22 The superstring

The theories we have described were motivated by thinking of a picture of a string moving in space–time. We arrived in this way at a description of strings in terms of two-dimensional quantum fields. The theories, so far, are theories of bosons only. But, in this more abstract picture, we can imagine adding two-dimensional fermionic fields as well. This possibility was first considered by Ramond, Neveu and Schwarz and leads to the superstring theories, Type I, Types IIA and IIB and the two heterotic string theories. We first develop the theories in the light cone gauge, where their spectra are readily exhibited. Then we discuss interactions.

22.1 Open superstrings

A priori there appears to be a great deal of freedom in how we introduce fermions: their number, their representations under the (space–time) Lorentz group and possibly other options. Various consistency conditions restrict these choices. In the case of open strings we have to introduce one fermion ψ^I for each coordinate X^I. For the action of the fermions we take

$$S_\psi = \frac{1}{2\pi}\int d^2\sigma\, i\bar{\psi}^I(\partial_\alpha\gamma^\alpha)\psi^I. \tag{22.1}$$

In two dimensions, a particularly simple choice for the γ-matrices is

$$\gamma^0 = \sigma_2, \quad \gamma^1 = i\sigma_1 \tag{22.2}$$

and the analog of γ_5 in four dimensions is

$$\gamma_3 = \sigma_3. \tag{22.3}$$

The Dirac equation in this basis is purely imaginary, so we can take the fermions to be real (Majorana). We can work with eigenfunctions of σ_3:

$$\psi^I = \begin{pmatrix}\psi^I_- \\ \psi^I_+\end{pmatrix}. \tag{22.4}$$

In this way, if we again introduce light cone coordinates on the world sheet,

$$\sigma^\pm = \tau \pm \sigma, \tag{22.5}$$

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the action becomes

$$S_\psi = \frac{1}{2\pi}\int d^2\sigma\,(\psi^I_+\partial_-\psi^I_+ + \psi^I_-\partial_+\psi^I_-). \tag{22.6}$$

We need to impose boundary conditions at the string end points. To determine suitable boundary conditions, we vary the Lagrangian to obtain the Euler–Lagrange equations. The surface terms which arise in the variation involve $\psi_+\delta\psi_+ - \psi_-\delta\psi_-$. So the boundary terms vanish if $\psi_+ = \pm\psi_-$. An overall sign doesn't matter, so we can take the plus sign at $\sigma = 0$:

$$\psi^I_+(0,\tau) = \psi^I_-(0,\tau) \tag{22.7}$$

This leaves two choices for the boundary conditions at $\sigma = \pi$:

$$\psi^I_+(\pi,\tau) = \pm\psi^I_-(\pi,\tau). \tag{22.8}$$

Fermions which obey the boundary condition with the plus sign are called Ramond fermions; those with the minus sign are called Neveu–Schwarz (NS) fermions. Corresponding to the Ramond case are the mode expansions

$$\psi^I_- = \frac{1}{\sqrt{2}}\sum_{n\in Z} d^I_n e^{-in(\tau-\sigma)}, \quad \psi^I_+ = \frac{1}{\sqrt{2}}\sum_{n\in Z} d^I_n e^{-in(\tau+\sigma)}. \tag{22.9}$$

In the NS case we have

$$\psi^I_- = \frac{1}{\sqrt{2}}\sum_{r\in Z+1/2} b^I_r e^{-ir(\tau-\sigma)}, \quad \psi^I_- = \frac{1}{\sqrt{2}}\sum_{r\in Z+1/2} b^I_r e^{-ir(\tau-\sigma)}. \tag{22.10}$$

Now we quantize these fields:

$$\{\psi^I(\sigma,\tau)_\pm, \psi^J(\sigma',\tau)_\pm\} = \pi\delta(\sigma-\sigma')\delta^{IJ}\delta_{\pm\pm} \tag{22.11}$$

This gives, for the modes:

$$\left\{b^I_r, b^J_s\right\} = \delta^{IJ}\delta_{r+s}, \quad \left\{d^I_m, d^J_n\right\} = \delta^{IJ}\delta_{m+n}. \tag{22.12}$$

The Hamiltonian in light cone gauge, for the Ramond sector, is

$$H = \vec{p}^{\,2} + N_\alpha + N_d. \tag{22.13}$$

Here the Ns are the various number operators:

$$N_\alpha = \sum_{m=1}^{\infty}\alpha^I_{-m}\alpha^I_m, \quad N_d = \sum_{m=1}^{\infty} m d^I_{-m} d^I_m. \tag{22.14}$$

For the NS sector, N_d is replaced by N_b:

$$N_b = \sum_{r=1/2}^{\infty} m b_{-r} b_r. \tag{22.15}$$

Each of these Hamiltonian contributions has a normal-ordering constant. We will determine these shortly. The states of the theory are the eigenstates of the fermion number operators

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$b_n^\dagger b_n$, $d_n^\dagger d_n$ etc. for non-zero n. The eigenvalues can take the values 0 or 1 in each case. The zero modes, which arise in the Ramond sector, are special. They give rise to space–time fermions.

22.2 Quantization in the Ramond sector: the appearance of space–time fermions

Usually, we do field theory at infinite volume but here we are considering field theory at a finite volume ($0 < \sigma < \pi$), and this has introduced some new features. For the bosonic fields X^I we have already seen that there are zero modes, which gave rise to the coordinates and momenta of space–time. For the fermions we now have the new feature that there are two sectors, with two independent Hilbert spaces. It is tempting to simply keep one sector, but it turns out that when we consider string interactions it is necessary to include both: even if we attempted to exclude, say, the Ramond states, they would appear in string loop diagrams.

There is another feature: the appearance of fermion zero modes d_0^I in the Ramond sector. These are not conventional creation and annihilation operators. They obey the commutation relations

$$\left\{d_0^I, d_0^J\right\} = \delta^{IJ}. \tag{22.16}$$

These are, up to a factor 2, the anticommutation relations of the Dirac gamma matrices for a $(D-2)$-dimensional space, i.e. they are associated with the group $O(D-2)$. Anticipating the fact that $D = 10$, we are interested in the Dirac matrices of $O(8)$. Before giving a construction of the spinor representations of $O(8)$, let us first simply state the basic result: $O(8)$ has two spinor representations, 8_s and $8_s'$, and a vector representation, 8_v, all eight-dimensional. So we can realize the commutation relations, not on a Fock space, but on a space corresponding to one of the eight-dimensional representations of $O(8)$. Labeling these states a, $\dot{a}$, then

$$\langle \dot{a}|d_0^I|a\rangle = \frac{1}{\sqrt{2}}\gamma^I_{\dot{a}a}. \tag{22.17}$$

We can construct an explicit representation for these matrices in various ways. A simple and easy to remember construction is to think of $O(8)$ as acting on eight coordinates x^I. Group these into complex coordinates:

$$z^1 = x^1 + ix^2, \quad z^2 = x^3 + ix^4, \quad z^3 = x^5 + ix^6, \quad z^4 = x^7 + ix^8 \tag{22.18}$$

and their complex conjugates. This defines an embedding of $U(4)$ in O(8). Correspondingly, we define

$$a^1 = d_0^1 + id_0^2, \tag{22.19}$$

etc. The a^is obey the commutation relations

$$\{a^i, a^{j\dagger}\} = \delta^{ij}, \tag{22.20}$$

all others vanishing. These are just the conventional anticommutation relations of fermion creation and annihilation operators (but remember that for this discussion they are just matrices and should not be confused with the d_ns, which are genuinely creation and annihilation operators). Among products of these operators we can distinguish two classes: those built from an even number of as and those built from an odd number. In four dimensions the analogous distinction corresponds to the eigenvalue (± 1) of γ^5.

Now we define a state, $|0\rangle$, annihilated by the a^is. We can then form two sets of states, those with even fermion number and those with odd fermion number. The even states are

$$|0\rangle, \quad a^{i\dagger}a^{j\dagger}|0\rangle, \quad a^{1\dagger}a^{2\dagger}a^{3\dagger}a^{4\dagger}|0\rangle. \tag{22.21}$$

These states form one of the eight representations, say 8_s. The second is formed by the states of odd fermion number. States are now labeled $|p^I, a, \{\text{oscillators}\}\rangle$.

What we have learned is that the states in the Ramond sector are space–time *fermions*; the states in the NS sector are space–time bosons.

22.3 Type II theory

For closed strings we still have two-component fields ψ, but the possible choices of boundary conditions are somewhat different. We still require that the fermion surface terms vanish, but we also require that currents such as $\psi_+^I\psi_+^J$ be periodic. (These currents are part of the generators of rotations in space–time.) So we impose the Ramond and Neveu–Schwarz boundary conditions independently on the left and right movers. Recalling that the Lagrangian for the fermions breaks up into left- and right-moving parts, we treat the left- and right-moving fermions as independent fields. The fermions have the mode expansions

$$\psi^I = \sum_{n\epsilon Z} d_n^I e^{-2in(\tau-\sigma)}, \quad \psi^I = \sum_{n\epsilon Z+1/2} b_r^I e^{-2ir(\tau-\sigma)} \tag{22.22}$$

in the Ramond and NS sectors, respectively, and

$$\tilde{\psi}^I = \sum \tilde{d}_n^I e^{-2in(\tau+\sigma)}\tilde{\psi}^I = \sum \tilde{b}_r^I e^{-2ir(\tau+\sigma)}. \tag{22.23}$$

The light cone Hamiltonian is now

$$H = p^2 + N_\alpha + \tilde{N}_\alpha + N_d + \tilde{N}_d - a. \tag{22.24}$$

In constructing the spectrum, this must be supplemented with the condition of invariance under shifts in σ; in the covariant formulation this was the $L_0 = \tilde{L}_0$ constraint (see the discussion after Eq. (21.84)).

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22.4 World-sheet supersymmetry

Before considering the spectra of superstring theories, we consider the question of supersymmetry. The theory we are considering is supersymmetric in two dimensions. Just as we decomposed the fermions into left and right movers, we can introduce a two-component anticommuting parameter θ:

$$\theta = \begin{pmatrix} \theta_- \\ \theta_+ \end{pmatrix}. \tag{22.25}$$

Then we define the superfield

$$Y^I = X^I + \bar{\theta}\psi^I + \frac{1}{2}\bar{\theta}\theta B^I. \tag{22.26}$$

We will see shortly that B^I is an auxiliary field, which in the case of strings in flat space we can set to zero by its equations of motion. The supersymmetry generators are

$$Q_A = \frac{\partial}{\partial \bar{\theta}_A} + i(\gamma^\alpha \theta)_A \partial_\alpha \tag{22.27}$$

(we are using the capital letter A for two-dimensional spinor indices here to distinguish them from lower case a, which we used for $O(8)$ spinor indices, and from α, which we used for two-dimensional vector indices). As in four dimensions, we can introduce a covariant derivative operator which anticommutes with the supersymmetry generators:

$$D = \frac{\partial}{\partial \bar{\theta}} - i\gamma^\alpha \theta \partial_\alpha. \tag{22.28}$$

In terms of the superfields, the action may be written in a manifestly invariant way:

$$\begin{aligned} S &= \frac{i}{4\pi} \int d^2\sigma\, d^2\theta\, \bar{D}Y^\mu DY_\mu \\ &= \frac{-1}{2\pi} \int d^2\sigma\, (\partial_\alpha X^I \partial^\alpha X^I - i\bar{\psi}^I \gamma^\alpha \partial_\alpha \psi^I_\mu - B^I B^I). \end{aligned} \tag{22.29}$$

Note that B^I vanishes by its equations of motion.

Finally, note that, in the NS sectors, the boundary conditions explicitly break the world-sheet supersymmetry; they map bosonic fields into fermionic fields and vice versa, and these fields obey different boundary conditions. The Ramond sector is supersymmetric.

In the covariant formulation, this supersymmetry is essential to an understanding of the full set of constraints on the states. But it is important to stress that it is a symmetry of the world-sheet theory; its implications for the theory in space–time are subtle.

22.5 The spectra of the superstrings

We have, so far, considered first the world-sheet structure of the superstring theories. We have not yet explored their spectra in detail. As in the case of the bosonic string, we will see

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that these theories possess a massless graviton. We will also find that they have a massless spin-$3/2$ particle, the gravitino. For the couplings of such a particle to be consistent requires that the space–time theory is supersymmetric.

22.5.1 The normal-ordering constants

First, we give a general formula for the normal-ordering constant. This is related to the algebra of the energy–momentum tensor we discussed in Section 21.4. For a left- or right-moving boson, with modes which differ from an integer by η (e.g. the modes are $1-\eta$, $2-\eta$ etc.), the contribution to the normal-ordering constant is

$$\Delta = -\frac{1}{24} + \frac{1}{4}\eta(1-\eta). \tag{22.30}$$

For fermions, the contribution is the opposite. So we can recover some familiar results. In the bosonic string, with 24 transverse degrees of freedom, we see that the normal-ordering constant is -1. For the superstring, in the NS–NS sector (see below) we have a contribution of $-1/24$ for each boson and $1/24 - 1/16$ for each of the eight fermions on the left (and similarly on the right). So the normal-ordering constant is $-1/2$. For the RR sector, the normal-ordering constant vanishes.

There are simple derivations of the above formula, whose justification requires careful consideration of conformal field theory. The normal-ordering constant is just the vacuum energy of the corresponding two-dimensional free-field theory. So we need

$$f(\eta) = \frac{1}{2}\sum_{1}^{\infty}(n+\eta). \tag{22.31}$$

Ignoring the fact that the sum is ill-defined, we can shift n by one and compensate by a change in η:

$$f(\eta) = f(\eta+1) + \frac{\eta}{2}. \tag{22.32}$$

If we assume that the result is quadratic in η, we recover the formula above, up to a constant. We can "calculate" this constant by the following trick, known as zeta function regularization. For $\eta = 0$ we need

$$\sum_{n=1}^{\infty} n = \lim_{s\to -1}\sum_{n=1}^{\infty} n^{-s}. \tag{22.33}$$

The object on the right-hand side of this equation is $\zeta(s)$, the *Riemann zeta function*. The analytic structure of this function is something of great interest to mathematicians, but one well-known fact is that its singularities lie off the real axis. Using integral representations one can derive a standard result: $\zeta(-1) = -1/12$. This fixes the constant as $-1/24$. This argument may (or should) appear questionable to the reader. The real justification comes from considering questions in conformal field theory.

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22.5.2 The different sectors of the Type II theory

In the Type II theory there are four possible choices of boundary condition: NS for both left and right movers, Ramond for both left and right movers, Ramond for left and NS for right and NS for left and R for right. We will refer to these as the NS–NS, R–R, R–NS and NS–R sectors. Consider, first, the NS–NS sector. There are no zero-mode fermions, so we just have a normal (unique) ground state for the oscillators. From our computation of the normal ordering constants in the previous section, we see that $a = -1/2$ for both left and right movers. The lowest state is simply the state $|\vec{p}\,\rangle$. It has mass-squared -1 (in units with $\alpha' = 2$). Since no oscillators are excited, the $L_0 = \tilde{L}_0$ condition is satisfied. Now consider the first excited states; again, we must have invariance under σ translations, so these are the states

$$\psi^I_{-1/2}\tilde{\psi}^J_{-1/2}|\vec{p}\,\rangle. \tag{22.34}$$

Because $a = -1/2$ for both left and right movers, these states are massless. The symmetric combination here contains a scalar and a massless spin-2 particle, the graviton; the antisymmetric combination is an antisymmetric tensor field. At the next level we can create massive states using four space–time fermions or two bosons or two fermions and one boson.

Now let us turn to the other sectors. Consider, first, the R–NS sector, where ψ is Ramond and $\tilde{\psi}$ is NS. Now, the left-moving normal-ordering constant is zero, while the right-moving constant is $-1/2$. So we can satisfy the level-matching condition (invariance under σ translations) if we take the left movers to be in their ground state and take the right-moving NS state to be an excitation with a single fermion operator above the ground state, i.e.

$$\left|\Psi^I_a\right\rangle = \tilde{\psi}^I_{-1/2}|a\,\vec{p}\,\rangle. \tag{22.35}$$

From the space–time viewpoint, these are particles of spin-$3/2$ and $1/2$. In the NS–R sector, we have another spin-$3/2$ particle.

Just as a massless spin-2 particle requires that the underlying theory be generally covariant, a massless spin-$3/2$ particle, as we discussed in the context of four-dimensional field theories, requires *space–time* supersymmetry. But now we seem to have a paradox. With space–time supersymmetry we cannot have tachyons, yet our lowest state in the NS–NS sector, $|\vec{p}\,\rangle$, is a tachyon.

The solution to this paradox was discovered by Gliozzi, Scherk and Olive, who argued that it is necessary to project out states, i.e. to keep only states in the spectrum which satisfy a particular condition. This projection, which yields a consistent supersymmetric theory, is known as the GSO projection. Note, first, that we have been a bit sloppy with the fermion indices on the ground states. We have two types of fermion indices, a and $\dot{a}$, corresponding to the two spinor representations of $O(8)$. So we do the following. We keep only states on the left which are odd under the left-moving world-sheet fermion number; we do the same on the right but we include in the definition of the world-sheet fermion

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number the chirality of the zero-mode states. We take

$$(-1)^F = \exp\left(i\pi\gamma^9\right)\,\exp\left(i\pi\sum_{1/2}^{\infty}\psi_n\psi_{-n}\right). \tag{22.36}$$

In the R–NS sector we make a similar set of projections. Here we have a choice, however, in which chirality we take. If we project on states of opposite $(-1)^F$ then we get the Type IIA theory; if we take the same chirality, we get the Type IIB theory.

Returning to the NS–NS sector we make a similar projection, keeping only states which are odd under both left- and right-moving fermion number. In this way we eliminate the would-be tachyon in this sector.

Somewhat more puzzling is the R–R sector in each theory. Here both the left- and right-moving ground states are spinors. So, in space–time the states are bosons. We can organize them as tensors by constructing antisymmetric products of γ-matrices, $\gamma^{ijk\cdots}$. As we know from our experience in four dimensions these form irreducible representations, in this case of the little group $O(8)$. Thinking of our construction of the γ-matrices in terms of the as, we can see that γs with even numbers of indices connect states of opposite chirality while those with odd numbers of indices connect states with the same chirality. Which tensors appear depends on whether we consider the IIA or IIB theories. In the IIA case, only the tensors of even rank are non-vanishing. These tensors correspond to field strengths (one can consider an analogy with the magnetic moment coupling in electrodynamics, $\bar{\psi}\gamma^{\mu\nu}\psi$). So, in the IIA theory one has second- and fourth-rank tensors; the sixth- and eighth-rank field strengths are dual to these. In terms of gauge fields there are a one-index tensor (a vector) and a third-rank antisymmetric tensor. In the IIB theory, there are a scalar, a second-rank tensor and a fourth-rank tensor. In string perturbation theory, because the couplings are through the field strengths, there are no objects carrying the fundamental charge. Later we will see that there are non-perturbative objects, *D-branes*, which do carry these charges.

22.5.3 Other possibilities: modular invariance and the GSO projection

The reader may feel that the choices of projections, and for that matter the choices of representations for the two-dimensional fermions, seem rather arbitrary. It turns out that the possible choices, at least for flat background space–times, are highly restricted. There are only a few consistent theories. Those we have described are the only ones without tachyons but with both left- and right-moving supersymmetries on the world sheet.

In the bosonic string theory, we saw that it is crucial that the theory be formulated in 26 dimensions. One problem with the theory outside 26 dimensions is that it is not modular invariant. This means that it is not invariant under certain global two-dimensional general coordinate transformations. This world-sheet anomaly is correlated with anomalies in space–time. As for the gauge anomalies in field theories, these lead to breakdown of unitarity, Lorentz invariance or both.

For the superstring theories we will now explain why modular invariance demands a projection like the GSO projection. The point is that modular transformations relate sectors with different choices of boundary condition.

In our discussion of string theories up to this point, path integrals have appeared only occasionally, but they are extremely useful in discussing string perturbation theory. The propagation of strings can be described by a two-dimensional path integral over the string coordinates, $X_\mu(\sigma, \tau)$, weighted by e^{-S}, where S is the string action. At tree level the closed-string world sheet has the topology of a sphere. At one loop it has the topology of a torus. So, at one loop, string amplitudes can be described as path integrals of a two-dimensional field theory on a torus. Note that we need here the full path integral, not simply the generator of the Green's function for the field theory. The path integral on the torus, with no insertion of vertex operators, yields the partition function of the two-dimensional field theory. To understand this, let us consider the fermion partition function. Actually, there are several fermion partition functions. We begin with a single right-moving Majorana fermion and take, first, Neveu–Schwarz boundary conditions. There are two sorts of partition function we might define. First,

$$\mathrm{Tr}\, q^{L_0} = \prod_{r=1/2}^{\infty} (1 + q^r). \tag{22.37}$$

Alternatively,

$$\mathrm{Tr}\, (-1)^F q^{L_0} = \prod_{r=1/2}^{\infty} (1 - q^r). \tag{22.38}$$

From a path integral point of view, the first expression is like a standard thermal partition function. It can be represented as a path integral with antiperiodic boundary conditions in the time direction. The second integral corresponds to a path integral with even boundary conditions for fermions in the time direction. We can represent the torus as in Fig. 21.2. Taking the vertical direction to be the time direction and the horizontal direction the space direction, we can indicate the boundary conditions with plus and minus signs along the sides of the square. Recalling the action of modular transformations on the torus, however, we see that the modular group mixes up the various boundary conditions. Not only does it mix the temporal boundary conditions, it mixes the spatial boundary conditions as well.

It will be convenient for much of our later analysis to group the fermions in complex pairs. In the present case this grouping is rather arbitrary, say $\Psi^1 = \psi^1 + i\psi^2$ and so on. Then the partition functions can be conveniently written in terms of ϑ functions. These functions, which have been extensively studied by mathematicians, transform nicely under modular transformations:

$$\vartheta \begin{bmatrix} \theta \\ \phi \end{bmatrix} (0, \tau) = \eta(\tau) e^{2\pi i \theta \phi} q^{\theta^2/2 - 1/24} \prod_{m=1}^{\infty} \left(1 + e^{2\pi i \phi} q^{m + \theta - 1/2}\right) \times \left(1 + e^{-2\pi i \phi} q^{m - \theta - 1/2}\right). \tag{22.39}$$

Under $\tau \to \tau + 1$,

$$\vartheta \begin{bmatrix} \theta \\ \phi \end{bmatrix} (0, \tau + 1) = e^{i\theta^2 - \theta - \theta\phi} \vartheta \begin{bmatrix} \theta \\ \phi - \theta \end{bmatrix} (0, \tau), \tag{22.40}$$

while, under $\tau \to -1/\tau$,

$$\vartheta \begin{bmatrix} \theta \\ \phi \end{bmatrix} (0, 1/\tau) = e^{2\pi i \theta \phi} \vartheta \begin{bmatrix} -\phi \\ \theta \end{bmatrix} (0, \tau). \tag{22.41}$$

These transformation properties have a physical interpretation. Returning to Eqs. (21.125)–(21.127), the transformation $\tau \to -1/\tau$ exchanges the time and space directions of the torus. So these transformations interchange sectors with a given projection (the multiplication of states by a given phase) with states with a twist in the space direction. This is precisely what one would expect from a path integral, where boundary conditions in the time direction correspond to the weighting of states with (symmetry) phases.

Setting

$$Z^{\alpha}_{\beta}(\tau) = \frac{1}{\eta(\tau)} \vartheta \begin{bmatrix} \alpha/2 \\ \beta/2 \end{bmatrix} (0, \tau), \tag{22.42}$$

the partition function for the eight fermions in the NS sector is $(Z^1_0)^4$, for example. If we include a factor $(-1)^F$, this is replaced by $(Z^1_1)^4$. We can work out similar expressions for the Ramond sector. From our expression for the transformation of the ϑ functions, it is clear that none of these is modular invariant by itself, as we would expect from our path integral arguments. So it is necessary to combine them and include also the eight bosons. When we do, we have the possibility of including minus signs (in more general situations, as we will see later, we will have more complicated possible phase choices). There are a finite number of possible choices. Two that work are

$$Z^{\pm} = \frac{1}{2} \left[Z^0_0(\tau)^4 - Z^0_1(\tau)^4 + Z^1_0(\tau)^4 \mp Z^1_1(\tau)^4 \right]. \tag{22.43}$$

These transform simply under the modular transformations; all the terms transform to each other, up to an overall factor. There is a similar factor from the left-moving fermions (where one need not, a priori, take the same phase). Recall that the bosonic partition function is

$$Z_X(\tau) = (4\pi\alpha'\tau_2)^{-1/2} |\eta(q)|^{-2}. \tag{22.44}$$

Here the η function comes from the oscillators. The τ_2 factors come from the integration over the momenta. There are two additional such factors, coming from the integrals over the two light cone momenta. So the full partition function is

$$Z = C \int \frac{d^2\tau}{\tau_2^2} Z_X^8 Z^+(\tau) Z^{\pm}(\tau)^*. \tag{22.45}$$

It is not hard to check that this expression is modular invariant.

If we examine the partition function carefully, we see that we have uncovered the GSO projection. Consider the first two terms in $Z^{\pm}$. They amount to just

$$\mathrm{Tr}[1 - (-1)^F]_{\mathrm{NS}}, \tag{22.46}$$

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i.e. the physical states of the theory, in the NS sector, are only those of odd fermion number. There is a similar projector in the Ramond sector. The two possible choices of left- relative to right-moving Zs correspond precisely to the two possible supersymmetric string theories. Our original argument for the GSO projector was consistency in space–time, but here we have a more direct, world-sheet, consistency argument.

These are the only choices of phases which lead to supersymmetric strings in ten dimensions. However, there are other choices which lead to non-supersymmetric strings. These give what has come to be known as the Type 0 superstring. We will leave consideration of these theories to the exercises.

22.5.4 More on the Type I theory: gauge groups

In our discussion of the bosonic string theory, we mentioned that one can obtain non-Abelian gauge groups by allowing charges at the ends of the strings. There is an infinite set of possibilities, which we have not explore, as all these theories have other problematic features if one is trying to describe nature.

In the case of open superstrings, it turns out that the possible structures are quite constrained. First, it is necessary to include closed strings as well, in order to obtain a unitary theory. This can be seen by considering the scattering of four open strings. By stretching the diagram of Fig. 22.1 one can see that closed strings appear in intermediate states. These strings cannot be oriented. This leads to a different structure in the closed string sector from what we saw in the IIA or IIB theories. It is necessary to require that states be symmetric under the exchange of left- and right-moving quantum numbers. We will discuss the required projection later when we talk about D-branes and orientifold planes.

Second, it turns out that the absence of anomalies fixes uniquely the gauge symmetry as $O(32)$. From the point of view of our experience with four-dimensional anomalies this is somewhat surprising, but it turns out that in ten dimensions supergravity by itself can be anomalous, and this is the case for the open string. Allowing for charges at the end of the string leads to a set of additional mixed gauge and gravitational anomalies. Almost miraculously, if one takes the ends of the string to lie in the vector representation of $O(32)$, all anomalies cancel.

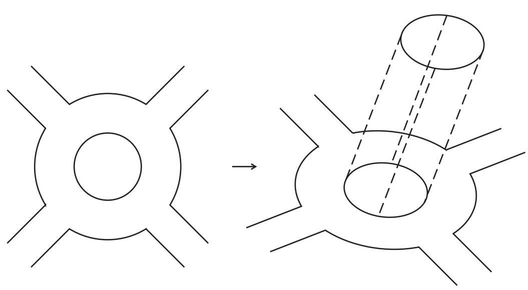

Fig. 22.1 Deforming the diagram for open-string scattering reveals an intermediate closed-string state.

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22.6 Manifest space–time supersymmetry: the Green–Schwarz formalism

In the Ramond–Neveu–Schwarz formalism, space–time supersymmetry is obscure. It only arises after imposing the GSO projector. The supersymmetry operators must connect the different sectors, which are essentially different two-dimensional field theories. Such operators can be constructed, although we will not do that in this text. Instead, we consider in this section a different formalism, the *Green–Schwarz formalism*, in which the space–time supersymmetry is manifest. This formalism is best understood in the light cone gauge.

In the Green–Schwarz formalism one still has the bosonic coordinates X^I, but the eight fermionic coordinates ψ^I in the vector representation of $O(8)$ are replaced by eight fermionic coordinates in a spinor representation of $O(8)$ (we have already seen that $O(8)$ possesses two spinor representations, of opposite chirality). These are usually written as $S^a(\sigma,\tau)$. Their Lagrangian is

$$\mathcal{L}_{\rm gs} = \frac{i}{2\pi}\bar{S}^a\rho^\alpha\partial_\alpha S^a, \tag{22.47}$$

where we have written the Ss as two component fermions and ρ^α denotes the two-dimensional γ-matrices. The S_as can be taken as real (Majorana). They can be decomposed into left and right movers, $S_\pm$. Unlike the case of RNS fermions, for both closed and open strings one has only one boundary condition. As in the case of the RNS fermions, for open strings the boundary conditions relates the left and right movers:

$$S_+^a(0,\tau) = S_-^a(0,\tau), \quad S_+^a(\pi,\tau) = S_-^a(\pi,\tau). \tag{22.48}$$

For closed strings one simply has a periodicity condition,

$$S_\pm^a(\sigma+\pi,\tau) = S_\pm^a(\sigma,\tau). \tag{22.49}$$

The mode expansions, in the case of closed strings, are

$$\begin{aligned} S_+^a &= \sum_{-\infty}^{\infty} S_n^a e^{-2in(\tau-\sigma)}, \\ S_-^a &= \sum_{-\infty}^{\infty} \tilde{S}_n^a e^{-2in(\tau+\sigma)}. \end{aligned} \tag{22.50}$$

The S_ns obey the anticommutation relations

$$\left\{S_n^a, S_m^b\right\} = \delta^{ab}\delta_{m+n}, \quad \left\{\tilde{S}_n^a, \tilde{S}_m^b\right\} = \delta^{ab}\delta_{m+n}. \tag{22.51}$$

For non-zero n these are canonical fermion creation-and-annihilation-operator anticommutation relations. Because of their quantum numbers, the Ss, acting on space–time bosonic states, produce fermionic states and vice versa.

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The light cone Hamiltonian, in terms of these fields, takes the form:

$$H = \frac{1}{2p^+}[(p^I)^2 + N + \tilde{N}], \tag{22.52}$$

where

$$N = \sum_{m=1}^{\infty} \left(\alpha^I_{-m}\alpha^I_m + mS^a_{-m}S^a_m\right), \quad \tilde{N} = \sum_{m=1}^{\infty} \left(\tilde{\alpha}^I_{-m}\tilde{\alpha}^I_m + m\tilde{S}^a_{-m}\tilde{S}^a_m\right). \tag{22.53}$$

Note that there is no normal-ordering constant; more precisely, the normal-ordering constants associated with the left- and right-moving fields vanish, because the contributions of the bosonic and fermionic fields cancel (as they do in the Ramond sector of the superstring).

As in the Ramond sectors of the superstring theories, the anticommutation relations of the zero modes are important and interesting:

$$\left\{S^a_0, S^b_0\right\} = \delta^{ab}. \tag{22.54}$$

Again they are similar to the anticommutation relations of Dirac γ-matrices, but now the indices are different from the RNS case. The solution is to allow S_0 to act on 16 states, eight of which carry spinor labels, $\dot{b}$, and eight of which carry $O(8)$ vector labels, I. Then

$$\langle I|S^a_0|\dot{b}\rangle = \gamma^I_{a\dot{b}}. \tag{22.55}$$

We will leave the verification of this relation for the exercises and proceed directly to the identification of the massless states of the closed-string theories. The IIA and IIB theories are distinguished by the relative helicities of the S and $\tilde{S}$ fields. In the IIA case they are opposite; in the IIB case, the same. The massless fields are obtained just by tensoring the left and right states of the zero modes. The states

$$\epsilon_{IJ}|I\rangle|J\rangle \tag{22.56}$$

are the graviton, B-field and dilaton; the states where $I \to a$ or $J \to a$ are the two gravitini of the theory; those where both I and J are replaced by spinor indices are the states that we discovered in the Ramond–Ramond sector of the superstring theories.

In this formalism the space–time supersymmetry is manifest. There are two sets of supersymmetry generators. One generates not only space–time supersymmetries, but world-sheet supersymmetries as well. This is as it should be; the world-sheet Hamiltonian in the light cone gauge is also the space–time Hamiltonian,

$$Q^{\dot{a}} = \frac{1}{\sqrt{P^+}}\gamma^I_{a,\dot{a}}\sum_{-\infty}^{\infty} S^a_{-n}\alpha^I_n. \tag{22.57}$$

The second set is built of the zero modes alone:

$$Q^a = \sqrt{2P^+}S^a_0. \tag{22.58}$$

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The supersymmetry generators obey the commutation relations:

$$\{Q^a, Q^b\} = 2P^+\delta^{ab}, \tag{22.59}$$

$$\{Q^a, Q^{\dot{a}}\} = \sqrt{2}\gamma^I_{a\dot{a}}P^I, \tag{22.60}$$

$$\{Q^{\dot{a}}, Q^{\dot{b}}\} = 2H\delta^{\dot{a}\dot{b}}. \tag{22.61}$$

The manifest supersymmetry and the close connection between world-sheet and space–time supersymmetries make the Green–Schwarz formalism a powerful tool, both conceptually and computationally, despite its lack of manifest Lorentz invariance.

22.7 Vertex operators

Because there are more world-sheet fields in the superstring than in the bosonic string, the vertex operators are more complicated. In the RNS formalism, the supersymmetry on the world sheet is a relic of a larger, local, supersymmetry, much as conformal invariance is a relic of the general coordinate invariance of the two-dimensional supersymmetry. The resulting superconformal symmetry provides constraints on vertex operators beyond those of the Virasoro algebra. These constraints can be implemented in a variety of ways, depending on how one treats the superconformal ghosts. In the simplest version, the vertex operators must be supersymmetric. In the case of the Type II theories, the vertex operators must respect both the left- and right-moving supersymmetries. For the massless fields of the Type II theory, for example,

$$V = \epsilon_{\mu\nu}(\bar{\partial}X^\mu - ik_\rho\psi^\rho\psi^\mu)(\bar{\partial}X^\nu - ik_\sigma\tilde{\psi}^\sigma\tilde{\psi}^\nu)e^{ik\cdot x}. \tag{22.62}$$

Here ϵ is subject to the constraint $k^\mu\epsilon_{\mu\nu} = 0$. Depending on the symmetries of ϵ, the vertex operator describes the production of gravitons, dilatons or antisymmetric tensor fields. It is straightforward to check that the coupling of three gravitons is that expected from the Einstein Lagrangian.

In the Green–Schwarz formalism, it is Lorentz invariance which governs the form of the vertex operators. As in the covariant formulation, the vertex operators in the Type II theory are products of separate vertex operators for the left and the right movers, with $e^{ik\cdot x}$ factors. These products have the structure

$$V_B = \zeta_{\mu\nu}B^\mu\tilde{B}^\nu e^{ik\cdot X}, \tag{22.63}$$

where

$$B^I = \partial X^I - R^{IJ}k^J, \quad B^+ = p^+ \tag{22.64}$$

and, from the light cone gauge condition, $\zeta^{\mu+} = 0$. Here

$$R^{IJ} = \frac{1}{4}\gamma^{IJ}_{ab}S^aS^b. \tag{22.65}$$

In the Green–Schwarz approach, it is no more difficult to deal with vertex operators for fermions than those for bosons. The polarizations $\zeta_{\mu\nu}$ are replaced by polarizations with one or two spinor indices. Then, as appropriate, one replaces the B^{μ}s with fermionic operators, F^{a} and $F^{\dot{a}}$. We will not give these here, as we will not need them in the text, but they can be found in the references. In the covariant approach, more conformal field theory machinery is required to construct fermion emission operators.

Suggested reading

The superstring is well treated in various textbooks. Green *et al.* (1987) focuses heavily on the light cone formulation; Polchinski (1998) focuses on the RNS formulation. Both provide a great deal of additional detail, including the construction of vertex operators and S-matrices in the two formalisms. A concise and quite readable introduction to the problem of fermion vertex operators in the RNS formulation is provided by the lectures of Peskin (1987).

Exercises

(1) Consider the R–R sectors of the IIA and IIB theories, and study the objects

$$\bar{u}\gamma^{IJK\cdots}u.$$

Show that, in the IIA case, only even-rank tensors are non-vanishing while in the IIB theory only the odd-rank tensors are non-vanishing. Phrase this in the language of ten dimensions rather than the eight light cone dimensions. To do this consider a particle moving along the direction x^9, and show that the Dirac equation correlates chirality in ten dimensions with chirality in eight. To do this, you may want to make the following choice of Γ-matrices:

$$\Gamma^0 = \sigma_2 \otimes I_{16}, \quad \Gamma^i = i\sigma_1 \otimes \gamma^i, \quad \Gamma^9 = i\sigma_3 \otimes I_{16}. \tag{22.66}$$

(2) Write down the Green–Schwarz Lagrangian in a superspace formulation. Show that $Q^{\dot{a}}$ is the supersymmetry generator expected in this approach. Construct the symmetry generated by Q^{a}, and show that this has the structure of a non-linearly realized (spontaneously broken) supersymmetry. Can you offer some interpretation?

(3) Verify that, with the choice of Eq. (22.55), the zero modes of the Green–Schwarz operators S^a obey the correct anticommutation relations.

(4) Verify the expression for the partition function for the Type II theories. Show that it is modular invariant. Consider a different choice, which defines the type-0 superstring,

$$\left|Z_0^0\right|^8 + \left|Z_1^0\right|^8 + \left|Z_0^1\right|^8 \mp \left|Z_1^1\right|^8. \tag{22.67}$$

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Attempt to verify that this is also modular invariant, but at least show that the spectrum does not include a spin-3/2 particle.

(5) Verify that the operator product of two graviton vertex operators in the RNS formalism yields the correct on-shell coupling of three gravitons. Remember the gauge condition in this analysis. The three-graviton vertex in Einstein's theory can be found, for example, in Sannan (1986).

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