5

The Dirac equation and the Dirac field

The Standard Model is a quantum field theory. In Chapter 4 we discussed the classical electromagnetic field. The transition to a quantum field will be made in Chapter 8. In this chapter we begin our discussion of the *Dirac equation*, which was invented by Dirac as an equation for the relativistic quantum wave function of a single electron. However, we shall regard the Dirac wave function as a field, which will subsequently be quantised along with the electromagnetic field. The Dirac equation will be regarded as a field equation. The transition to a quantum field theory is called *second quantisation.* The field, like the Dirac wave function, is complex. We shall show how the Dirac field transforms under a Lorentz transformation, and find a Lorentz invariant Lagrangian from which it may be derived.

On quantisation, the electromagnetic fields $A_\mu(x)$, $F_{\mu\nu}(x)$ become space- and time-dependent operators. The expectation values of these operators in the environment described by the quantum states are the classical fields. The Dirac fields $\psi(x)$ also become space- and time-dependent operators on quantisation. However, there are no corresponding measurable classical fields. This difference reflects the Pauli exclusion principle, which applies to fermions but not to bosons. In this chapter and in the following two chapters, the properties of the Dirac fields as operators are rarely invoked: for the most part the manipulations proceed as if the Dirac fields were ordinary complex functions, and the fields can be thought of as single-particle Dirac wave functions.

5.1 The Dirac equation

Dirac invented his equation in seeking to make Schrödinger's equation for an electron compatible with special relativity. The Schrödinger equation for an electron wave function ψ is

$$\mathrm{i}\frac{\partial \psi}{\partial t} = H\psi.$$

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To secure a symmetry between space and time, Dirac postulated the Hamiltonian for a free electron to be of the form

$$H_D = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m = -\mathrm{i}\alpha \cdot \boldsymbol{\nabla} + \beta m, \tag{5.1}$$

where m is the mass of the electron, $\mathbf{p}$ its momentum, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, and $\alpha_1, \alpha_2, \alpha_3$ and β are matrices. ψ is a column vector, and the Schrödinger equation becomes the multicomponent Dirac equation:

$$(\mathrm{i}\partial/\partial t + \mathbf{i}\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} - \beta m)\psi = 0. \tag{5.2}$$

If this equation is to describe a free electron of mass m, its solutions should also satisfy the Klein–Gordon equation of Section 3.5. Multiplying the Dirac equation on the left by the operator $(\mathrm{i}\partial/\partial t - \mathrm{i}\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m)$, we obtain

$$\Big[-\partial^2/\partial t^2 + \sum_i \alpha_i^2 \partial_i \partial_i + \sum_{i<j} (\alpha_i \alpha_j + \alpha_j \alpha_i)\partial_i \partial_j + \mathrm{i}m \sum_i (\alpha_i \beta + \beta \alpha_i)\partial_i - \beta^2 m^2\Big]\psi = 0,$$

where $\partial_i = \partial/\partial x^i$. This equation is identical to the Klein–Gordon equation if

$$\begin{aligned} &\beta^2 = 1, \quad \alpha_1^2 = \alpha_2^2 = \alpha_3^2 = 1, \\ &\alpha_i \alpha_j + \alpha_j \alpha_i = 0, \quad i \neq j; \quad \alpha_i \beta + \beta \alpha_i = 0, \quad i = 1, 2, 3. \end{aligned} \tag{5.3}$$

The reader may recall that similar equations are satisfied by the set of 2×2 Pauli spin matrices $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$, where it is conventional to take

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.4}$$

We shall also find it useful to write

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for the 2×2 unit matrix.

However, here we have four anticommuting matrices, the α_i and β, to represent. It proves necessary to introduce a second set of Pauli matrices and represent the α_i and β by 4×4 matrices. The representation is not unique: different choices are appropriate for illuminating different properties of the Dirac equation. We shall use the so-called *chiral representation*, in which

$$\alpha^i = \begin{pmatrix} -\sigma^i & \mathbf{0} \\ \mathbf{0} & \sigma^i \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbf{0} & \sigma^0 \\ \sigma^0 & \mathbf{0} \end{pmatrix}, \tag{5.5}$$

writing the matrices in 2×2 'block' form. Here

$$\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and the 4×4 identity matrix may be written

$$\mathrm{I} = \begin{pmatrix} \sigma^0 & \mathbf{0} \\ \mathbf{0} & \sigma^0 \end{pmatrix}.$$

It can easily be checked that these matrices satisfy the conditions (5.3). (The block multiplication of matrices is described in Appendix A.)

Since the α_i and β are 4×4 matrices, the Dirac wave function ψ is a four-component column matrix. Regarded as a relativistic Schrödinger equation, the Dirac equation has, as we shall see, remarkable consequences: it describes a particle with intrinsic angular momentum $(\hbar/2)\sigma$ and intrinsic magnetic moment $(q\hbar/2m)\boldsymbol{\sigma}$ if the particle carries charge q, and there exist 'negative energy' solutions, which Dirac interpreted as antiparticles.

A Lagrangian density that yields the Dirac equation from the action principle is

$$\begin{aligned} \mathcal{L} &= \psi^\dagger(\mathrm{i}\partial/\partial t + \mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} - \beta m)\psi \\ &= \psi_a^*(I_{ab}\mathrm{i}\partial/\partial t + \mathrm{i}\alpha_{ab}\cdot\nabla\beta_{ab}m)\psi_b, \end{aligned} \tag{5.6}$$

where we have written in the matrix indices. ψ_a^* is a row matrix, the Hermitian conjugate $\psi^\dagger = \psi^{\mathrm{T}*}$ of ψ. Instead of varying the real and imaginary parts of ψ_a independently, it is formally equivalent to treat ψ_a and its complex conjugate ψ_a^* as independent fields (cf. Section 3.7). The condition that $S = \int \mathcal{L}\,\mathrm{d}^4x$ be stationary for an arbitrary variation $\delta\psi_a^*$ then gives the Dirac equation immediately, since $\mathcal{L}$ does not depend on the derivatives of ψ_a^*.

5.2 Lorentz transformations and Lorentz invariance

The chiral representation (5.5) of the matrices α^i and β is particularly convenient for discussing the way in which the Dirac field must transform under a Lorentz transformation. We have written the Dirac matrices in blocks of 2×2 matrices, and it is natural to write similarly the four-component Dirac field as a pair of two-component fields

$$\psi = \begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{pmatrix} = \begin{pmatrix} \psi_{\mathrm{L}} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \psi_{\mathrm{R}} \end{pmatrix}, \tag{5.7}$$

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where ψ_{L} and ψ_{R} are, respectively, the top and bottom two components of the four-component Dirac field:

$$\psi_{\mathrm{L}} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \ \psi_{\mathrm{R}} = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}. \tag{5.8}$$

The Dirac equation (5.2) becomes

$$\mathrm{i}\begin{pmatrix} \sigma^0 & \mathbf{0} \\ \mathbf{0} & \sigma^0 \end{pmatrix}\begin{pmatrix} \partial_0\psi_{\mathrm{L}} \\ \partial_0\psi_{\mathrm{R}} \end{pmatrix} + \mathrm{i}\begin{pmatrix} -\sigma^i & \mathbf{0} \\ \mathbf{0} & \sigma^i \end{pmatrix}\begin{pmatrix} \partial_i\psi_{\mathrm{L}} \\ \partial_i\psi_{\mathrm{R}} \end{pmatrix} - m\begin{pmatrix} \mathbf{0} & \sigma^0 \\ \sigma^0 & \mathbf{0} \end{pmatrix}\begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{pmatrix} = 0. \tag{5.9}$$

Block multiplication then gives two coupled equations for ψ_{L} and ψ_{R}:

$$\begin{aligned} &\mathrm{i}\sigma^0\partial_0\psi_{\mathrm{L}} - \mathrm{i}\sigma^i\partial_i\psi_{\mathrm{L}} - m\psi_{\mathrm{R}} = 0, \\ &\mathrm{i}\sigma^0\partial_0\psi_{\mathrm{R}} + \mathrm{i}\sigma^i\partial_i\psi_{\mathrm{R}} - m\psi_{\mathrm{L}} = 0. \end{aligned} \tag{5.10}$$

We shall find it highly convenient for displaying the Lorentz structure to define

$$\sigma^\mu = (\sigma^0, \sigma^1, \sigma^2, \sigma^3), \quad \tilde{\sigma}^\mu = (\sigma^0, -\sigma^1, -\sigma^2, -\sigma^3).$$

With this notation, the equations (5.10) may be written

$$\begin{aligned} &\mathrm{i}\tilde{\sigma}^\mu\partial_\mu\psi_{\mathrm{L}} - m\psi_{\mathrm{R}} = 0, \\ &\mathrm{i}\sigma^\mu\partial_\mu\psi_{\mathrm{R}} - m\psi_{\mathrm{L}} = 0. \end{aligned} \tag{5.11}$$

To obtain the Lagrangian density (5.6) in terms of ψ_{L} and ψ_{R}, we need to multiply the expression on the left-hand side of (5.9) by the row matrix $(\psi_{\mathrm{L}}^\dagger, \psi_{\mathrm{R}}^\dagger)$, where the Hermitian conjugate fields are $\psi_{\mathrm{L}}^\dagger = (\psi_1^*, \psi_2^*)$, $\psi_{\mathrm{R}}^\dagger = (\psi_3^*, \psi_4^*)$. Block multiplication gives

$$\mathcal{L} = \mathrm{i}\psi_{\mathrm{L}}^\dagger\tilde{\sigma}^\mu\partial_\mu\psi_{\mathrm{L}} + \mathrm{i}\psi_{\mathrm{R}}^\dagger\sigma^\mu\partial_\mu\psi_{\mathrm{R}} - m(\psi_{\mathrm{L}}^\dagger\psi_{\mathrm{R}} + \psi_{\mathrm{R}}^\dagger\psi_{\mathrm{L}}). \tag{5.12}$$

Variations $\delta\psi_{\mathrm{L}}^*$ and $\delta\psi_{\mathrm{R}}^*$ in the action give the field equations (5.11).

To show that the Lagrangian has the same form in every frame of reference, we must relate the field $\psi'(x')$ in the frame K' to $\psi(x)$ in the frame K, when x' and x refer to the same point in space-time, and are related by a proper Lorentz transformation

$$x'^\mu = L^\mu{}_\nu x^\nu. \tag{5.13}$$

The operator ∂_μ transforms like a covariant vector, so that

$$\partial'_\mu = L_\mu{}^\nu\partial_\nu,$$

which has the inverse

$$\partial_\mu = L^\nu{}_\mu\partial'_\nu. \tag{5.14}$$

(See Problem 2.2.)

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It is shown in Appendix B (equations (B.17) and (B.18)) that with this Lorentz transformation we can associate 2×2 matrices $\mathbf{M}$ and $\mathbf{N}$ with determinant 1 and with the properties

$$\mathbf{M}^{\dagger}\tilde{\sigma}^{\nu}\mathbf{M} = L^{\nu}{}_{\mu}\tilde{\sigma}^{\mu}, \tag{5.15}$$

$$\mathbf{N}^{\dagger}\sigma^{\nu}\mathbf{N} = L^{\nu}{}_{\mu}\sigma^{\mu}. \tag{5.16}$$

The matrices $\mathbf{M}$ and $\mathbf{N}$ are related by (B.19):

$$\mathbf{M}^{\dagger}\mathbf{N} = \mathbf{N}^{\dagger}\mathbf{M} = 1. \tag{5.17}$$

In the frame K' the Lagrangian density (5.12) can be written

$$\mathcal{L} = \mathrm{i}\psi_{\mathrm{L}}^{\dagger}\mathbf{M}^{\dagger}\tilde{\sigma}^{\nu}\mathbf{M}\partial'_{\nu}\psi_{\mathrm{L}} + \mathrm{i}\psi_{\mathrm{R}}^{\dagger}\mathbf{N}^{\dagger}\sigma^{\nu}\mathbf{N}\partial'_{\nu}\psi_{\mathrm{R}} - m(\psi_{\mathrm{L}}^{\dagger}\psi_{\mathrm{R}} + \psi_{\mathrm{R}}^{\dagger}\psi_{\mathrm{L}}), \tag{5.18}$$

where we have used (5.14) along with (5.15) and (5.16) in the first two terms.

We must define

$$\psi'_{\mathrm{L}}(x') = \mathbf{M}\psi_{\mathrm{L}}(x), \tag{5.19}$$

$$\psi'_{\mathrm{R}}(x') = \mathbf{N}\psi_{\mathrm{R}}(x), \tag{5.20}$$

to give

$$\mathcal{L} = \mathrm{i}\psi_{\mathrm{L}}'^{\dagger}\tilde{\sigma}^{\nu}\partial'_{\nu}\psi'_{\mathrm{L}} + \mathrm{i}\psi_{\mathrm{R}}'^{\dagger}\sigma^{\nu}\partial'_{\nu}\psi'_{\mathrm{R}} - m(\psi_{\mathrm{L}}'^{\dagger}\psi'_{\mathrm{R}} + \psi_{\mathrm{R}}'^{\dagger}\psi'_{\mathrm{L}})$$

(noting that $\psi_{\mathrm{L}}'^{\dagger}\psi'_{\mathrm{R}} = \psi_{\mathrm{L}}^{\dagger}\mathbf{M}^{\dagger}\mathbf{N}\psi_{\mathrm{R}} = \psi_{\mathrm{L}}^{\dagger}\psi_{\mathrm{R}}$, since $\mathbf{M}^{\dagger}\mathbf{N} = \mathbf{I}$, and similarly $\psi_{\mathrm{R}}'^{\dagger}\psi'_{\mathrm{L}} = \psi_{\mathrm{R}}^{\dagger}\psi_{\mathrm{L}}$).

With the transformations (5.19) and (5.20) the Lagrangian, and hence the field equations, take the same form in every inertial frame. The way to construct an $\mathbf{M}$ and an $\mathbf{N}$ for any Lorentz transformation is given in Appendix B.

An example of a rotation is

$$L^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.21}$$

This is a rotation of the coordinate axes through an angle θ about the z-axis and is equivalent to equations (2.1). The corresponding matrix $\mathbf{M}$ is unitary:

$$\mathrm{M} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\theta/2} \end{pmatrix}. \tag{5.22}$$

Hence, from (5.17), $\mathbf{N} = (\mathbf{M}^{\dagger})^{-1} = \mathrm{M}$, since $\mathbf{M}\mathbf{M}^{\dagger} = \mathbf{1}$. The reader may verify that (5.15) and (5.16) hold. $\mathbf{M}$ is unitary (and hence equal to $\mathbf{N}$) for all rotations.

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An example of a Lorentz boost is

$$L^{\mu}{}_{v} = \begin{pmatrix} \cosh\theta & 0 & 0 & -\sinh\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\theta & 0 & 0 & \cosh\theta \end{pmatrix}. \tag{5.23}$$

This is a boost with velocity $v/c = \tanh\theta$ along the z-axis and is equivalent to equations (2.3). The corresponding matrix **M** is

$$\mathrm{M} = \begin{pmatrix} e^{\theta/2} & 0 \\ 0 & e^{-\theta/2} \end{pmatrix}, \quad \text{and} \quad \mathbf{N} = (\mathbf{M}^{\dagger})^{-1} = \begin{pmatrix} e^{-\theta/2} & 0 \\ 0 & e^{\theta/2} \end{pmatrix} = \mathbf{M}^{-1}. \tag{5.24}$$

5.3 The parity transformation

The Lagrangian density (5.12) can also be made invariant under space inversion of the axes. Denoting by a prime the space coordinates of a point as seen from the inverted axes, we have

$$\mathbf{r}' = -\mathbf{r} \quad \text{and} \quad \nabla' = -\nabla. \tag{5.25}$$

Hence, from the definitions (5.10) of σ^{μ} and $\tilde{\sigma}^{\mu}$,

$$\tilde{\sigma}^{\mu}\partial'_{\mu} = \sigma^{\mu}\partial_{\mu}, \quad \sigma^{\mu}\partial'_{\mu} = \tilde{\sigma}^{\mu}\partial_{\mu}. \tag{5.26}$$

Our Lagrangian density (5.12) is evidently invariant if $\psi(\mathbf{r}) \to \psi^{P}(\mathbf{r}')$ where

$$\psi_{\mathrm{L}}^{P}(\mathbf{r}') = \psi_{\mathrm{R}}(\mathbf{r}), \quad \psi_{\mathrm{R}}^{P}(\mathbf{r}') = \psi_{\mathrm{L}}(\mathbf{r}). \tag{5.27}$$

Actually the Lagrangian density would also retain the same form if we were to take, for example,

$$\psi_{\mathrm{L}}^{P}(\mathbf{r}') = e^{i\alpha}\psi_{\mathrm{R}}(\mathbf{r}), \quad \psi_{\mathrm{R}}^{P}(\mathbf{r}') = e^{i\alpha}\psi_{\mathrm{L}}(\mathbf{r}),$$

for any real α. It is the standard convention to adopt the form (5.27) for the field transformation under space inversion.

5.4 Spinors

Two-component complex quantities that transform under a Lorentz transformation according to the rules (5.19) and (5.20) are called *left-handed spinors* and *right-handed spinors,* respectively. Our subscripts L and R anticipated this. The four-component Dirac field is often called a *Dirac spinor.*

Spinors have the remarkable property that they can be combined in pairs to make Lorentz scalars, pseudoscalars, four-vectors, pseudovectors and higher order tensors. For example, $(\psi_{\mathrm{L}}^{\dagger}\psi_{\mathrm{R}} + \psi_{\mathrm{R}}^{\dagger}\psi_{\mathrm{L}})$ is a Lorentz invariant real scalar and $i(\psi_{\mathrm{L}}^{\dagger}\psi_{\mathrm{R}} - \psi_{\mathrm{R}}^{\dagger}\psi_{\mathrm{L}})$ is a real pseudoscalar; it is invariant under proper Lorentz

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transformations but changes sign under space inversion. Using (5.15), (5.16) and (5.27), we can see that $(\psi_L^\dagger \tilde{\sigma}^\mu \psi_L + \psi_R^\dagger \sigma^\mu \psi_R)$ is a four-vector, the space-like components of which change sign under space inversion (since $\tilde{\sigma}^i = -\sigma^i$), and $(\psi_L^\dagger \tilde{\sigma}^\mu \psi_L - \psi_R^\dagger \sigma^\mu \psi_R)$ is an axial four-vector, the space-like components of which are unchanged under space inversion.

5.5 The matrices γ^μ

The separation of the Dirac spinor into left-handed and right-handed components will be particularly appropriate when we discuss the weak interaction. For describing the electromagnetic interactions of fermions it is convenient to introduce 4×4 matrices γ^μ defined by

$$\gamma^0 = \beta; \quad \gamma^i = \beta\alpha_i, \quad i = 1, 2, 3. \tag{5.28}$$

It follows from the properties of the β and α^i matrices that

$$\begin{aligned} &(\gamma^0)^2 = \mathbf{I}; \quad (\gamma^i)^2 = -\mathbf{I}, \quad i = 1, 2, 3; \\ &\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \mathbf{0}, \quad \mu \neq \nu. \end{aligned} \tag{5.29}$$

In the chiral representation,

$$\gamma^0 = \begin{pmatrix} \mathbf{0} & \sigma^0 \\ \sigma^0 & \mathbf{0} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}. \tag{5.30}$$

Written with the γ^μ matrices, the Lagrangian density (5.6) becomes

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi, \tag{5.31}$$

where $\bar{\psi}$ is the row matrix $\bar{\psi} = \psi^\dagger\gamma^0$, and the Dirac equation takes the symmetrical form

$$(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi = 0. \tag{5.32}$$

Another useful matrix $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$. In the chiral representation,

$$\gamma^5 = \begin{pmatrix} -\sigma^0 & \mathbf{0} \\ \mathbf{0} & \sigma^0 \end{pmatrix}.$$

The matrices $\frac{1}{2}(\mathbf{I} - \gamma^5)$, $\frac{1}{2}(\mathbf{I} + \gamma^5)$ are projection operators giving the left-handed and right-handed parts of a Dirac spinor:

$$\frac{1}{2}(\mathbf{I} - \gamma^5)\psi = \begin{pmatrix} \sigma^0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_L \\ \mathbf{0} \end{pmatrix}, \tag{5.33}$$

$$\frac{1}{2}(\mathbf{I} + \gamma^5)\psi = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sigma^0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \psi_R \end{pmatrix}. \tag{5.34}$$

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It is straightforward to verify that the Lorentz scalars and vectors constructed in Section 5.4 from two-component spinors can be written:

$$
\begin{aligned}
\psi_{\mathrm{L}}^{\dagger}\psi_{\mathrm{R}} + \psi_{R}^{\dagger}\psi_{L} &= \bar{\psi}\psi \text{ (scalar)}\\
\mathrm{i}(\psi_{\mathrm{L}}^{\dagger}\psi_{\mathrm{R}} - \psi_{\mathrm{R}}^{\dagger}\psi_{\mathrm{L}}) &= i\bar{\psi}\gamma^{5}\psi \text{ (pseudoscalar)}\\
\psi_{\mathrm{L}}^{\dagger}\tilde{\sigma}^{\mu}\psi_{\mathrm{L}} + \psi_{\mathrm{R}}^{\dagger}\sigma^{\mu}\psi_{\mathrm{R}} &= \bar{\psi}\gamma^{\mu}\psi \text{ (contravariant four-vector)}\\
\psi_{\mathrm{L}}^{\dagger}\tilde{\sigma}^{\mu}\psi_{\mathrm{L}} - \psi_{\mathrm{R}}^{\dagger}\sigma^{\mu}\psi_{\mathrm{R}} &= \bar{\psi}\gamma^{5}\gamma^{\mu}\psi \text{ (contravariant axial vector).}
\end{aligned}
$$

Note that these quantities are all real.

5.6 Making the Lagrangian density real

A potential problem with our Lagrangian density (5.6) or (5.12) is that it is not real. Regarding ψ as a wave function, $\mathcal{L}$ is a complex function; regarding ψ as an operator, $\mathcal{L}$ is not Hermitian. As a consequence, the energy–momentum tensor is complex. Indeed, to apply Hamilton's principle, the variation δS in the action must be real. The term $-m(\psi_{\mathrm{L}}^{\dagger}\psi_{\mathrm{R}} + \psi_{\mathrm{R}}^{\dagger}\psi_{\mathrm{L}})$ in (5.12) is real, and the imaginary part of $\mathcal{L}$ may be written

$$
\begin{aligned}
&(1/2\mathrm{i})[\mathrm{i}\psi_{\mathrm{L}}^{\dagger}\tilde{\sigma}^{\mu}\partial_{\mu}\psi_{\mathrm{L}} + \mathrm{i}\psi_{\mathrm{R}}^{\dagger}\sigma^{\mu}\partial_{\mu}\psi_{\mathrm{R}} - (\mathrm{i}\psi_{\mathrm{L}}^{\dagger}\tilde{\sigma}^{\mu}\partial_{\mu}\psi_{\mathrm{L}} + \mathrm{i}\psi_{\mathrm{R}}^{\dagger}\sigma^{\mu}\partial_{\mu}\psi_{\mathrm{R}})^{\dagger}]\\
&\quad = (1/2\mathrm{i})[\mathrm{i}\psi_{\mathrm{L}}^{\dagger}\tilde{\sigma}^{\mu}\partial_{\mu}\psi_{\mathrm{L}} + \mathrm{i}\psi_{\mathrm{R}}^{\dagger}\sigma^{\mu}\partial_{\mu}\psi_{\mathrm{R}} + \mathrm{i}(\partial_{\mu}\psi_{\mathrm{L}}^{\dagger})\tilde{\sigma}^{\mu}\psi_{\mathrm{L}} + \mathrm{i}(\partial_{\mu}\psi_{\mathrm{R}})^{\dagger}\sigma^{\mu}\psi_{\mathrm{R}}],
\end{aligned}
$$

(where we have used the Hermitian property of the matrices σ^{μ} and $\tilde{\sigma}^{\mu}$). The last expression is just

$$
(1/2)\partial_{\mu}(\psi_{\mathrm{L}}^{\dagger}\tilde{\sigma}^{\mu}\psi_{\mathrm{L}} + \psi_{\mathrm{R}}^{\dagger}\sigma^{\mu}\psi_{\mathrm{R}}).
$$

This is a sum of derivatives, which give only irrelevant end-point contributions to the action (cf. Section 3.1). Hence δS is real. The imaginary part of $\mathcal{L}$ can be discarded, and we can take

$$
\begin{aligned}
\mathcal{L} = \frac{1}{2}[(\mathrm{i}\psi_{\mathrm{L}}^{\dagger}\tilde{\sigma}^{\mu}\partial_{\mu}\psi_{\mathrm{L}} + \mathrm{i}\psi_{\mathrm{R}}^{\dagger}\sigma^{\mu}\partial_{\mu}\psi_{\mathrm{R}}) \qquad (5.35)\\
+ \text{Hermitian conjugate}] - m(\psi_{\mathrm{L}}^{\dagger}\psi_{\mathrm{R}} + \psi_{\mathrm{R}}^{\dagger}\psi_{\mathrm{L}}). \qquad (5.36)
\end{aligned}
$$

For further interesting discussion of this question see Olive (1997).

Problems

5.1 Show that the matrix $\mathbf{M} = \mathbf{N}$ of equation (5.22) when inserted into equations (5.15) and (5.16) generates the rotation matrix (5.21).

5.2 Show that the matrices $\mathbf{M}$ and $\mathbf{N} = \mathbf{M}^{-1}$ given by equation (5.24) when inserted into equations (5.15) and (5.16) generate the Lorentz boost of equation (5.23).

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5.3 Show that $\psi_R^\dagger \psi_L$ and $\psi_L^\dagger \psi_R$ are invariant under proper Lorentz transformations.

Show that $\psi_R^\dagger \sigma^\mu \psi_R$ and $\psi_L{}^\dagger \tilde{\sigma}^\mu \psi_L$ are contravariant four-vectors under proper Lorentz transformations.

Show that $\psi_R^\dagger \sigma^\mu \tilde{\sigma}^\nu \psi_L$ and $\psi_L^\dagger \tilde{\sigma}^\mu \sigma^\nu \psi_R$ are contravariant tensors under proper Lorentz transformations.

5.4 Demonstrate the equivalence of the expressions (5.6) and (5.31) for the Lagrangian density.

5.5 Show that γ^5 has the properties

$$(\gamma^5)^2 = \mathbf{I}; \quad \gamma^\mu \gamma^5 = -\gamma^5 \gamma^\mu; \quad \mu = 0, 1, 2, 3.$$

5.6 Show that $\mathrm{i}\bar{\psi}\gamma^5\psi$ is a pseudoscalar field and $\bar{\psi}\gamma^5\gamma^\mu\psi = -\bar{\psi}\gamma^\mu\gamma^5\psi$ is an axial vector field.

5.7 Show that $(\gamma^0)^\dagger = \gamma^0$, $(\gamma^i)^\dagger = -\gamma^i$.