbb{Z}) \longrightarrow \mathcal{H}^2(P, \mathcal{O}_P).$$

We show that this is exact.

Let a be an element of $\tau_*\mathcal{H}^2(\tau^{-1}(P), \mathbb{Z})$ which vanishes in $\mathcal{H}^2(P, \mathcal{O}_P)$. We first show that a belongs to $\tau_* \operatorname{fil}^1$. In fact, this follows from the fact that $f'_*R^2g_*\mathbb{Z} \to f'_*R^2g_*(\mathcal{O}_P)$ is injective, for this map is rewritten as

$$\mathcal{H}^0(P, \textstyle\bigwedge^2(\mathbb{G}_{m,\log,P}/\mathbb{G}_{m,\log,S,P})) \longrightarrow \mathcal{H}^0(P, \mathcal{O}_P \otimes_{\mathbb{Z}} \textstyle\bigwedge^2(\mathbb{G}_{m,\log,P}/\mathbb{G}_{m,\log,S,P})).$$

Next consider the exact sequence

$$\mathcal{H}^1(P, \mathbb{G}_{m,\log,P}) \longrightarrow \mathcal{H}^1(P, \mathbb{G}_{m,\log,P}/\mathbb{G}_{m,\log,S,P}) \longrightarrow \mathcal{H}^2(P, \mathbb{G}_{m,\log,S,P}).$$

Since the composites

$$\tau_* \operatorname{fil}^1 \longrightarrow \mathcal{H}^1(P, \mathbb{G}_{m,\log,P}/\mathbb{G}_{m,\log,S,P}) \longrightarrow \mathcal{H}^2(P, \mathbb{G}_{m,\log,S,P})$$

and

$$\tau_* \operatorname{fil}^1 \longrightarrow \mathcal{H}^2(P, \mathcal{O}_P) \xrightarrow{\exp} \mathcal{H}^2(P, \mathbb{G}_{m,\log,S,P})$$

coincide, the image of a in gr^1 comes from $\mathcal{H}^1(P, \mathbb{G}_{m,\log,P})$. Hence we may assume that

$$a \in \tau_* \operatorname{fil}^2 = \operatorname{Image}(\mathcal{H}^2(P, \mathbb{Z}) \to \tau_*\mathcal{H}^2(\tau^{-1}(P), \mathbb{Z})).$$

But then by the exact sequence

$$\mathcal{H}^1(P, \mathbb{G}_{m,P}) \longrightarrow \mathcal{H}^2(P, \mathbb{Z}) \longrightarrow \mathcal{H}^2(P, \mathcal{O}_P),$$

a comes from $\mathcal{H}^1(P, \mathbb{G}_{m,P})$.

Together with 6.1, we have 8.3.

§9. When is P/S good for $\mathcal{H}^m$?

Here we discuss when P/S is good for $\mathcal{H}^m$.

9.1. In the following, let $f: P \to S$ be a proper, separated and log smooth morphism of fs log analytic spaces.

In this section, assume that f is *vertical* (2.3). (Otherwise, in general, one cannot expect that f yields a pure log Hodge structure, but a mixed log Hodge structure.)

In this section, assume also that for any $p \in P$, the cokernel of $M^{\mathrm{gp}}_{S,f(p)}/\mathcal{O}^{\times}_{S,f(p)} \to M^{\mathrm{gp}}_{P,p}/\mathcal{O}^{\times}_{P,p}$ is torsion free. (Otherwise, it is suitable to discuss goodness on the ket sites ([5], [12]), not on the usual sites.)

We say that P/S is *good* (or f is good) if P/S is good for $\mathcal{H}^m$ for any m.

9.2. Roughly speaking, one can expect an f as above is good if f is projective (or satisfies some Kähler conditions) and satisfies either one of the following two conditions:

(a) S is log smooth;

(b) f is exact.

We will discuss this in the rest of this section in detail. Recall here that for an f satisfying either (a) or (b), the condition (i) in the definition of the goodness in 7.1 is satisfied by the functoriality of the unipotent log Riemann-Hilbert correspondences ([5] Theorem (6.3); for this, the projectiveness is not necessary).

9.3. In the rest of this section, let f be as in 9.1.

As for the case (a) in 9.2, by [12] 8.11, f is good if f is projective locally over S and if S is log smooth over $\mathbb{C}$.

The projectiveness can be replaced with the log Kählerness, that is, f is good if, locally on S, f is log Kähler and if S is log smooth over $\mathbb{C}$. Here we say that f is log Kähler if there exists a log Kähler metric ([12] 6.3) on P over S. (It is plausible that projectiveness implies that f is log Kähler locally on S; it is valid if the log structure of P is given by a divisor with simple normal crossings having only finite number of components ([12] 6.4).)

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The proof of the above goodness for such a log Kähler f is the same as in [12]. What was shown in [12] was the functoriality of $\mathbb{Z}$-VPLHs under the projectiveness assumption. The same proof works for the functoriality of $\mathbb{R}$-VPLHs under the log Kählerness. Note that if f is Kähler in the usual sense, then f is log Kähler. One can expect that a model of a log complex torus should be always log Kähler, but we have not proved it.

9.4. On the other hand, in general, a good P/S yields a (limit) mixed Hodge structure at each point of S where the log structure is non-trivial. In this sense, if an f as in 9.1 yields a (limit) mixed Hodge structure, then this suggests that such an f may be good. For example, if f is (multi-)semistable or weakly semistable, it is proved that ([23], [2], [16]) f yields a mixed Hodge structure under some Kähler or algebraizability conditions. Hence we can expect that these f may be good under the same kinds of conditions, which may not be covered by the results in 9.3. But the authors have not proved such a result. Note that in these cases both (a) and (b) in 9.2 are satisfied.

9.5. Case (b). As for the case where the base is not necessarily log smooth, we cannot describe here any non-trivial example of a good f. But, again under the Kähler or algebraizability conditions on P (for example, [22] assumes that there exists an element of $H^2(P, \mathbb{Z}(1))$ whose restriction to each irreducible component of the underlying analytic space of P is the class of an ample divisor), f yields a mixed Hodge structure if f is semi-stable over an fs log point ([17], [22], [3]). Note that the condition (b) in 9.2 is satisfied for these f.

9.6. When neither (a) nor (b) are satisfied, it is not always valid that P/S is good (even if f is projective). A counter example is as follows. Let S be $\mathrm{Spec}(\mathbb{C}[x, y]/(x^2, y^2))$ endowed with the log structure defined by $\mathbb{N}^2$; $e_1 \mapsto x, e_2 \mapsto y$, where (e_1, e_2) is the canonical basis of $\mathbb{N}^2$. Let P be the log blowing up of S along the log ideal (e_1, e_2). Then $f \colon P \to S$ is not good for $\mathcal{H}^0$. (In fact, $f_*^{\log}\mathbb{Z} = \mathbb{Z}$. Hence, if f were good, then $f_*\mathcal{O}_P = \mathcal{O}_S$. But $\mathcal{O}_S \to f_*\mathcal{O}_P$ is not injective.)

For this P/S, for any log abelian variety B over S of dimension > 0, the map $B(S) \to B(P)$ is not injective.

§10. On log Albanese maps and log Abel-Jacobi maps

In this section, we discuss some open problems.

10.1. Assume that P/S is proper, separated and log smooth and good for $\mathcal{H}^1$. Let $A_{P/S}$ be the log Albanese variety of P/S.

For a log complex torus B over S and a morphism $f : P \to B$ over S, a homomorphism $\mathcal{E}xt^1(\tau^{-1}(B), \mathbb{Z}) \to \mathcal{H}^1(\tau^{-1}(P), \mathbb{Z})$ of sheaves on $(\mathrm{fs}/S)^{\log}$ is defined by pulling back via f. By 6.2, we see that it preserves the Hodge filtrations so that we have a homomorphism of log Hodge structures $H^* \to \mathcal{H}^1(P)$, where H denotes the log Hodge structure associated to B. This homomorphism induces a homomorphism of log Hodge structures $\mathcal{H}^1(P)^* \to H$ and hence a homomorphism of log complex tori $A_{P/S} \to B$.

We propose a problem.

PROBLEM 10.2. *Prove the following: Assume that P/S is good for $\mathcal{H}^0$ and $\mathcal{H}^1$ and assume that all fibers of P over S are connected.*

(1) *For a log complex torus B over S, the following sequence is exact.*

$$0 \longrightarrow B(S) \longrightarrow B(P) \longrightarrow \mathrm{Hom}(A_{P/S}, B).$$

The last arrow is surjective if P/S has a section.

(2) *Let the assumption be as above. Assume that P/S has a section e. Then for a log complex torus B over S, we have a canonical bijection*

$$\{\textit{morphism } f : P \to B \textit{ over } S \mid f(e) = 0\} \longrightarrow \mathrm{Hom}(A_{P/S}, B).$$

Note that (2) follows from (1).

See 9.6 for a counter example in the case we drop the assumption that P/S is good for $\mathcal{H}^0$.

10.3. In 10.2 (2), the morphism $\psi_e : P \to A_{P/S}$ corresponding to the identity map of $A_{P/S}$ should be called the *log Albanese map* of P/S with respect to e. It is the log version of 1.5. Another interesting problem is to find a log version of the classical definition of the Albanese map as an integral.

10.4. In the case where P is a model of a log complex torus A over S, assuming that P/S is good for $\mathcal{H}^0$ and $\mathcal{H}^1$, the log Albanese variety of P/S is A (4.2), and the log Albanese map $\psi_e : P \to A$ would be nothing but $a \mapsto a - e$.

Here we give a partial answer to the above problem.

PROPOSITION 10.5. *The problem* 10.2 *is solved affirmatively when S is log smooth over $\mathbb{C}$, and f is projective locally on S, vertical, and for any $p \in P$, the cokernel of $M^{\mathrm{gp}}_{S,f(p)}/\mathcal{O}^{\times}_{S,f(p)} \to M^{\mathrm{gp}}_{P,p}/\mathcal{O}^{\times}_{P,p}$ is torsion free.* (*Note that such an f is good as is explained in* 9.3.)

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Proof. (2) follows from (1). We prove (1). Let H be the log Hodge structure over S corresponding to B. Consider the sheaf

$$\mathcal{E}(H)_P = H_{\mathbb{Z}} \backslash (\mathcal{O}_P^{\log} \otimes_{\mathbb{Z}} H_{\mathbb{Z}})/F^0(\mathcal{O}_P^{\log} \otimes_{\mathbb{Z}} H_{\mathbb{Z}})$$

on $P^{\log}$. Let u be the morphism $P^{\log} \to S$. Then

$$B(P) = H^0(P, \mathcal{E}xt^1(\mathbb{Z}, H)) \subset H^0(P, \mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H)) = H^0(S, u_*(\mathcal{E}(H)_P)).$$

We have exact sequences of sheaves on S

$$\begin{aligned} 0 \longrightarrow \tau_* H_{\mathbb{Z}} \longrightarrow H_{\mathcal{O}}/F^0 H_{\mathcal{O}} \longrightarrow u_*(\mathcal{E}(H)_P) \longrightarrow R^1 u_* H_{\mathbb{Z}} \\ \longrightarrow \mathcal{H}^1(P, \mathcal{O}_P) \otimes_{\mathcal{O}_S} H_{\mathcal{O}}/F^0 H_{\mathcal{O}}, \end{aligned}$$

$$\begin{aligned} 0 \longrightarrow R^1 \tau_* H_{\mathbb{Z}} \longrightarrow R^1 u_* H_{\mathbb{Z}} \longrightarrow \tau_*(\mathcal{H}^1(\tau^{-1}(P), \mathbb{Z}) \otimes_{\mathbb{Z}} H_{\mathbb{Z}}) \\ \longrightarrow R^2 \tau_* H_{\mathbb{Z}} \longrightarrow R^2 u_* H_{\mathbb{Z}}. \end{aligned}$$

Here τ is the τ of S. (Note that we use here the assumption of connected fibers and of the goodness.) The last arrow is injective if P/S has a section. Note that we have a homomorphism

$$\tau_*(\mathcal{H}^1(\tau^{-1}(P), \mathbb{Z}) \otimes_{\mathbb{Z}} H_{\mathbb{Z}}) \longrightarrow \mathcal{H}^1(P, \mathcal{O}_P) \otimes_{\mathcal{O}_S} H_{\mathcal{O}}/F^0 H_{\mathcal{O}}$$

whose kernel coincides with $\mathcal{H}om(\mathcal{H}^1(P)^*, H) = \mathcal{H}om(A_{P/S}, B)$. (This is seen by regarding the source as the group of the homomorphisms of the lattices of log Hodge structures.) Hence the above exact sequences give an exact sequence

$$0 \longrightarrow \mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H) \longrightarrow \mathcal{H}^0(P, \mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H)) \longrightarrow \mathcal{H}om(A_{P/S}, B)$$

whose last arrow is surjective if P/S has a section. From this we easily obtain the desired exact sequence in (1).

To see the last statement of (1), we take a specific section of the surjection $\mathcal{H}^0(P, \mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, H)) \to \mathcal{H}om(A_{P/S}, B)$ when P/S has a section, explained as follows.

In the following, we assume that P/S has a section e. Then, we can define the map $\psi_e : P \to \mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, \mathcal{H}^1(P)^*)$ as follows, which gives the above-mentioned section, and which would be expected to give a direct definition of the log Albanese map with respect to e in case whenever the problem is solved. Let S' be an fs log analytic space over S. Let a be a section of $P_{S'}/S'$. Consider the complex of sheaves $C = [\mathbb{Z} \to a_*^{\log}\mathbb{Z} \oplus e_*^{\log}\mathbb{Z}]$ on $P_{S'}^{\log}$, where the first $\mathbb{Z}$ is put in degree 0 and $a_*^{\log}\mathbb{Z} \oplus e_*^{\log}\mathbb{Z}$ is put in degree 1. By applying $Rf_*^{\log}$, where f denotes $P_{S'} \to S'$, we obtain an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathcal{H}^1(\tau^{-1}(P_{S'}), C) \longrightarrow \mathcal{H}^1(\tau^{-1}(P_{S'}), \mathbb{Z}) \longrightarrow 0 \quad \text{on } S'.$$

By identifying the cokernel of the diagonal map $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ with $\mathbb{Z}$ via $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}\,;\,(x, y) \mapsto x - y$, we obtain an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{H}^1(\tau^{-1}(P_{S'}), C) \longrightarrow \mathcal{H}^1(\tau^{-1}(P_{S'}), \mathbb{Z}) \longrightarrow 0 \quad \text{on } S'.$$

This exact sequence gives the section $\psi_e(a)$ of the $\mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, \mathcal{H}^1(P)^*)$ over S', so that we obtain $\psi_e \colon P \to \mathcal{E}xt^1_{\text{naive}}(\mathbb{Z}, \mathcal{H}^1(P)^*)$.

The rest is to show that the image of $\psi_e(a)$ is contained in $\mathcal{E}xt^1(\mathbb{Z}, \mathcal{H}^1(P)^*)$, that is, we claim that the above extension is admissible. If this claim is proved, then we have a section of the last map in the exact sequence in (1), which completes the proof.

To see this claim, we may assume that S' is an fs log point. Now we use the assumption that S is log smooth. (Note that we do not use it until here.) Since P is log smooth by this assumption, we may assume that there is another log smooth S_1 such that $S' \to P$ factors as $S' \to S_1 \to P$ with the first arrow being strict. Regard S_1 as an S-fs log analytic space by the composite $S_1 \to P \to S$. Replacing S by S_1 and P by P_{S_1}, we may assume further that $S = S_1$. Thus we may assume that $S' \to S$ is strict. Then the desired admissibility is nothing but the admissibility of the degeneration of the variation of the mixed Hodge structure over the largest open subspace of S where the log structure is trivial. This is known. The case where the underlying analytic space of S is smooth is explained in [15] 12.10. The general case can be reduced to this case by log blowing ups of S because the admissibility can be checked by seeing only smooth points of the support divisor of $M_S/\mathcal{O}_S^*$ in virtue of a theorem of Kashiwara ([11], Theorem 4.4.1, cf. [15] 5.4). ∎

Remark. In case when S is one-dimensional, there is another proof using an integral as in the definition of the Albanese map in the classical situation.

Next, we propose another problem.

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PROBLEM 10.6. *Prove the following: Let P/S be as in* 8.2. *Then we have the inclusion*

$$\mathcal{H}^1(P, \mathbb{G}_{m,\log})_0 \cap \operatorname{Image}(\mathcal{H}^1(P, \mathbb{G}_m) \to \mathcal{H}^1(P, \mathbb{G}_{m,\log})) \subset A^*_{P/S}.$$

In case where this statement is valid, then the inclusion map should be called the *log Abel-Jacobi map*.

10.7. The above two problems 10.2 and 10.6 are contained in more general problem to define a *log Abel-Jacobi map of degree r* which should be a map into the *log intermediate Jacobian* $\mathcal{E}xt^1(\mathbb{Z}, \mathcal{H}^{2r-1}(P)(r))$ of degree r, where we assume that P/S is good for $\mathcal{H}^{2r-1}$ and $\mathcal{E}xt^1$ is taken in the category of log mixed Hodge structures. (The problem 10.2 is the case where r is the relative dimension of P/S, and 10.6 is the case $r = 1$.) We do not give here the precise formulation of this problem.

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