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THE CRITICAL GALTON–WATSON PROCESS WITHOUT FURTHER POWER MOMENTS

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Abstract

In this paper we prove a conditional limit theorem for a critical Galton–Watson branching process $\{Z_n; n \geq 0\}$ with offspring generating function $s + (1-s)L((1-s)^{-1})$, where $L(x)$ is slowly varying. In contrast to a well-known theorem of Slack (1968), (1972) we use a functional normalization, which gives an exponential limit. We also give an alternative proof of Sze's (1976) result on the asymptotic behavior of the nonextinction probability.

Keywords: Critical Galton–Watson process; conditional theorem; slowly varying function

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1. Introduction, statement of results, and discussion

Let $Z = \{Z_n; n \geq 0\}$ be a critical Galton–Watson process initiated by a single particle. The main purpose of this note is to study processes with an offspring generating function $f(s)$ satisfying the condition

$$f(s) = s + (1-s)L((1-s)^{-1}) \quad \text{for some slowly varying } L(x). \tag{1}$$

Note that $L(x) \to 0$ as $x \to \infty$ by the assumed criticality of our process.

Evidently, $\mathrm{E}\, Z_n^{1+\delta} = \infty$ for every $\delta > 0$, provided that (1) holds. For critical branching processes with this property there are only a few papers. Zubkov [11] proved limit theorems for the distance to the common nearest ancestor under some additional restrictions on the function $L(x)$. In [10] the asymptotic behavior of the nonextinction probability $Q_n := \mathrm{P}(Z_n > 0)$ was studied. Bondarenko and Topchii [1] obtained lower and upper bounds for the expectation of the maximum $M_n := \max_{k \leq n} Z_k$ under the condition that $\mathrm{E}\, Z_1 \log^{\beta}(1 + Z_1) < \infty$ for some $\beta > 0$.

We begin with the following general result for critical Galton–Watson processes, which was proven by Slack [8], [9].

Theorem 1. (Slack [8], [9].) *For a critical Galton–Watson process the following four assertions are equivalent.*

(a) *The sequence of distributions* $F_n(x) := \mathrm{P}(Q_n Z_n < x \mid Z_n > 0)$ *converges weakly to some nondegenerate limit.*

(b) *We have* $f(s) = s + (1-s)^{1+\alpha} L((1-s)^{-1})$ *for some* $\alpha \in (0, 1]$.

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(c) *There exists a slowly varying function* $L^*(x)$ *such that* $Q_n = n^{-1/\alpha} L^*(n)$ *for some* $\alpha \in (0, 1]$.

(d) *The Laplace transform of the limit of the sequence* F_n *is* $\lambda \mapsto 1 - \lambda(1 + \lambda^{\alpha})^{-1/\alpha}$ *for some* $\alpha \in (0, 1]$.

Therefore, the sequence $F_n(x)$ cannot have a nondegenerate limit if (1) holds. In other words, the normalization with the nonextinction probability does not work in the present case, and we need to find an alternative way to normalize the branching process Z_n.

For a general offspring generating function $f(s)$, we set

$$H(x) := x(f(1 - x^{-1}) - 1 + x^{-1}), \qquad x \geq 1, \tag{2}$$

and

$$V(y) := \int_0^{1-1/y} \frac{\mathrm{d}s}{f(s) - s} = \int_1^y \frac{\mathrm{d}x}{xH(x)}, \qquad y \geq 1. \tag{3}$$

Note that $H(x) \equiv L(x)$ if (1) holds.

The following conditional limit theorem is our main result.

Theorem 2. *Assume that* $f(s)$ *satisfies (1). Then, for all* $x > 0$,

$$\lim_{n\to\infty} \mathrm{P}(H(Q_n^{-1})V(Z_n) < x \mid Z_n > 0) = 1 - \mathrm{e}^{-x}. \tag{4}$$

Nonextinction probabilities are in a sense natural norming constants for critical branching processes, since

$$\mathrm{E}\{Q_n Z_n \mid Z_n > 0\} \equiv 1$$

always. But under condition (1) the expectation overnormalizes Z_n.

Corollary 1. *Under the assumptions of Theorem 2,*

$$\lim_{n\to\infty} \mathrm{P}(Q_n Z_n < x \mid Z_n > 0) = 1$$

for every $x > 0$.

It is well known that for supercritical Galton–Watson processes the normalization with the expectation leads to a nondegenerate limit if and only if $\mathrm{E}\, Z_1 \log Z_1 < \infty$. Furthermore, if $\mathrm{E}\, Z_1 \log Z_1 = \infty$ then we can find a sequence $c_n > 0$ such that $c_n Z_n$ converges almost surely. Consequently, in this irregular case, a linear normalization is possible. In contrast to the supercritical case, it follows, from (4), that there is no linear normalization for Z_n satisfying (1).

Darling [3] was the first to use the functional normalization for proving limit theorems. In [3] the limit behavior of a sum of independent and identically distributed random variables with slowly varying right tails was studied. For branching processes, this type of normalization is usually used if the expectation of the number of offspring is infinite. The first contribution to this area was also made by Darling [4]. He has shown that under some additional assumptions on $f(s)$ there exists $\gamma \in (0, 1)$ such that the sequence $\mathrm{P}(\gamma^n \log(1+Z_n) < x)$ converges to a proper distribution function $\Psi(x)$. Hudson and Seneta [5] gave sufficient conditions for the weak convergence of $\gamma^n L(Z_n)$ for some slowly varying function $L(x)$ and some $\gamma \in (0, 1)$. Schuh and Barbour [6] proved that for every Galton–Watson process with infinite mean there exists a norming function $U(x)$ such that $\mathrm{e}^{-n} U(Z_n)$ converges almost surely to some nondegenerate random variable.

The functional normalization $V(x)$ in Theorem 2 is individual; for processes with different offspring generating functions we have different normalizations. In order to compare the limiting behavior of Z_n for different functions $L(x)$ in (1), we must reduce individual normalizations to a common one. Below we give some examples of the reduction to the logarithmic normalizing function. In each example we have a limit theorem of the following form. There exist a centering sequence A_n and a norming sequence B_n such that

$$\lim_{n\to\infty} \mathrm{P}\biggl(\frac{\log Z_n - A_n}{B_n} < x \biggm| Z_n > 0\biggr) = F(x), \tag{5}$$

where $F(x)$ is a distribution function.

Example 1. Assume that

$$L(x) = \beta^{-1}(\log^{1-\beta} x)\exp\{-\log^{\beta} x\}(1+o(1)) \quad \text{as } x \to \infty, \tag{6}$$

where $\beta \in (0, 1)$. Then, recalling that $H(x) \equiv L(x)$ under (1) and using the definition of $V(x)$, (3), we have

$$V(y) = \int_1^y \frac{\mathrm{d}x}{xL(x)} = \exp\{\log^{\beta} y\}(1+o(1)) \quad \text{as } y \to \infty. \tag{7}$$

Because of continuity of the limiting distribution in (4), we may replace H and V by their asymptotic equivalents given in (6) and (7), respectively. Thus,

$$\lim_{n\to\infty} \mathrm{P}(\beta^{-1}(\log^{1-\beta} Q_n^{-1})\exp\{\log^{\beta} Z_n - \log^{\beta} Q_n^{-1}\} < x \mid Z_n > 0) = 1 - \mathrm{e}^{-x}$$

under (6). Substituting $x = \beta^{-1}\mathrm{e}^{y}$ and taking the logarithm, we obtain

$$\lim_{n\to\infty} \mathrm{P}(\log^{\beta} Z_n - \log^{\beta} Q_n^{-1} + (1-\beta)\log\log Q_n^{-1} < y \mid Z_n > 0) = 1 - \exp\biggl(-\frac{\mathrm{e}^{y}}{\beta}\biggr).$$

Therefore,

$$\lim_{n\to\infty} \mathrm{P}(\log Z_n < (b_n + y)^{1/\beta} \mid Z_n > 0) = 1 - \exp\biggl(-\frac{\mathrm{e}^{y}}{\beta}\biggr), \tag{8}$$

where

$$b_n := \log^{\beta} Q_n^{-1} - (1-\beta)\log\log Q_n^{-1}. \tag{9}$$

Noting that

$$(b_n + y)^{1/\beta} = b_n^{1/\beta} + \frac{y}{\beta} b_n^{1/\beta - 1}(1+o(1)) \quad \text{as } n \to \infty,$$

and taking into account the continuity of the right-hand side of (8), we conclude that

$$\lim_{n\to\infty} \mathrm{P}\biggl(\frac{\log Z_n - b_n^{1/\beta}}{b_n^{1/\beta-1}} < \frac{y}{\beta} \biggm| Z_n > 0\biggr) = 1 - \exp\biggl(-\frac{\mathrm{e}^{y}}{\beta}\biggr). \tag{10}$$

The next equalities follow from the definition of b_n, (9),

$$b_n^{1/\beta} = \log Q_n^{-1} - (\beta^{-1} - 1)(\log^{1-\beta} Q_n^{-1})\log\log Q_n^{-1} + o(\log^{1-\beta} Q_n^{-1}),$$
$$b_n^{1/\beta - 1} = \log^{1-\beta} Q_n^{-1}(1+o(1)) \quad \text{as } n \to \infty.$$

Substituting these expressions for $b_n^{1/\beta}$ and $b_n^{1/\beta-1}$ into (10) and dropping the $o(1)$ term under the P-symbol, we observe that (5) holds with $F(x) := 1 - \exp(-\mathrm{e}^{\beta x}/\beta)$,

$$A_n := \log Q_n^{-1} - (\beta^{-1} - 1)(\log^{1-\beta} Q_n^{-1}) \log\log Q_n^{-1}, \quad \text{and} \quad B_n := \log^{1-\beta} Q_n^{-1}.$$

Using these formulas, we obtain, from (5), the relation

$$\lim_{n\to\infty} \mathrm{P}\biggl(\biggl|\frac{\log Z_n}{\log Q_n^{-1}} - 1\biggr| > \varepsilon \biggm| Z_n > 0\biggr) = 0 \tag{11}$$

for every $\varepsilon > 0$.

In the next two examples the process $\log Z_n$ converges without centering, i.e. $A_n \equiv 0$ in (5).

Example 2. If $L(x) \sim \log^{-\beta} x$ as $x \to \infty$ for some $\beta > 0$ then

$$V(y) = (\beta + 1)^{-1} \log^{\beta+1} x(1 + o(1)).$$

As we have already mentioned, we may insert the asymptotic equivalents of H and V into (4). Thus, we have

$$\lim_{n\to\infty} \mathrm{P}\biggl(\frac{\log^{\beta+1} Z_n}{(\beta+1)\log^{\beta} Q_n^{-1}} < x \biggm| Z_n > 0\biggr) = 1 - \mathrm{e}^{-x},$$

which is equivalent to

$$\lim_{n\to\infty} \mathrm{P}(\log Z_n < x \log^{\beta/(\beta+1)} Q_n^{-1} \mid Z_n > 0) = 1 - \exp\biggl(-\frac{x^{\beta+1}}{\beta+1}\biggr).$$

Roughly speaking, here $\log Z_n$ grows as $\log^{\beta/(\beta+1)} Q_n^{-1}$. This is slower than for the process from the previous example; see (11).

Example 3. Let $\log_{(1)} x := \log x$ and, for all $k \geq 1$, define recursively $\log_{(k+1)} x := \log(\log_{(k)} x)$. Suppose that $L(x) \sim (\log_{(k)} x)^{-1}$ for some $k \geq 2$. For this choice of $L(x)$,

$$V(y) = \int_0^{\log y} \frac{\mathrm{d}x}{L(\mathrm{e}^x)} = (\log y)\log_{(k)} y(1 + o(1)) \quad \text{as } y \to \infty.$$

Hence, by Theorem 2,

$$\lim_{n\to\infty} \mathrm{P}(\log Z_n \log_{(k)} Z_n < x \log_{(k)} Q_n^{-1} \mid Z_n > 0) = 1 - \mathrm{e}^{-x}. \tag{12}$$

Taking the logarithm, we obtain

$$\lim_{n\to\infty} \mathrm{P}(\log_{(2)} Z_n + \log_{(k+1)} Z_n < \log_{(k+1)} Q_n^{-1} + \log x \mid Z_n > 0) = 1 - \mathrm{e}^{-x}.$$

Since $\mathrm{P}(Z_n > N \mid Z_n > 0) \to 1$ for every fixed N, and $\log_{(k+1)} Q_n^{-1} \to \infty$ as $n \to \infty$, we infer that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathrm{P}\biggl(\biggl|\frac{\log_{(2)} Z_n}{\log_{(k+1)} Q_n^{-1}} - 1\biggr| > \varepsilon \biggm| Z_n > 0\biggr) = 0.$$

Therefore, for every $k \geq 2$,

$$\lim_{n\to\infty} \mathrm{P}\biggl(\biggl|\frac{\log_{(k)} Z_n}{\log_{(2k-1)} Q_n^{-1}} - 1\biggr| > \varepsilon \biggm| Z_n > 0\biggr) = 0.$$

This allows us to replace $\log_{(k)} Z_n$ in (12) by $\log_{(2k-1)} Q_n^{-1}$. As a result, we obtain

$$\lim_{n\to\infty} \mathrm{P}\biggl(\log Z_n < x\frac{\log_{(k)} Q_n^{-1}}{\log_{(2k-1)} Q_n^{-1}} \biggm| Z_n > 0\biggr) = 1 - \mathrm{e}^{-x}.$$

This example shows that the process $\log Z_n$ can grow with an arbitrarily small speed.

Next we turn again to the situation that was described in Theorem 1. Assume that

$$f(s) = s + (1-s)^{1+\alpha} L((1-s)^{-1}),$$

where $\alpha \in (0, 1]$ and $L(x)$ is a slowly varying function. Then, by definitions (2) and (3),

$$H(x) = x^{-\alpha} L(x) \tag{13}$$

and

$$V(y) = \int_1^y \frac{\mathrm{d}x}{x^{1-\alpha}L(x)} = \frac{y^\alpha}{\alpha L(y)}(1 + o(1)) \quad \text{as } y \to \infty. \tag{14}$$

Since $V(y)$ increases, we have

$$\mathrm{P}(Q_n Z_n < x \mid Z_n > 0) = \mathrm{P}(V(Z_n) < V(xQ_n^{-1}) \mid Z_n > 0).$$

Normalizing the random variable $V(Z_n)$ with $H(Q_n^{-1})$ and taking into account (13) and (14), we arrive at the identity

$$\mathrm{P}(Q_n Z_n < x \mid Z_n > 0) = \mathrm{P}(H(Q_n^{-1})V(Z_n) < \alpha^{-1}x^\alpha + \varepsilon_n(x) \mid Z_n > 0),$$

where $\varepsilon_n(x) \to 0$ as $n \to \infty$. Combining this equality with Theorem 1, we conclude that the sequence of distributions

$$\mathrm{P}(H(Q_n^{-1})V(Z_n) < x \mid Z_n > 0)$$

converges weakly to some nondegenerate limit. Thus, we can combine Theorems 1 and 2 to obtain the following result. If

$$f(s) = s + (1-s)^{1+\alpha} L((1-s)^{-1})$$

for some $0 \leq \alpha \leq 1$ and some slowly varying $L(x)$ then

$$\lim_{n\to\infty} \mathrm{P}(H(Q_n^{-1})V(Z_n) < x \mid Z_n > 0) = F^{(\alpha)}(x),$$

where $F^{(\alpha)}(x)$ is a distribution function.

The rest of this paper is organized as follows. Section 2 is devoted to the proof of Theorem 2. Section 3 contains an alternative proof of Sze's result on the asymptotic behavior of the nonextinction probability and some remarks related to this.

2. Proof of the main result

In this section we prove Theorem 2 and Corollary 1.

2.1. Auxiliary results

An essential step in our method is to connect the weak convergence of the functional normalized sequence $V(Z_n)$ with the convergence of Laplace transforms of Z_n.

Lemma 1. *Let $V(x)$ be a continuous, increasing, slowly varying function. We denote the inverse function of $V(x)$ by $G(x)$. If there exist a continuous function $\varphi(x)$ and a sequence $a_n > 0$ such that, for all $x > 0$,*

$$\lim_{n\to\infty} \mathrm{E}\biggl\{\exp\biggl(-\frac{Z_n}{G(a_n x)}\biggr) \biggm| Z_n > 0\biggr\} = \varphi(x), \tag{15}$$

then, for all $x > 0$,

$$\lim_{n\to\infty} \mathrm{P}(a_n^{-1} V(Z_n) < x \mid Z_n > 0) = \varphi(x). \tag{16}$$

Proof. We can easily verify that, for all x, $\varepsilon > 0$ and an arbitrary sequence $\{a_n\}$, the following estimates hold:

$$\begin{aligned}\mathrm{E}\biggl\{\exp\biggl(-\frac{Z_n}{G(a_n x)}\biggr) \biggm| Z_n > 0\biggr\} &\le \mathrm{P}(Z_n < G(a_n(x+\varepsilon)) \mid Z_n > 0) \\ &\qquad + \exp\biggl(-\frac{G(a_n(x+\varepsilon))}{G(a_n x)}\biggr), \\ \mathrm{E}\biggl\{\exp\biggl(-\frac{Z_n}{G(a_n x)}\biggr) \biggm| Z_n > 0\biggr\} &\ge \mathrm{P}(Z_n < G(a_n(x-\varepsilon)) \mid Z_n > 0) \\ &\qquad \times \exp\biggl(-\frac{G(a_n(x-\varepsilon))}{G(a_n x)}\biggr).\end{aligned}$$

Since $V(y)$ is increasing and slowly varying, then by Theorem 1.11 of [7] we have

$$\lim_{x\to\infty} \frac{G(x)}{G(cx)} = 0$$

for every constant $c > 1$. Thus, for any given $\varepsilon > 0$ and all $x > 0$,

$$\begin{aligned}\limsup_{n\to\infty} \mathrm{P}(Z_n < G(a_n(x-\varepsilon)) \mid Z_n > 0) &\le \limsup_{n\to\infty} \mathrm{E}\biggl\{\exp\biggl(-\frac{Z_n}{G(a_n x)}\biggr) \biggm| Z_n > 0\biggr\}, \\ \liminf_{n\to\infty} \mathrm{E}\biggl\{\exp\biggl(-\frac{Z_n}{G(a_n x)}\biggr) \biggm| Z_n > 0\biggr\} &\le \liminf_{n\to\infty} \mathrm{P}(Z_n < G(a_n(x+\varepsilon)) \mid Z_n > 0),\end{aligned}$$

or, equivalently,

$$\limsup_{n\to\infty} \mathrm{P}(a_n^{-1} V(Z_n) < x-\varepsilon \mid Z_n > 0) \le \limsup_{n\to\infty} \mathrm{E}\biggl\{\exp\biggl(-\frac{Z_n}{G(a_n x)}\biggr) \biggm| Z_n > 0\biggr\}, \tag{17}$$

$$\liminf_{n\to\infty} \mathrm{E}\biggl\{\exp\biggl(-\frac{Z_n}{G(a_n x)}\biggr) \biggm| Z_n > 0\biggr\} \le \liminf_{n\to\infty} \mathrm{P}(a_n^{-1} V(Z_n) < x+\varepsilon \mid Z_n > 0). \tag{18}$$

If (15) holds, then (17) and (18) imply that

$$\limsup_{n\to\infty} \mathrm{P}(a_n^{-1} V(Z_n) < x \mid Z_n > 0) \le \varphi(x+\varepsilon)$$

and

$$\liminf_{n\to\infty} \mathrm{P}(a_n^{-1}V(Z_n) < x \mid Z_n > 0) \geq \varphi(x-\varepsilon).$$

Letting $\varepsilon \to 0$ and taking into account the continuity of $\varphi(x)$, we obtain (16).

Remark 1. The above proof of Lemma 1 is in the spirit of the proof of Lemma 1 of [5], but essentially simpler.

Lemma 2. *Let the sequence* $\{y_k; k \geq 0\}$ *be recursively defined by*

$$y_{k+1} := y_k - y_k l(y_k), \qquad y_0 \in (0, 1], \tag{19}$$

where $l(y)$ *is an increasing function on* $(0, 1]$, $0 < l(y) < 1$ *for all* y, *and* $\lim_{y\downarrow 0} l(y) = 0$. *Then*

$$\lim_{k\to\infty} \frac{y_{k+1}}{y_k} = 1. \tag{20}$$

Proof. Since y_n decreases, the limit $y^* := \lim_{n\to\infty} y_n$ exists, and y^* is the root of the equation $y = (1 - l(y))y$. But, under the condition that $l(y) < 1$ the latter equation has the unique solution $y = 0$. Thus, $y^* = 0$, i.e. the sequence y_n converges to 0. Therefore,

$$\frac{y_{n+1}}{y_n} = 1 - l(y_n) \to 1 \quad \text{as } n \to \infty. \tag{21}$$

This completes the proof.

Lemma 3. *Let the function* $l(y)$ *be slowly varying as* $y \downarrow 0$ *and satisfy the conditions of Lemma 2. Then there exists the sequence* $\alpha_k \downarrow 0$ *such that, for every* $n > 0$ *and* $j \geq 1$,

$$j < \int_{y_{n+j}}^{y_n} \frac{\mathrm{d}y}{yl(y)} < j + \sum_{n}^{n+j-1} \alpha_k.$$

Proof. Obviously,

$$\frac{y_k - y_{k+1}}{y_k l(y_k)} \equiv 1. \tag{22}$$

Since $yl(y)$ increases, it follows that

$$1 = \frac{y_k - y_{k+1}}{y_k l(y_k)} < \int_{y_{k+1}}^{y_k} \frac{\mathrm{d}y}{yl(y)} \tag{23}$$

and

$$\int_{y_{k+1}}^{y_k} \frac{\mathrm{d}y}{yl(y)} < \frac{y_k - y_{k+1}}{y_{k+1} l(y_{k+1})}. \tag{24}$$

Furthermore, in view of (22), it follows that

$$\frac{y_k - y_{k+1}}{y_{k+1} l(y_{k+1})} = \frac{y_k l(y_k)}{y_{k+1} l(y_{k+1})}. \tag{25}$$

Now applying (21), we obtain

$$\frac{y_k l(y_k)}{y_{k+1} l(y_{k+1})} = \frac{l(y_k)}{l(y_{k+1})}(1 - l(y_k))^{-1}. \tag{26}$$

Since $l(y)$ is slowly varying, it follows, from (20), that

$$\lim_{k\to\infty}\frac{l(y_{k+1})}{l(y_k)}=1.$$

Returning to (25) and taking into account (23), (24), and (26), we conclude that, for some sequence $\alpha_k\to 0$,

$$j<\int_{y_{n+j}}^{y_n}\frac{\mathrm{d}y}{yl(y)}<\sum_{k=n}^{n+j-1}\frac{y_k-y_{k+1}}{y_{k+1}l(y_{k+1})}<j+\sum_{k=n}^{n+j-1}\alpha_k.$$

This completes the proof.

Corollary 2. *If* $(n+j)\to\infty$ *then*

$$j^{-1}\int_{y_{n+j}}^{y_n}\frac{\mathrm{d}y}{yl(y)}\to 1. \tag{27}$$

Set

$$W(x):=\int_x^1\frac{\mathrm{d}y}{yl(y)},\qquad 0<x<1, \tag{28}$$

where $l(y)$ is as defined in Lemma 2.

Lemma 4. *Let* $l(y)$ *and* $\{y_k\}$ *be as defined in Lemma 3, the sequence* b_n *be decreasing and satisfying the condition*

$$W(b_n)=a_nx(1+o(1))\quad \text{as } n\to\infty, \tag{29}$$

where $x\in(0,\infty)$*,* $a_n:=1/l(y_n)$*, and* $k_n:=\min\{k\colon y_k<b_n\}$*. Then*

$$\lim_{n\to\infty}\frac{y_{n+k_n}}{y_n}=\mathrm{e}^{-x}.$$

Proof. First of all, we note that b_n converges to 0 as $n\to\infty$. Indeed, since y_n converges to 0 and $\lim_{y\to 0}l(y)=0$, the sequence a_n tends to ∞. According to condition (29), $b_n\to 0$ as $n\to\infty$. Hence, k_n tends to ∞.

On the one hand, using Corollary 2 with $n=0$ and $j=k_n$, we have

$$W(y_{k_n})\sim k_n\quad \text{as } n\to\infty. \tag{30}$$

Using the definition of k_n and the monotonicity of b_n gives

$$W(y_{k_{n-1}})<W(b_n)\le W(y_{k_n}).$$

On the other hand, setting $j=1$ in (27), we obtain

$$W(y_{k_n})-W(y_{k_n-1})=\int_{y_{k_n}}^{y_{k_n-1}}\frac{\mathrm{d}y}{yl(y)}=1+o(1)\quad \text{as } n\to\infty.$$

Hence,

$$W(y_{k_n})\sim W(b_n)\quad \text{as } n\to\infty. \tag{31}$$

Finally, combining (30), (31), and (29), we have

$$k_n \sim W(y_{k_n}) \sim W(b_n) \sim a_n x \quad \text{as } n \to \infty. \tag{32}$$

Since $l(y)$ increases

$$\int_{y_{n+k_n}}^{y_n} \frac{\mathrm{d}y}{yl(y)} > \frac{1}{l(y_n)} \log \frac{y_n}{y_{n+k_n}}.$$

Conversely, by Lemma 3,

$$\int_{y_{n+k_n}}^{y_n} \frac{\mathrm{d}y}{yl(y)} \sim k_n \quad \text{as } n \to \infty. \tag{33}$$

As a result, we have

$$\frac{1}{l(y_n)} \log \frac{y_n}{y_{n+k_n}} < k_n(1 + o(1)).$$

Taking into account (32) and recalling that $a_n = 1/l(y_n)$, we obtain, for every $\varepsilon \in (0, 1)$ and all large n, the inequality

$$x > (1 - \varepsilon) \log \frac{y_n}{y_{n+k_n}};$$

whence,

$$y_n < \mathrm{e}^{x/(1-\varepsilon)} y_{n+k_n} < \mathrm{e}^{x/(1-\varepsilon)} y_n.$$

Since $l(x)$ is slowly varying, it follows, from the previous inequalities, that

$$\frac{l(y)}{l(y_n)} \to 1 \quad \text{as } n \to \infty$$

uniformly for $y \in [y_n, y_{n+k_n}]$. This, together with (33), gives

$$k_n \sim \int_{y_{n+k_n}}^{y_n} \frac{\mathrm{d}y}{yl(y)} \sim \frac{1}{l(y_n)} \log \frac{y_n}{y_{n+k_n}} \quad \text{as } n \to \infty.$$

Conversely, it follows, from (32), that $k_n \sim a_n x$. Hence, recalling that $a_n = 1/l(y_n)$, we obtain

$$\lim_{n\to\infty} \log \frac{y_n}{y_{n+k_n}} = x.$$

This completes the proof.

2.2. Proof of Theorem 2

The function $V(x)$ defined in (3) satisfies the conditions of Lemma 1. Thus, to prove Theorem 2 it suffices to show that (15) holds with $a_n = [H(1/Q_n)]^{-1}$ and $\varphi(x) = 1 - \mathrm{e}^{-x}$.

For a critical Galton–Watson process the sequence Q_n satisfies the recursion equation

$$Q_{k+1} = 1 - f(1 - Q_k) = Q_k\left(1 - H\left(\frac{1}{Q_k}\right)\right), \tag{34}$$

i.e. the sequence $\{Q_k\}$ coincides with $\{y_k\}$ defined in Lemma 2 for $l(x) := H(1/x)$. Furthermore, we can easily verify that the function $(f(s) - s)/(1 - s)$ is decreasing. Recalling the definition of $H(x)$, (2), we see that $l(x)$ is increasing. By (1), $H(x) = L(x)$ varies slowly at ∞.

Therefore, $l(x)$ varies slowly at 0. We note, finally, that $l(x) \le l(1) = p_0 < 1$ for $x \in [0, 1]$. Summarizing, we conclude that $l(x) = H(1/x)$ satisfies all the conditions of Lemma 3.

Let

$$s_n = s_n(x) = \exp\left\{-\frac{1}{G(a_n x)}\right\}.$$

Evidently, s_n increases and $(1 - s_n)^{-1} \sim G(a_n x)$ as $n \to \infty$. Hence,

$$V\left(\frac{1}{1 - s_n}\right) \sim a_n x \quad \text{as } n \to \infty. \tag{35}$$

Noting that $V(1/x) = W(x)$ for all $x \in [0, 1]$, we can rewrite (35) in the following form:

$$W(1 - s_n) \sim a_n x \quad \text{as } n \to \infty.$$

Consequently, all the conditions of Lemma 4 are fulfilled with $l(x) = H(1/x)$, $y_n = Q_n$, and $b_n = 1 - s_n$. Therefore,

$$\lim_{n\to\infty} \frac{Q_{n+k_n}}{Q_n} = \mathrm{e}^{-x}, \tag{36}$$

where

$$k_n := \min\{k\colon Q_k < 1 - s_n\}. \tag{37}$$

From this definition of k_n it follows that $f_{k_n-1}(0) \le s_n < f_{k_n}(0)$. Thus,

$$f_{n+k_n-1}(0) \le f_n(s_n) < f_{n+k_n}(0). \tag{38}$$

Using the inequality

$$1 - f(f_j(0)) = \int_{f_j(0)}^{1} f'(y)\,\mathrm{d}y > f'(f_j(0))(1 - f_j(0)),$$

we obtain

$$f'(f_j(0))(1 - f_j(0)) < 1 - f_{j+1}(0) < 1 - f_j(0) \tag{39}$$

for every critical Galton–Watson process. Since $\lim_{j\to\infty} f'(f_j(0)) = 1$, we conclude, from (39), that $(1 - f_{j-1}(0)) \sim (1 - f_j(0))$ as $j \to \infty$. Combining this with (38) yields

$$1 - f_n(s_n) \sim Q_{n+k_n} \quad \text{as } n \to \infty. \tag{40}$$

From this relation and (36) we find that

$$\lim_{n\to\infty} \mathrm{E}\left\{\exp\left(-\frac{Z_n}{G(a_n x)}\right) \;\middle|\; Z_n > 0\right\} = \lim_{n\to\infty}\left(1 - \frac{1 - f_n(s_n)}{1 - f_n(0)}\right) = 1 - \mathrm{e}^{-x}.$$

This completes the proof.

Remark 2. The reduction of $(1 - f_n(s_n))$ to $(1 - f_{n+k_n}(0))$ with a proper k_n which is realized in the proof of Theorem 2, was proposed in [8]; see also [2]. If the asymptotic behavior of the nonextinction probability Q_n is known, we can immediately derive the corresponding limit theorem; see Theorems 1 and 2 of [2]. Assume, for example, that $Q_n \sim n^{-1/\alpha}$ as $n \to \infty$ for some $\alpha \in (0, 1]$. Letting $s_n = 1 - xQ_n$ and recalling the definition of k_n, (37), we see that $k_n \sim n/x^\alpha$ as $n \to \infty$. Therefore, by (40), we have

$$1 - f_n(s_n) \sim Q_{n+k_n} \sim (1 + x^{-\alpha})^{-1/\alpha} Q_n \quad \text{as } n \to \infty.$$

Finally,

$$\lim_{n\to\infty} \mathrm{E}\{s_n^{Z_n} \mid Z_n > 0\} = 1 - (1 + x^{-\alpha})^{-1/\alpha},$$

i.e. we obtain assertions (a) and (d) of Theorem 1.

In contrast to [2], the approach we use to prove a limit theorem for Z_n does not require any information on Q_n. Instead, we consider the ratio Q_{n+k_n}/Q_n, which is in the spirit of [8].

Remark 3. An asymptotic expression for $1 - f_n(s)$ can be found under the additional condition

$$L(x) = o(\log^{-1} x). \tag{41}$$

By Theorem 1 of [10], this condition is sufficient for the validity of the relation

$$1 - f_n(s) \sim [G(n + V((1-s)^{-1}))]^{-1} \quad \text{as } n \to \infty.$$

In particular,

$$Q_n \sim [G(n)]^{-1} \quad \text{as } n \to \infty. \tag{42}$$

Using the method described in Remark 2, we can derive Theorem 2 from (42). Note that assumption (41) is superfluous.

2.3. Proof of Corollary 1

Obviously,

$$\mathrm{P}(Q_n Z_n < x \mid Z_n > 0) = \mathrm{P}(H(Q_n^{-1})V(Z_n) < H(Q_n^{-1})V(xQ_n^{-1}) \mid Z_n > 0). \tag{43}$$

It follows, from the definition of $V(x)$, (3), that, for arbitrary $\varepsilon \in (0, 1)$,

$$V(y) \geq \int_{\varepsilon y}^{y} \frac{\mathrm{d}x}{xH(x)} = \frac{\log \varepsilon^{-1}}{H(y)}(1 + o(1)) \quad \text{as } y \to \infty. \tag{44}$$

This means that $H(y)V(y) \to \infty$ as $y \to \infty$. Since $H(x)$ $(= L(x))$ is slowly varying,

$$L(Q_n^{-1})V(xQ_n^{-1}) \sim L(xQ_n^{-1})V(xQ_n^{-1}) \quad \text{as } n \to \infty. \tag{45}$$

Combining (44) and (45), we conclude that $L(Q_n^{-1})V(xQ_n^{-1}) \to \infty$ as $n \to \infty$. This relation, together with (43) and (4), proves the corollary.

3. On the nonextinction probability

In Subsection 3.1 we give sufficient conditions when the sequence y_j, defined in Lemma 2, is asymptotically, as $j \to \infty$, equivalent to $W^{-1}(j)$; see Lemma 7, below. An application of this result to the sequence Q_n gives us the asymptotic behavior of Q_n. In Subsection 3.2 we discuss the influence of the function $L(x)$ of (1) on the nonextinction probability.

3.1. On the inversion problem for the function $W(x)$

Let $\{y_j\}$ be the sequence defined in Lemma 2. It follows, from (28) and Corollary 2 with $n = 0$, that

$$W(y_j) = j + \psi_j$$

and $\psi_j = o(j)$ as $j \to \infty$. Hence,

$$y_j = W^{-1}(j + \psi_j).$$

Fix $\alpha > 0$ and let $l(x) = x^{\alpha}$ in (28). Then $W(x) = \alpha^{-1}(x^{-\alpha} - 1)$, and, consequently,

$$W^{-1}(x + o(x)) \sim W^{-1}(x) \quad \text{as } x \to \infty. \tag{46}$$

In particular, $y_j \sim W^{-1}(j)$ as $j \to \infty$. But, if $W(x)$ is defined by a slowly varying $l(x)$, then (46) is not true in general. However, if $l(x)$ goes to 0 sufficiently fast then $W^{-1}(j + \psi_j) \sim W^{-1}(j)$, as Lemma 7, below, shows. The proof of Lemma 7 is based on the following auxiliary results.

Lemma 5. *Let $l(y)$ satisfy the conditions of Lemma 2 and, in addition, let*

$$l'(y) < c\frac{l(y)}{y} \tag{47}$$

for some positive constant c. Then there exists a constant $C > 0$ such that, for every $n \ge 0$ and $j \ge 1$,

$$j < \int_{y_{n+j}}^{y_n} \frac{\mathrm{d}y}{yl(y)} < j + C\sum_{n}^{n+j-1} l(y_k). \tag{48}$$

Proof. It follows, from (47), that, for any $x > y$,

$$\log\frac{l(x)}{l(y)} < c\ln\frac{x}{y} < c\frac{x-y}{y}.$$

Hence, by applying (21), we obtain

$$\frac{l(y_k)}{l(y_{k+1})} < \exp\biggl\{c\biggl(\frac{1}{1-l(y_k)} - 1\biggr)\biggr\} = 1 + O(l(y_k)).$$

Comparing this bound with (25) and (26), we conclude that there exists a $C > 0$ such that, for any $n \ge 0$ and $j \ge 1$,

$$\sum_{n}^{n+j-1} \frac{y_k - y_{k+1}}{y_{k+1}l(y_{k+1})} < j + C\sum_{n}^{n+j-1} l(y_k).$$

Applying this bound to (24) and taking into account (23), we complete the proof.

Lemma 6. *Let the function $f(x)$ have an increasing second derivative, $f(1) = f'(1) = 1$, and set*

$$l(x) := x^{-1}(f(1-x) + x - 1).$$

Then

$$l'(x) < \frac{l(x)}{x}.$$

Proof. Let

$$g(x) = f(1-x) + x - 1.$$

Obviously,

$$g'(x) = 1 - f'(1-x), \qquad g'(0) = 0,\ g(0) = 0,$$

and

$$g''(x) = f''(1-x).$$

Furthermore,

$$g'(x) = \int_0^x g''(y)\,\mathrm{d}y > g''(x)x,$$

since $g''(x)$ decreases. Hence,

$$2g'(x) > g''(x)x + g'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(xg'(x)).$$

Integrating both sides, we obtain

$$xg'(x) < 2g(x),$$

i.e.

$$\frac{g'(x)}{x} - \frac{g(x)}{x^2} < \frac{g(x)}{x^2}.$$

It remains to note that

$$\frac{g(x)}{x^2} = \frac{l(x)}{x}$$

and

$$\frac{g'(x)}{x} - \frac{g(x)}{x^2} = l'(x).$$

This completes the proof.

We are now ready to state the main result of this subsection.

Lemma 7. *If $l(x)$ satisfies the conditions of Lemma 5, and, in addition, if $l(x) = o(\log^{-1} x)$, then, as $j \to \infty$,*

$$y_j = W^{-1}(j)(1 + o(1)).$$

Proof. Letting $n = 0$ and $y_0 = 1$ in (48) and taking into account the definition of $W(x)$, (28), we have

$$j < W(y_j) < j + C \sum_{k=0}^{j-1} l(y_k). \tag{49}$$

It follows, from the left inequality and monotonicity of $W(x)$, that $y_j < W^{-1}(j)$. Since $l(x)$ increases, this bound implies that

$$\sum_{k=0}^{j-1} l(y_k) < \sum_{k=0}^{j-1} l(W^{-1}(k)).$$

Noting that $l(W^{-1}(x))$ decreases, we have, for every $k \geq 1$, the bound

$$l(W^{-1}(k)) \leq \int_{k-1}^{k} l(W^{-1}(x))\,\mathrm{d}x.$$

Therefore,

$$\sum_{k=1}^{j-1} l(W^{-1}(k)) \leq \int_0^j l(W^{-1}(x))\,\mathrm{d}x.$$

Substituting $x = W(y)$ in the last integral and taking into account the equality

$$W'(x) = -\frac{1}{xl(x)}, \tag{50}$$

we can easily verify that

$$\int_0^j l(W^{-1}(x))\,\mathrm{d}x = \int_1^{W^{-1}(j)} l(y)W'(y)\,\mathrm{d}y = -\int_1^{W^{-1}(j)} y^{-1}\,\mathrm{d}y = -\log W^{-1}(j).$$

Thus,

$$\sum_{k=1}^{j-1} l(W^{-1}(k)) \le -\log W^{-1}(j).$$

Substituting this bound into (49), we obtain, for some $C < \infty$, the inequality

$$j < W(y_j) < j - C\log W^{-1}(j),$$

or, equivalently,

$$W^{-1}(j - C\log W^{-1}(j)) < y_j < W^{-1}(j). \tag{51}$$

To show that these bounds for y_j are asymptotically equivalent, we consider the difference

$$\log W^{-1}(j) - \log W^{-1}(j - C\log W^{-1}(j)).$$

Since $\log W^{-1}(j)$ and $(\log W^{-1})'(x)$ are negative, this difference is positive and does not exceed

$$C\log W^{-1}(j) \inf_{x\in[j,\,j-C\log W^{-1}(j)]} (\log W^{-1})'(x) < \infty. \tag{52}$$

Applying (50), we obtain

$$(\log W^{-1})'(x) = \frac{(W^{-1})'(x)}{W^{-1}(x)} = \frac{1}{W^{-1}(x)W'(W^{-1}(x))} = -l(W^{-1}(x)).$$

Since $l(W^{-1}(x))$ is decreasing, the left-hand side of (52) equals

$$-Cl(W^{-1}(j))\log W^{-1}(j).$$

This means that

$$\lim_{j\to\infty}(\log W^{-1}(j) - \log W^{-1}(j - C\log W^{-1}(j))) = 0$$

if $l(x) = o(\log^{-1} x)$ as $x \to 0$. The statement of the lemma follows from (51) and the last relation.

Now we use Lemma 7 to determine the asymptotic behavior of the nonextinction probability of Z_n. In view of (34) the recursion formula (19) holds for $y_n = Q_n$ if

$$l(x) = H\left(\frac{1}{x}\right) = x^{-1}(f(1-x) - 1 + x). \tag{53}$$

It is easily seen that the generating function $f(x)$ of every critical Galton–Watson process satisfies the conditions of Lemma 6. Thus, all the conditions of Lemma 5 are fulfilled for $l(x)$ defined by (53). Now applying Lemma 7 with $y_n = Q_n$, we obtain

$$Q_n \sim W^{-1}(n) \quad \text{as } n \to \infty,$$

provided that $H(1/x) = o(\log^{-1} x)$ as $x \to 0$. This result was obtained in [10] (see Theorem 1 and Corollary 2 therein) using another method.

We conclude this subsection with an example, which shows that the condition $l(x) = o(\log^{-1} x)$ is close to being necessary for the validity of the statement of Lemma 7.

Example 4. Assume that $l(x) = \log^{-\alpha} x^{-1}$ for some $\alpha \in (0, 1]$. In this case,

$$y_{n+1} = y_n\left(1 - \log^{-\alpha} \frac{1}{y_n}\right) \quad \text{for } y_0 < \mathrm{e}^{-1}.$$

Taking the logarithm of both sides, we arrive at the equality

$$\log \frac{1}{y_{n+1}} = \log \frac{1}{y_n} - \log\left(1 - \log^{-\alpha} \frac{1}{y_n}\right).$$

Using Taylor's expansion for $\log(1 - x)$, we obtain

$$\log \frac{1}{y_{n+1}} = \log \frac{1}{y_n} + \sum_{j=1}^{\infty} \frac{1}{j} \log^{-j\alpha} \frac{1}{y_n}.$$

Hence,

$$\begin{aligned}\log^{\alpha+1} \frac{1}{y_{n+1}} &= \log^{\alpha+1} \frac{1}{y_n}\left(1 + \log^{-\alpha-1} \frac{1}{y_n} + \frac{1}{2}\log^{-2\alpha-1} \frac{1}{y_n} + O\left(\log^{-3\alpha-1} \frac{1}{y_n}\right)\right)^{\alpha+1} \\ &= \log^{\alpha+1} \frac{1}{y_n} + (\alpha+1) + \frac{\alpha+1}{2}\log^{-\alpha} \frac{1}{y_n} + O\left(\log^{-2\alpha} \frac{1}{y_n}\right).\end{aligned}$$

Letting $x_n := \log^{\alpha+1} y_n^{-1}$, we have

$$x_{n+1} = x_n + (\alpha+1) + \frac{\alpha+1}{2} x_n^{-\alpha/(\alpha+1)} + O(x_n^{-2\alpha/(\alpha+1)}).$$

Hence,

$$x_n = x_0 + (\alpha+1)n + \sum_{j=0}^{n-1} \frac{\alpha+1}{2} x_j^{-\alpha/(\alpha+1)} + O\left(\sum_{j=0}^{n-1} x_j^{-2\alpha/(\alpha+1)}\right). \tag{54}$$

Clearly, $x_n = (\alpha+1)n + o(n)$ as $n \to \infty$. Therefore,

$$x_j^{-\alpha/(\alpha+1)} = (\alpha+1)^{-\alpha/(\alpha+1)} j^{-\alpha/(\alpha+1)}(1 + o(1)) \quad \text{as } j \to \infty.$$

Summing over $j \in [0, n)$, we obtain

$$\sum_{j=0}^{n-1} \frac{\alpha+1}{2} x_j^{-\alpha/(\alpha+1)} = c(\alpha) n^{1/(\alpha+1)}(1 + o(1)) \quad \text{as } n \to \infty,$$

where $c(\alpha) = 2^{-1}(\alpha+1)^{1+1/(\alpha+1)}$. Substituting this equality into (54), we have

$$x_n = (\alpha+1)n + c(\alpha)n^{1/(\alpha+1)}(1+o(1)) \quad \text{as } n \to \infty.$$

Recalling the definition of x_n, we obtain

$$y_n = \exp\{-((\alpha+1)n)^{1/(\alpha+1)} - c'(\alpha)n^{(1-\alpha)/(1+\alpha)}(1+o(1))\} \quad \text{as } n \to \infty, \tag{55}$$

where $c'(\alpha) = 2^{-1}(\alpha+1)^{(1-\alpha)/(\alpha+1)}$.

Conversely, $W(x) = (\alpha+1)^{-1}\log^{\alpha+1} x^{-1}$ if $l(x) = \log^{-\alpha} x^{-1}$. Thus,

$$W^{-1}(n) = \exp\{-((\alpha+1)n)^{1/(\alpha+1)}\}. \tag{56}$$

Comparing (55) and (56), we see that y_n and $W^{-1}(n)$ are not asymptotically equivalent for all $\alpha \in (0, 1]$.

3.2. On the connection between the asymptotics of $L(x)$ and Q_n

To use (42) we need to determine the asymptotic behavior of $G(x) = V^{-1}(x)$. But this is not easy, because of the slow variation of $V(x)$. We will demonstrate it with the following example. Assume that

$$V(x) = a\log^{\theta} x + b\log^{\theta-\beta} x + o(\log^{\theta-1} x), \tag{57}$$

where a and b are positive, $\theta > 1$, and $\beta \in (0, 1]$. To find the asymptotic behavior of the function $G(x)$, we consider the equation

$$az^{\theta} + bz^{\theta-\beta} + g(z) = x, \tag{58}$$

where $g(z) = o(z^{\theta-1})$. It is easily seen that $z = (x/a)^{1/\theta}(1+o(1))$ as $x \to \infty$. Letting $z = (x/a)^{1/\theta}(1+\delta(x))$ in (58), we have

$$x(1+\theta\delta(x) + O(\delta^2(x))) + b\left(\frac{x}{a}\right)^{1-\beta/\theta}(1+(\theta-\beta)\delta(x) + O(\delta^2(x))) + o(x^{1-1/\theta}) = x.$$

Therefore, as $x \to \infty$,

$$\delta(x) = -\frac{b}{a\theta}\left(\frac{x}{a}\right)^{-\beta/\theta}(1+o(1))$$

and

$$z = \left(\frac{x}{a}\right)^{1/\theta} - \frac{b}{a\theta}\left(\frac{x}{a}\right)^{(1-\beta)/\theta}(1+o(1)).$$

This means that under (57), we have

$$G(x) = \exp\left\{\left(\frac{x}{a}\right)^{1/\theta} - \frac{b}{a\theta}\left(\frac{x}{a}\right)^{(1-\beta)/\theta}(1+o(1))\right\} \quad \text{as } x \to \infty.$$

Therefore, in order to find the asymptotics of $G(x)$, it is not enough to know only the main term of the asymptotics of $V(x)$. Consequently, if $Z^{(i)}$, $i = 1, 2$, are Galton–Watson processes satisfying (1) with slowly varying functions $L^{(i)}$ and $L^{(1)}(x) \sim L^{(2)}(x)$, then it may happen that $Q_n^{(1)}$ and $Q_n^{(2)}$ are *not* asymptotically equivalent.

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References

[1] Bondarenko, E. M. and Topchii, V. A. (2001). Estimates for expectation of the maximum of a critical Galton–Watson process on a finite interval. *Siberian Math. J.* **42,** 209–216.

[2] Borovkov, K. A. (1988). A method for the proof of limit theorems for branching processes. *Teor. Verojatn. i Primenen* **33,** 115–123.

[3] Darling, D. A. (1952). The influence of the maximum term in the addition of independent random variables. *Trans. Amer. Math. Soc.* **73,** 95–107.

[4] Darling, D. A. (1970). The Galton–Watson process with infinite mean. *J. Appl. Prob.* **7,** 455–456.

[5] Hudson, I. L. and Seneta, E. (1977). A note on simple branching processes with infinite mean. *J. Appl. Prob.* **14,** 836–842.

[6] Schuh, H.-J. and Barbour, A. D. (1977). On the asymptotic behaviour of branching processes with infinite mean. *Adv. Appl. Prob.* **9,** 681–723.

[7] Seneta, E (1976). *Regularly Varying Functions* (Lecture Notes Math. **508**). Springer, Berlin.

[8] Slack, R. S. (1968). A branching process with mean one and possibly infinite variance. *Z. Wahrscheinlichkeitsth.* **9,** 139–145.

[9] Slack, R. S. (1972). Further notes on branching process with mean one. *Z. Wahrscheinlichkeitsth.* **25,** 31–38.

[10] Sze, M. (1976). Markov processes associated with critical Galton–Watson processes with application to extinction probability. *Adv. Appl. Prob.* **8,** 278–295.

[11] Zubkov, A. M. (1975). Limit distributions of the distance to the nearest common ancestor. *Teor. Verojatn. i Primenen* **20,** 614–623.