3
General Relativity

In order to discuss the occurrence of singularities and the possible breakdown of General Relativity, it is important to have a precise statement of the theory and to indicate to what extent it is unique. We shall therefore present the theory as a number of postulates about a mathematical model for space–time.

In § 3.1 we introduce the mathematical model and in § 3.2 the first two postulates, local causality and local energy conservation. These postulates are common to both Special and General Relativity, and thus may be regarded as tested by the many experiments that have been performed to check the former. In § 3.3 we derive the equations of the matter fields and obtain the energy–momentum tensor from a Lagrangian.

The third postulate, the field equations, is given in § 3.4. This is not so well established experimentally as the first two postulates, but we shall see that any alternative equations would seem to have one or more undesirable properties, or else require the existence of extra fields which have not yet been detected experimentally.

3.1 The space–time manifold

The mathematical model we shall use for space–time, i.e. the collection of all events, is a pair $(\mathscr{M}, \mathbf{g})$ where $\mathscr{M}$ is a connected four-dimensional Hausdorff C^∞ manifold and $\mathbf{g}$ is a Lorentz metric (i.e. a metric of signature $+2$) on $\mathscr{M}$.

Two models $(\mathscr{M}, \mathbf{g})$ and $(\mathscr{M}', \mathbf{g}')$ will be taken to be equivalent if they are *isometric*, that is if there is a diffeomorphism $\theta: \mathscr{M} \to \mathscr{M}'$ which carries the metric $\mathbf{g}$ into the metric $\mathbf{g}'$, i.e. $\theta_* \mathbf{g} = \mathbf{g}'$. Strictly speaking then, the model for space–time is not just one pair $(\mathscr{M}, \mathbf{g})$ but a whole equivalence class of all pairs $(\mathscr{M}', \mathbf{g}')$ which are equivalent to $(\mathscr{M}, \mathbf{g})$. We shall normally work with just one representative member $(\mathscr{M}, \mathbf{g})$ of the equivalence class, but the fact that this pair is defined only up to equivalence is important in some situations, in particular in the discussion of the Cauchy problem in chapter 7.

The manifold $\mathscr{M}$ is taken to be connected since we would have no knowledge of any disconnected component. It is taken to be Hausdorff since this seems to accord with normal experience. However in chapter 5 we shall consider an example in which one might dispense with this condition. Together with the existence of a Lorentz metric, the Hausdorff condition implies that $\mathscr{M}$ is paracompact (Geroch (1968*c*)).

A manifold corresponds naturally to our intuitive ideas of the continuity of space and time. So far this continuity has been established for distances down to about 10^{-15} cm by experiments on pion scattering (Foley *et al.* (1967)). It may be difficult to extend this to much smaller lengths as to do so would require a particle of such high energy that several other particles might be created and confuse the experiment. Thus it may be that a manifold model for space–time is inappropriate for distances less than 10^{-15} cm and that we should use theories in which space–time has some other structure on this scale. However such breakdowns of the manifold picture would not be expected to affect General Relativity until the typical gravitational length scale became of that order. This would happen when the density became about 10^{58} gm cm^{-3}, which is a condition so extreme as to be completely beyond our present knowledge. Nevertheless, by adopting a manifold model for space–time, and making certain other reasonable assumptions, we shall show in chapters 8–10 that some breakdowns of General Relativity must occur. It may be the field equations that go wrong, or it may be that quantization of the metric is needed, or it may be a breakdown of the manifold structure itself that occurs.

The metric $\mathbf{g}$ enables the non-zero vectors at a point $p \in \mathscr{M}$ to be divided into three classes: a non-zero vector $\mathbf{X} \in T_p$ being said to be *timelike*, *spacelike* or *null* according to whether $g(\mathbf{X}, \mathbf{X})$ is negative, positive or zero respectively (cf. figure 5).

The order of differentiability, r, of the metric ought to be sufficient for the field equations to be defined. They can be defined in a distributional sense if the metric coordinate components g_{ab} and g^{ab} are continuous and have locally square integrable generalized first derivatives with respect to the local coordinates. (A set of functions $f_{;a}$ on R^n are said to be the generalized derivatives of a function f on R^n if, for any C^∞ function ψ on R^n with compact support,

$$\int f_{;a}\psi \, \mathrm{d}^n x = -\int f(\partial\psi/\partial x^a)\, \mathrm{d}^n x.)$$

However this condition is too weak, since it guarantees neither the existence nor the uniqueness of geodesics, for which a C^{2-} metric is required. (A C^{2-} metric is one for which the first coordinate derivatives of the metric coordinate components satisfy a local Lipschitz condition, see § 2.1.) We shall in fact assume for most of the book that the metric is at least C^2. This allows the field equations (which involve the second derivatives of the metric) to be defined at every point. In § 8.4 we shall weaken the condition on the metric to C^{2-} and show that this does not affect the results on the occurrence of singularities.

In chapter 7, we use a different kind of differentiability condition in order to show that the time development of the field equations is determined by suitable initial conditions. We require there that the metric components and their generalized first derivatives up to order $m (m \geqslant 4)$ are locally square integrable. This would certainly be true if the metric were C^4.

In fact, the order of differentiability of the metric is probably not physically significant. Since one can never measure the metric exactly, but only with some margin of error, one could never determine that there was an actual discontinuity in its derivatives of any order. Thus one can always represent one's measurements by a C^∞ metric.

If the metric is assumed to be C^r, the atlas of the manifold must be C^{r+1}. However, one can always find an analytic subatlas in any C^s atlas $(s \geqslant 1)$ (Whitney (1936), cf. Munkres (1954)). Thus it is no restriction to assume from the start that the atlas is analytic, even though one could physically determine only a C^{r+1} atlas if the metric were C^r.

We have to impose some condition on our model $(\mathscr{M}, \mathbf{g})$ to ensure that it includes all the non-singular points of space–time. We shall say that the C^r pair $(\mathscr{M}', \mathbf{g}')$ is a *C^r-extension* of $(\mathscr{M}, \mathbf{g})$ if there is an isometric C^r imbedding $\mu: \mathscr{M} \to \mathscr{M}'$. If there were such an extension $(\mathscr{M}', \mathbf{g}')$ we should have to regard points of $\mathscr{M}'$ as also being points of space–time. We therefore require that the model $(\mathscr{M}, \mathbf{g})$ is *C^r-inextendible*, that is there is no C^r extension $(\mathscr{M}', \mathbf{g}')$ of $(\mathscr{M}, \mathbf{g})$ where $\mu(\mathscr{M})$ does not equal $\mathscr{M}'$.

As an example of a pair $(\mathscr{M}_1, \mathbf{g}_1)$ which is not inextendible, consider two-dimensional Euclidean space with the x-axis removed between $x_1 = -1$ and $x_1 = +1$. The obvious way to extend this would simply be to replace the missing points, but one could also extend it by taking another copy $(\mathscr{M}_2, \mathbf{g}_2)$ of the space, and identifying the bottom side of the x_1-axis for $|x_1| < 1$ with the top side of the x_2-axis for $|x_2| < 1$, and also identifying the top side of the x_1-axis for $|x_1| < 1$ with the bottom side of the x_2-axis for $|x_2| < 1$. The resultant space $(\mathscr{M}_3, \mathbf{g}_3)$ is inextendible but not complete as we have left out the points $x_1 = \pm 1$, $y_1 = 0$. We cannot put these points back in because we were perverse enough to extend the top and bottom sides of the x-axis on different sheets. If however one takes the subset $\mathscr{U}$ of $\mathscr{M}_3$ defined by $1 < x_1 < 2$, $-1 < y_1 < 1$, then one could extend the pair $(\mathscr{U}, \mathbf{g}_3|_{\mathscr{U}})$ and put back the point $x_1 = 1, y_1 = 0$. This motivates a rather stronger definition of inextendibility: a pair $(\mathscr{M}, \mathbf{g})$ is said to be *C^r-locally inextendible* if there is no open set $\mathscr{U} \subset \mathscr{M}$ with non-compact closure in $\mathscr{M}$, such that the pair $(\mathscr{U}, \mathbf{g}|_{\mathscr{U}})$ has an extension $(\mathscr{U}', \mathbf{g}')$ in which the closure of the image of $\mathscr{U}$ is compact.

3.2 The matter fields

There will be various fields on $\mathscr{M}$, such as the electromagnetic field, the neutrino field, etc., which describe the matter content of space–time. These fields will obey equations which can be expressed as relations between tensors on $\mathscr{M}$ in which all derivatives with respect to position are covariant derivatives with respect to the symmetric connection defined by the metric $\mathbf{g}$. This is so because the only relations defined by a manifold structure are tensor relations, and the only connection defined so far is that given by the metric. If there were another connection on $\mathscr{M}$, the difference between the two connections would be a tensor and could be regarded as another physical field. Similarly another metric on $\mathscr{M}$ could be regarded as a further physical field. (The equations of the matter fields are sometimes expressed as relations between spinors on $\mathscr{M}$. We do not deal with such relations in this book, as they are not needed for the problems we wish to consider. In fact, all spinor equations can be replaced by rather more complicated tensor equations; see e.g. Ruse (1937).)

The theory one obtains depends on what matter fields one incorporates in it. One should of course include all such fields which have been experimentally observed, but one might postulate the existence of as yet undetected fields. Thus for example Brans and Dicke (Dicke (1964), appendix 7) postulate the existence of a long range scalar field which is weakly coupled to the trace of the energy–momentum tensor. In the form given in Dicke (1964) appendix 2, the Brans–Dicke theory can be regarded simply as General Relativity with an extra scalar field. Whether this scalar field has been experimentally detected or not is at present under dispute.

We shall denote the matter fields included in the theory by $\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}$, where the subscript (i) numbers the fields considered. The following two postulates on the nature of the equations obeyed by the $\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}$ are common to both the Special and the General Theories of Relativity.

Postulate (a): *Local causality*

The equations governing the matter fields must be such that if $\mathscr{U}$ is a convex normal neighbourhood and p and q are points in $\mathscr{U}$ then a signal can be sent in $\mathscr{U}$ between p and q if and only if p and q can be joined by a C^1 curve lying entirely in $\mathscr{U}$, whose tangent vector is everywhere non-zero and is either timelike or null; we shall call such a curve, *non-spacelike*. (Our formulation of relativity excludes the possibility of particles such as tachyons, which move on spacelike curves.) Whether the signal is sent from p to q or from q to p will depend on the direction of time in $\mathscr{U}$. The problem of whether a consistent direction of time can be assigned at all points of space–time will be considered in § 6.2.

A more precise statement of this postulate can be given in terms of the Cauchy problem of the matter fields. Let $p \in \mathscr{U}$ be such that every non-spacelike curve through p intersects the spacelike surface $x^4 = 0$ within $\mathscr{U}$. Let $\mathscr{F}$ be the set of points in the surface $x^4 = 0$ which can be reached by non-spacelike curves in $\mathscr{U}$ from p. Then we require that the values of the matter fields at p must be uniquely determined by the values of the fields and their derivatives up to some finite order on $\mathscr{F}$, and that they are not uniquely determined by the values on any proper subset of $\mathscr{F}$ to which it can be continuously retracted. (For a fuller discussion of the Cauchy problem, see chapter 7.)

It is this postulate which sets the metric $\mathbf{g}$ apart from the other fields on $\mathscr{M}$ and gives it its distinctive geometrical character. If $\{x^a\}$ are normal coordinates in $\mathscr{U}$ about p, it is intuitively fairly obvious (and is proved in chapter 4) that the points which can be reached from p by non-spacelike curves in $\mathscr{U}$ are those whose coordinates satisfy

$$(x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2 \leqslant 0.$$

The boundary of these points is formed by the image of the null cone of p under the exponential map, that is the set of all null geodesics through p. Thus by observing which points can communicate with p, one can determine the null cone N_p in T_p. Once N_p is known, the metric at p may be determined up to a conformal factor. This may be seen as

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follows: let $\mathbf{X}, \mathbf{Y} \in T_p$ be respectively timelike and spacelike vectors. The equation

$$g(\mathbf{X}+\lambda\mathbf{Y}, \mathbf{X}+\lambda\mathbf{Y}) = g(\mathbf{X},\mathbf{X}) + 2\lambda g(\mathbf{X},\mathbf{Y}) + \lambda^2 g(\mathbf{Y},\mathbf{Y})$$
$$= 0$$

will have two real roots λ_1 and λ_2 as $g(\mathbf{X},\mathbf{X}) < 0$ and $g(\mathbf{Y},\mathbf{Y}) > 0$. If N_p is known, λ_1 and λ_2 may be determined. But

$$\lambda_1\lambda_2 = g(\mathbf{X},\mathbf{X})/g(\mathbf{Y},\mathbf{Y}).$$

Thus the ratio of the magnitudes of a timelike vector and a spacelike vector may be found from the null cone. Then if $\mathbf{W}$ and $\mathbf{Z}$ are any two non-null vectors at p,

$$g(\mathbf{W},\mathbf{Z}) = \tfrac{1}{2}(g(\mathbf{W},\mathbf{W}) + g(\mathbf{Z},\mathbf{Z}) - g(\mathbf{W}+\mathbf{Z}, \mathbf{W}+\mathbf{Z})).$$

Each of the magnitudes on the right-hand side may be compared with the magnitude of either $\mathbf{X}$ or $\mathbf{Y}$, and so $g(\mathbf{W},\mathbf{Z})/g(\mathbf{X},\mathbf{X})$ may be found. (If $\mathbf{W}+\mathbf{Z}$ is null, the corresponding expression involving $\mathbf{W}+2\mathbf{Z}$ could be used.) Thus observation of local causality enables one to measure the metric up to a conformal factor. In practice this measurement is performed most conveniently using the experimental fact that no signal has been observed to travel faster than electromagnetic radiation. This means that light must travel on null geodesics. This however is a consequence of the particular equations the electromagnetic field obeys, not of the theory of relativity itself. Causality will be considered further in chapter 6. Among other results, it will be shown that causal relations may be used to determine the topological structure of $\mathcal{M}$. The conformal factor in the metric may be determined using postulate (*b*) below; thus all the elements of the theory will be physically observable.

Postulate (*b*): *Local conservation of energy and momentum*

The equations governing the matter fields are such that there exists a symmetric tensor T^{ab}, called the energy–momentum tensor, which depends on the fields, their covariant derivatives, and the metric, and which has the properties:

(i) T^{ab} vanishes on an open set $\mathcal{U}$ if and only if all the matter fields vanish on $\mathcal{U}$,

(ii) T^{ab} obeys the equation

$$T^{ab}{}_{;b} = 0. \tag{3.1}$$

Condition (i) expresses the principle that all fields have energy. One might possibly object to the 'only if' on the grounds that there might be two non-zero fields, one of whose energy–momentum tensor exactly cancelled that of the other. This possibility is related to that of the existence of negative energy which will be discussed in § 3.3.

If the metric admits a Killing vector field $\mathbf{K}$, equations (3.1) can be integrated to give a conservation law. To see this, define P^a to be the vector whose components are $P^a = T^{ab}K_b$. Then,

$$P^a{}_{;a} = T^{ab}{}_{;a}K_b + T^{ab}K_{b;a}.$$

The first term is zero by the conservation equations, and the second vanishes as T^{ab} is symmetric and $2K_{(a;b)} = L_{\mathbf{K}}g_{ab} = 0$, since $\mathbf{K}$ is a Killing vector. Thus if $\mathscr{D}$ is a compact orientable region with boundary $\partial\mathscr{D}$, Gauss' theorem (§ 2.7) shows

$$\int_{\partial\mathscr{D}} P^b \,\mathrm{d}\sigma_b = \int_{\mathscr{D}} P^b{}_{;b}\,\mathrm{d}v = 0. \tag{3.2}$$

This may be interpreted as saying that the total flux over a closed surface of the $\mathbf{K}$-component of energy–momentum is zero.

When the metric is flat, as it is in the Special Theory of Relativity, one may choose coordinates $\{x^a\}$ in which the components of the metric are $g_{ab} = e_a\,\delta_{ab}$ (no summation) where δ_{ab} is the Kronecker delta and e_a is -1 if $a = 4$ and is $+1$ if $a = 1, 2, 3$. Then the following are Killing vectors:

$$\underset{\alpha}{\mathbf{L}} = \partial/\partial x^\alpha \quad (\alpha = 1,\, 2,\, 3,\, 4)$$

(these generate four translations) and

$$\underset{\alpha\beta}{\mathbf{M}} = e_\alpha x^\alpha \frac{\partial}{\partial x^\beta} - e_\beta x^\beta \frac{\partial}{\partial x^\alpha} \quad (\text{no summation; } \alpha, \beta = 1,\, 2,\, 3,\, 4)$$

(these generate six 'rotations' in space–time). These isometries form the ten-parameter Lie group of isometries of flat space–time known as the inhomogeneous Lorentz group. One may use them to define ten vectors $\underset{\alpha}{P^a}$ and $\underset{\alpha\beta}{P^a}$ which will obey (3.2). We may think of $\underset{4}{\mathbf{P}}$ as representing the flow of energy and $\underset{1}{\mathbf{P}}, \underset{2}{\mathbf{P}}, \underset{3}{\mathbf{P}}$ as the flow of the three components of linear momentum. The $\underset{\alpha\beta}{\mathbf{P}}$ can be interpreted as the flow of angular momentum.

If the metric is not flat there will not, in general, be any Killing vectors and so the above integral conservation laws will not hold. However, in a suitable neighbourhood of a point q one may introduce normal coordinates $\{x^a\}$. Then at q the components g_{ab} of the metric are $e_a \delta_{ab}$ (no summation), and the components $\Gamma^a{}_{bc}$ of the connection are zero. One may take a neighbourhood $\mathscr{D}$ of q in which the g_{ab} and $\Gamma^a{}_{bc}$ differ from their values at q by an arbitrarily small amount; then the $\underset{\alpha}{L}_{(a;b)}$ and $\underset{\alpha\beta}{M}_{(a;b)}$ will not exactly vanish in $\mathscr{D}$, but will in this neighbourhood differ from zero by an arbitrarily small amount. Thus

$$\int_{\partial\mathscr{D}} \underset{\alpha}{P}^b \, \mathrm{d}\sigma_b \quad \text{and} \quad \int_{\partial\mathscr{D}} \underset{\alpha\beta}{P}^b \, \mathrm{d}\sigma_b$$

will still be zero in the first approximation; that is to say, one still has approximate conservation of energy, momentum and angular momentum in a small region of space–time. Using this it can be shown that a small isolated body moves approximately on a timelike geodesic curve independent of its internal constitution provided that the energy density of matter in it is non-negative (for an account of the motion of a small body in relativity, see Dixon (1970)). This may be thought of as Galileo's principle that all bodies fall equally fast. In Newtonian terms one would say that the inertial mass (the m in $\mathbf{F} = m\mathbf{a}$) and the passive gravitational mass (the mass acted on by a gravitational field) are equal for all bodies. This has been verified to a high order of accuracy in experiments by Eötvos and by Dicke (1964).

Postulate (*a*) enables one to measure the metric up to a conformal factor at each point. Using postulate (*b*) one may relate these factors at different points, for the conservation equations $T^{ab}{}_{;b} = 0$ would not in general hold for a connection derived from a metric $\hat{\mathbf{g}} = \Omega^2\mathbf{g}$. One way of doing this would be to observe the paths of small 'test' particles and so to determine the timelike geodesic curves. Then if $\gamma(t)$ is such a curve with tangent vector $\mathbf{K} = (\partial/\partial t)_\gamma$, one has from (2.29)

$$\frac{\hat{\mathrm{D}}}{\partial t} K^a = \frac{\mathrm{D}}{\partial t} K^a + 2\Omega^{-1}\Omega_{;b} K^b K^a - \Omega^{-1}(K^b K^c \hat{g}_{bc}) \hat{g}^{ad} \Omega_{;d}.$$

Since $\gamma(t)$ is a geodesic with respect to the space–time metric $\mathbf{g}$, $K^{[b}(\mathrm{D}/\partial t) K^{a]} = 0$. Thus

$$K^{[b} \frac{\hat{\mathrm{D}}}{\partial t} K^{a]} = -(K^c K^d \hat{g}_{cd}) K^{[b} \hat{g}^{a]e} (\log \Omega)_{;e}. \tag{3.3}$$

Knowing the conformal structure, one can choose a metric $\hat{\mathbf{g}}$ which represents the conformal equivalence class of metrics and can evaluate the left-hand side of (3.3) for any test particle. Then the right-hand side of (3.3) determines $(\log \Omega)_{;b}$ up to the addition of a multiple of $K^a \hat{g}_{ab}$.

By considering another curve $\gamma'(t)$ whose tangent vector K'^a is not parallel to K^a, one can find $(\log \Omega)_{;b}$ and so can determine Ω everywhere up to a constant multiplying factor. This constant factor specifies one's units of measurement, and so can be chosen arbitrarily.

This is, of course, not the way one measures the conformal factor in practice; one makes use of the fact that there exist a large number of similar systems (such as the electronic states of atoms) whose internal motions define a number of events along the timelike curve which represents their position in space–time. The intervals between these events seem to be independent of their past history in the sense that the intervals measured by two nearby systems correspond. If one can effectively isolate them against external matter fields (so they must move on geodesic curves) and if one assumes their internal motion is independent of the curvature of space–time, then the only thing it can depend on is the metric. Thus the arc-length between two successive events on a curve must be the same for each pair of successive events on any such curve. If one takes this arc-length as one's unit of measurement, one can determine the conformal factor at any point of space–time.

In fact it may not be possible to isolate a system from external matter fields. Thus for example in the Brans–Dicke theory there is a scalar field which is non-zero everywhere. However the conformal factor can still be determined by the requirement that the conservation equation $T^{ab}{}_{;b} = 0$ should hold. Thus knowledge of the energy–momentum tensor T_{ab} determines the conformal factor.

3.3 Lagrangian formulation

The conditions (i) and (ii) of postulate (*b*) do not tell one how to construct the energy–momentum tensor for a given set of fields, or whether it is unique. In practice one relies heavily on one's intuitive knowledge of what energy and momentum are. However, there is a definite and unique formula for the energy–momentum tensor in the case that the equations of the fields can be derived from a Lagrangian.

Let L be the Lagrangian which is some scalar function of the fields $\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}$, their first covariant derivatives, and the metric. One obtains the equations of the fields by requiring that the action

$$I = \int_{\mathscr{D}} L \, \mathrm{d}v$$

be stationary under variations of the fields in the interior of a compact four-dimensional region $\mathscr{D}$. By a *variation of the fields* $\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}$ in $\mathscr{D}$ we mean a one-parameter family of fields $\mathbf{\Psi}_{(i)}(u,r)$ where $u \in (-\epsilon, \epsilon)$ and $r \in \mathscr{M}$, such that

(i) $\mathbf{\Psi}_{(i)}(0,r) = \mathbf{\Psi}_{(i)}(r)$,

(ii) $\mathbf{\Psi}_{(i)}(u,r) = \mathbf{\Psi}_{(i)}(r) \quad \text{when} \quad r \in \mathscr{M} - \mathscr{D}$.

We denote $\qquad \partial\mathbf{\Psi}_{(i)}(u,r)/\partial u|_{u=0} \quad \text{by} \quad \Delta\mathbf{\Psi}_{(i)}$.

Then

$$\left.\frac{\partial I}{\partial u}\right|_{u=0} = \sum_{(i)} \int_{\mathscr{D}} \left(\frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}} \Delta \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d} + \frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}} \Delta(\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}) \right) \mathrm{d}v,$$

where $\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}$ are the components of the covariant derivatives of $\mathbf{\Psi}_{(i)}$. But $\Delta(\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}) = (\Delta\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d})_{;e}$, thus the second term can be expressed as

$$\sum_{(i)} \int_{\mathscr{D}} \left[\left(\frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}} \Delta \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d} \right)_{;e} - \left(\frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}} \right)_{;e} \Delta \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d} \right] \mathrm{d}v.$$

The first term in this expression can be written as

$$\int_{\mathscr{D}} Q^a{}_{;a}\, \mathrm{d}v = \int_{\partial\mathscr{D}} Q^a\, \mathrm{d}\sigma_a,$$

where $\mathbf{Q}$ is a vector whose components are

$$Q^e = \sum_{(i)} \frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}} \Delta \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}.$$

This integral is zero as condition (ii) is the statement that $\Delta\mathbf{\Psi}_{(i)}$ vanish at the boundary $\partial\mathscr{D}$. Thus in order that $\partial I/\partial u|_{u=0}$ should vanish for all variations on all volumes $\mathscr{D}$, it is necessary and sufficient that the *Euler–Lagrange equations*,

$$\frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}} - \left(\frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}} \right)_{;e} = 0, \tag{3.4}$$

hold for all i. These are the equations of the fields.

We obtain the energy–momentum tensor from the Lagrangian by considering the change in the action induced by a change in the metric.

Suppose a variation $g_{ab}(u,r)$ leaves the fields $\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}$ unchanged but alters the components g_{ab} of the metric. Then

$$\left.\frac{\partial I}{\partial u}\right|_{u=0}=\int_{\mathscr{D}}\left(\sum_{(i)}\frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}}\Delta(\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e})+\frac{\partial L}{\partial g_{ab}}\Delta g_{ab}\right)\mathrm{d}v$$
$$+\int_{\mathscr{D}}L\frac{\partial(\mathrm{d}v)}{\partial g_{ab}}\Delta g_{ab}. \quad (3.5)$$

The last term arises because the volume measure $\mathrm{d}v$ depends on the metric, and so will vary when the metric is varied. To evaluate this term, recall that $\mathrm{d}v$ is in fact the four-form $(4!)^{-1}\boldsymbol{\eta}$ whose components are $\eta_{abcd}=(-g)^{\frac{1}{2}}4!\,\delta_{[a}{}^{1}\delta_{b}{}^{2}\delta_{c}{}^{3}\delta_{d]}{}^{4}$, where $g\equiv\det(g_{ab})$. Therefore

$$\begin{aligned}\frac{\partial\eta_{abcd}}{\partial g_{ef}}&=-\tfrac{1}{2}(-g)^{-\frac{1}{2}}\frac{\partial g}{\partial g_{ef}}4!\,\delta_{[a}{}^{1}\delta_{b}{}^{2}\delta_{c}{}^{3}\delta_{d]}{}^{4}\\&=-\tfrac{1}{2}(-g)^{-\frac{1}{2}}g^{ef}g4!\,\delta_{[a}{}^{1}\delta_{b}{}^{2}\delta_{c}{}^{3}\delta_{d]}{}^{4}\\&=\tfrac{1}{2}g^{ef}\eta_{abcd}.\end{aligned}$$

Thus

$$\frac{\partial(\mathrm{d}v)}{\partial g_{ab}}=\tfrac{1}{2}g^{ab}\,\mathrm{d}v.$$

The first term in (3.5) arises because $\Delta(\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e})$ will not necessarily be zero even though $\Delta\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}$ is, since the variation in the metric will induce a variation in the components $\Gamma^{a}{}_{bc}$ of the connection. As the difference between two connections transforms like a tensor, $\Delta\Gamma^{a}{}_{bc}$ may be regarded as the components of a tensor. They are related to the variation in the components of the metric by

$$\Delta\Gamma^{a}{}_{bc}=\tfrac{1}{2}g^{ad}\{(\Delta g_{db})_{;c}+(\Delta g_{dc})_{;b}-(\Delta g_{bc})_{;d}\}.$$

(The easiest way to derive this formula is to note that since it is a tensor relation, it must be valid in any coordinate system. In particular, one could choose normal coordinates about a point p. For these coordinates the components $\Gamma^{a}{}_{bc}$ and the coordinate derivatives of the components g_{ab} vanish at p. The formula given can then be verified to hold at p.) Using this relation, $\Delta\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}$ may be expressed in terms of $(\Delta g_{bc})_{;d}$ and the usual integration by parts employed to give an integrand involving Δg_{ab} only. Thus we may write $\partial I/\partial u$ as

$$\frac{1}{2}\int_{\mathscr{D}}(T^{ab}\Delta g_{ab})\,\mathrm{d}v,$$

where T^{ab} are the components of a symmetric tensor which is taken to be the energy–momentum tensor of the fields. (See Rosenfeld (1940) for the relation between this tensor and the so-called canonical energy–momentum tensor.)

This energy–momentum tensor satisfies the conservation equations as a consequence of the field equations obeyed by the $\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}$. For suppose one has a diffeomorphism $\phi\colon \mathscr{M}\to\mathscr{M}$ which is the identity everywhere except in the interior of $\mathscr{D}$. Then, by the invariance of integrals under a differential map,

$$I = \int_{\mathscr{D}} L\,\mathrm{d}v = \frac{1}{4!}\int_{\mathscr{D}} L\eta = \frac{1}{4!}\int_{\phi(\mathscr{D})} L\eta = \frac{1}{4!}\int_{\mathscr{D}} \phi^*(L\eta).$$

Thus

$$\frac{1}{4!}\int_{\mathscr{D}} (L\eta - \phi^*(L\eta)) = 0.$$

If the diffeomorphism ϕ is generated by a vector field $\mathbf{X}$ (non-zero only in the interior of $\mathscr{D}$) it follows that

$$\frac{1}{4!}\int_{\mathscr{D}} L_{\mathbf{X}}(L\eta) = 0.$$

But

$$\frac{1}{4!}\int_{\mathscr{D}} L_{\mathbf{X}}(L\eta) = \sum_{(i)}\int_{\mathscr{D}}\left(\frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}} - \left(\frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}}\right)_{;e}\right) \times L_{\mathbf{X}}\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}\,\mathrm{d}v + \frac{1}{2}\int_{\mathscr{D}} T^{ab}L_{\mathbf{X}}g_{ab}\,\mathrm{d}v.$$

The first term vanishes as a consequence of the field equations. In the second term, $L_{\mathbf{X}}g_{ab} = 2X_{(a;b)}$. Thus

$$\int_{\mathscr{D}} (T^{ab}L_{\mathbf{X}}g_{ab})\,\mathrm{d}v = 2\int_{\mathscr{D}} ((T^{ab}X_a)_{;b} - T^{ab}{}_{;b}X_a)\,\mathrm{d}v.$$

The first contribution may be transformed into an integral over the boundary of $\mathscr{D}$ which vanishes as $\mathbf{X}$ is zero there. Since the second term must therefore be zero for arbitrary $\mathbf{X}$, it follows that $T^{ab}{}_{;b} = 0$.

We shall now give as examples Lagrangians for some fields which will be of interest later.

Example 1: *A scalar field* ψ

This can represent, for example, the π^0-meson. The Lagrangian is

$$L = -\tfrac{1}{2}\psi_{;a}\psi_{;b}g^{ab} - \frac{1}{2}\frac{m^2}{\hbar^2}\psi^2$$

where m, $\hbar$ are constants. The Euler–Lagrange equations (3.4) are

$$\psi_{;ab}g^{ab} - \frac{m^2}{\hbar^2}\psi = 0.$$

The energy–momentum tensor is

$$T_{ab} = \psi_{;a}\psi_{;b} - \tfrac{1}{2}g_{ab}\left(\psi_{;c}\psi_{;d}g^{cd} + \frac{m^2}{\hbar^2}\psi^2\right). \tag{3.6}$$

Example 2: *The electromagnetic field*

This is described by a one-form **A**, called the potential, which is defined up to the addition of a gradient of a scalar function. The Lagrangian is

$$L = -\frac{1}{16\pi} F_{ab}F_{cd}g^{ac}g^{bd},$$

where the electromagnetic field tensor F is defined as $2\,\mathrm{d}\mathbf{A}$, i.e. $F_{ab} = 2A_{[b;a]}$. Varying A_a, the Euler–Lagrange equations (3.4) are

$$F_{ab;c}g^{bc} = 0.$$

This and $F_{[ab;c]} = 0$ (which is the equation $\mathrm{d}\mathbf{F} = \mathrm{d}(\mathrm{d}\mathbf{A}) = 0$) are the Maxwell equations for the source-free electromagnetic field. The energy–momentum tensor is

$$T_{ab} = \frac{1}{4\pi}(F_{ac}F_{bd}g^{cd} - \tfrac{1}{4}g_{ab}F_{ij}F_{kl}g^{ik}g^{jl}). \tag{3.7}$$

Example 3: *A charged scalar field*

This is really a combination of two real scalar fields ψ_1 and ψ_2. These are combined into a complex scalar field $\psi = \psi_1 + \mathrm{i}\psi_2$, which could represent, for example, π^+ and π^- mesons. The total Lagrangian of the scalar field and electromagnetic field is

$$L = -\tfrac{1}{2}(\psi_{;a} + \mathrm{i}eA_a\psi)g^{ab}(\bar{\psi}_{;b} - \mathrm{i}eA_b\bar{\psi}) - \frac{1}{2}\frac{m^2}{\hbar^2}\psi\bar{\psi} - \frac{1}{16\pi}F_{ab}F_{cd}g^{ac}g^{bd},$$

where e is a constant and $\bar{\psi}$ is the complex conjugate of ψ. Varying ψ, $\bar{\psi}$ and A_a independently, one obtains

$$\psi_{;ab}g^{ab} - \frac{m^2}{\hbar^2}\psi + \mathrm{i}eA_a g^{ab}(2\psi_{;b} + \mathrm{i}eA_b\psi) + \mathrm{i}eA_{a;b}g^{ab}\psi = 0,$$

and its complex conjugate, and

$$\frac{1}{4\pi}F_{ab;c}g^{bc} - \mathrm{i}e\psi(\bar{\psi}_{;a} - \mathrm{i}eA_a\bar{\psi}) + \mathrm{i}e\bar{\psi}(\psi_{;a} + \mathrm{i}eA_a\psi) = 0.$$

The energy–momentum tensor is

$$T_{ab} = \tfrac{1}{2}(\psi_{;a}\bar{\psi}_{;b} + \bar{\psi}_{;a}\psi_{;b}) + \tfrac{1}{2}(-\psi_{;a}\mathrm{i}eA_b\bar{\psi} + \bar{\psi}_{;b}\mathrm{i}eA_a\psi$$
$$+ \bar{\psi}_{;a}\mathrm{i}eA_b\psi - \psi_{;b}\mathrm{i}eA_a\bar{\psi}) + \frac{1}{4\pi}F_{ac}F_{bd}g^{cd} + e^2A_aA_b\psi\bar{\psi} + Lg_{ab}.$$

Example 4: *An isentropic perfect fluid*

The technique here is rather different. The fluid is described by a function ρ, called the density, and a congruence of timelike curves, called the flow lines. By a congruence of curves, is meant a family of curves, one through each point of $\mathscr{M}$. If $\mathscr{D}$ is a sufficiently small compact region, one can represent a congruence by a diffeomorphism γ: $[a,b]\times\mathscr{N}\to\mathscr{D}$ where $[a,b]$ is some closed interval of R^1 and $\mathscr{N}$ is some three-dimensional manifold with boundary. The curves are said to be timelike if their tangent vector $\mathbf{W} = (\partial/\partial t)_\gamma$, $t\in[a,b]$, is timelike everywhere. The tangent vector $\mathbf{V}$ is defined by $\mathbf{V} = (-g(\mathbf{W},\mathbf{W}))^{-\frac{1}{2}}\mathbf{W}$, so $g(\mathbf{V},\mathbf{V}) = -1$, and the fluid current vector is defined by $\mathbf{j} = \rho\mathbf{V}$. It is required that this is conserved, i.e. $j^a{}_{;a} = 0$. The behaviour of the fluid is determined by prescribing the elastic potential (or internal energy) ϵ as a function of ρ. The Lagrangian is taken to be

$$L = -\rho(1+\epsilon)$$

and the action I is required to be stationary when the flow lines are varied and ρ is adjusted to keep j^a conserved. A variation of the flow lines is a differentiable map γ: $(-\delta,\delta)\times[a,b]\times\mathscr{N}\to\mathscr{D}$ such that

$$\gamma(0,[a,b],\mathscr{N}) = \gamma([a,b],\mathscr{N})$$

and $\quad \gamma(u,[a,b],\mathscr{N}) = \gamma([a,b],\mathscr{N}) \quad$ on $\quad \mathscr{M}-\mathscr{D}, \quad (u\in(-\delta,\delta))$.

Then it follows that $\Delta\mathbf{W} = L_{\mathbf{K}}\mathbf{W}$ where the vector $\mathbf{K}$ is $\mathbf{K} = (\partial/\partial u)_\gamma$. This vector may be thought of as representing the displacement, under the variation, of a point of the flow line. It follows that

$$\Delta V^a = V^a{}_{;b}K^b - K^a{}_{;b}V^b - V^aV^bK_{b;c}V^c.$$

Using the fact that $\Delta(j^a{}_{;a}) = 0 = (\Delta j^a)_{;a}$, one has

$$(\Delta\rho)_{;a}V^a + \Delta\rho V^a{}_{;a} + \rho_{;a}\Delta V^a + \rho(\Delta V^a)_{;a} = 0.$$

Substituting for ΔV^a and integrating along the flow lines, one finds

$$\Delta\rho = (\rho K^b)_{;b} + \rho K_{b;c}V^bV^c.$$

Therefore the variation of the action integral is

$$\left.\frac{\partial I}{\partial u}\right|_{u=0} = -\int_{\mathscr{D}}\left\{((\rho K^b)_{;b} + \rho K_{b;c}V^bV^c)\left(1+\frac{\mathrm{d}(\rho\epsilon)}{\mathrm{d}\rho}\right)\right\}\mathrm{d}v.$$

Integrating by parts,

$$\left.\frac{\partial I}{\partial u}\right|_{u=0} = \int_{\mathscr{D}}\left\{\left(\rho\left(1+\frac{\mathrm{d}(\epsilon\rho)}{\mathrm{d}\rho}\right)\dot{V}^a + \rho\left(\frac{\mathrm{d}(\epsilon\rho)}{\mathrm{d}\rho}\right)_{;c}(g^{ca} + V^cV^a)\right)K_a\right\}\mathrm{d}v,$$

where $\dot{V}^a \equiv V^a{}_{;b} V^b$. If this is zero for all $\mathbf{K}$, it follows that

$$(\mu+p)\,\dot{V}^a = -p_{;b}(g^{ba}+V^bV^a),$$

where $\mu = \rho(1+\epsilon)$ is the energy density and $p = \rho^2(\mathrm{d}\epsilon/\mathrm{d}\rho)$ is the pressure. Thus $\dot{V}^a$, the acceleration of the flow lines, is given by the pressure gradient orthogonal to the flow lines.

To obtain the energy–momentum tensor one varies the metric. The calculations may be simplified by noting that the conservation of the current may be expressed as

$$(j^a)_{;a} = \frac{1}{(\sqrt{-g})}\frac{\partial}{\partial x^a}((\sqrt{-g})j^a) = 0.$$

Given the flow lines, the conservation equations determine j^a uniquely at each point on a flow line in terms of its initial value at some given point on the same flow line. Therefore $(\sqrt{-g})j^a$ is unchanged when the metric is varied. But

$$\rho^2 = g^{-1}((\sqrt{-g})j^a(\sqrt{-g})j^b)g_{ab},$$

so
$$2\rho\Delta\rho = (j^aj^b - j^cj_cg^{ab})\,\Delta g_{ab},$$

and thus
$$T^{ab} = \left\{\rho(1+\epsilon)+\rho^2\frac{\mathrm{d}\epsilon}{\mathrm{d}\rho}\right\}V^aV^b+\rho^2\frac{\mathrm{d}\epsilon}{\mathrm{d}\rho}g^{ab}$$

$$= (\mu+p)\,V^aV^b+pg^{ab}. \tag{3.8}$$

We shall call any matter whose energy–momentum tensor is of the above form (whether or not it is derived from a Lagrangian) a *perfect fluid*. From the energy and momentum conservation equations (3.1) applied to (3.8) one finds

$$\mu_{;a}V^a+(\mu+p)\,V^a{}_{;a} = 0, \tag{3.9}$$

$$(\mu+p)\,\dot{V}^a+(g^{ab}+V^aV^b)p_{;b} = 0. \tag{3.10}$$

These are the same as the equations derived from the Lagrangian. We shall call a perfect fluid *isentropic* if the pressure p is a function of the energy density μ only. In this case one can introduce a conserved density ρ and an internal energy ϵ and derive the equations and the energy–momentum tensor from a Lagrangian.

One may also give the fluid a conserved electric charge e (i.e. $J^a{}_{;a} = 0$ where $\mathbf{J} = e\,\mathbf{V}$ is the electric current). The Lagrangian for the charged fluid and the electromagnetic field is

$$L = -\frac{1}{16\pi}F_{ab}F_{cd}g^{ac}g^{bd}-\rho(1+\epsilon)-\tfrac{1}{2}J^aA_a.$$

The last term gives the interaction between the fluid and the field. Then varying **A**, the flow lines and the metric respectively, one finds

$$F^{ab}{}_{;b} = 4\pi J^a,$$

$$(\mu+p)\,\dot{V}^a = -p_{;b}(g^{ab}+V^aV^b)+F^a{}_bJ^b,$$

$$T^{ab} = (\mu+p)\,V^aV^b+pg^{ab}+\frac{1}{4\pi}(F^a{}_cF^{bc}-\tfrac{1}{4}g^{ab}F_{cd}F^{cd}).$$

3.4 The field equations

So far, the metric **g** has not been specified. In the Special Theory of Relativity, which does not include gravitational effects, it is taken to be flat. One might think that one could include gravitation by keeping the metric flat and by introducing an extra field on space–time. However, experiments have shown that light rays travelling near the sun are deflected. Since light rays are null geodesics, this shows that the space–time metric cannot be flat or even conformal to a flat metric. One therefore has to give some prescription for the curvature of space–time. It turns out that this prescription can be chosen so as to reproduce the results of Newtonian gravitation theory in the limit of small slowly varying curvature. It is therefore not necessary to introduce an extra field to describe gravitation. This is not to say that there could not be an additional field that produced part of the gravitational effects. Such a scalar field has been suggested by Jordan (1955), and Brans and Dicke (see Dicke (1964)). However, as mentioned before, such an additional field could be regarded as simply another matter field and included in the total energy–momentum tensor. We therefore adopt the view that the gravitational field is represented by the space–time metric itself. The problem then becomes one of finding field equations to relate the metric to the distribution of matter.

These equations should be tensor equations involving the matter only through its energy–momentum tensor, i.e. should not distinguish between two different matter fields which have the same distribution of energy and momentum. This can be regarded as a generalization of the Newtonian principle that the active gravitational mass of a body (the mass producing a gravitational field) is equal to the passive gravitational mass (the mass acted on by the gravitational field). This has been verified experimentally by Kreuzer (1968).

To determine what the field equations should be, we shall consider the Newtonian limit. Since the Newtonian gravitational field equation does not involve time, the correspondence with Newtonian theory should be made in a metric which is static. By a static metric is meant a metric which admits a timelike Killing vector field **K** which is orthogonal to a family of spacelike surfaces. These surfaces may be regarded as surfaces of constant time and may be labelled by the parameter t. We define the unit timelike vector **V** as f^{-1}**K**, where $f^2 = -K^a K_a$. Then $V^a{}_{;b} = -\dot{V}^a V_b$, where $\dot{V}^a = V^a{}_{;b} V^b = f^{-1} f_{;b} g^{ab}$ represents the departure from geodesity of the integral curves of **V** (which are of course also integral curves of **K**). Note that $\dot{V}^a V_a = 0$.

These integral curves define the static frame of reference, that is to say, the space–time metric seems to be independent of time to a particle whose history is one of these curves. A particle released from rest and following a geodesic would appear to have an initial acceleration of $-\dot{\mathbf{V}}$ with respect to the static frame. If f differs only slightly from unity the initial acceleration of a freely moving particle released from rest is approximately minus the gradient of f. This suggests that one should regard $f-1$ as the quantity analogous to the Newtonian gravitational potential.

One can derive an equation for this potential by considering the divergence of $\dot{V}^a$:

$$\dot{V}^a{}_{;a} = (V^a{}_{;b} V^b)_{;a} = V^a{}_{;b;a} V^b + V^a{}_{;b} V^b{}_{;a}$$
$$= R_{ab} V^a V^b + (V^a{}_{;a})_{;b} V^b + (V_b \dot{V}^b)^2 = R_{ab} V^a V^b.$$

But $$\dot{V}^a{}_{;a} = (f^{-1} f_{;b} g^{ab})_{;a} = -f^{-2} f_{;a} f_{;b} g^{ab} + f^{-1} f_{;ba} g^{ab}$$

and $$f_{;ab} V^a V^b = -f_{;a} V^a{}_{;b} V^b = -f^{-1} f_{;a} f_{;b} g^{ab},$$

so one finds $$f_{;ab}(g^{ab} + V^a V^b) = f R_{ab} V^a V^b.$$

The term on the left is the Laplacian of f with respect to the induced metric in the three-surface $\{t = \text{constant}\}$. If the metric is almost flat, this will correspond to the Newtonian Laplacian of the potential. One would therefore obtain agreement with Newtonian theory in the limit of a weak field (i.e. when $f \simeq 1$) if the term on the right is equal to $4\pi G$ times the matter density plus terms which are small in the weak field limit.

This will be the case if there is a relation of the form

$$R_{ab} = K_{ab}, \tag{3.11}$$

where K_{ab} is a tensorial function of the energy–momentum tensor and the metric, which is such that $(4\pi G)^{-1} K_{ab} V^a V^b$ is equal to the matter density plus terms which are small in the Newtonian limit. We shall for the moment assume a relation of this form.

Since R_{ab} satisfies the contracted Bianchi identities $R_a{}^b{}_{;b} = \frac{1}{2}R_{;a}$, (3.11) implies

$$K_a{}^b{}_{;b} = \tfrac{1}{2}K_{;b}. \tag{3.12}$$

This shows that the apparently natural equation $K_{ab} = 4\pi G T_{ab}$ cannot be correct, since (3.12) and the conservation equations $T_a{}^b{}_{;b} = 0$ would imply $T_{;a} = 0$. For a perfect fluid, for example, this would mean that $\mu - 3p$ was constant throughout space–time, which is clearly not satisfied by a general fluid.

In fact in general, the only first order identities satisfied by the energy–momentum tensor are the conservation equations. From this it follows that the only tensorial function K_{ab} of the energy-momentum tensor and the metric which obeys the identities (3.12) for all energy–momentum tensors, is

$$K_{ab} = \kappa(T_{ab} - \tfrac{1}{2}Tg_{ab}) + \Lambda g_{ab}, \tag{3.13}$$

where κ and Λ are constants. The values of these constants can be determined from the Newtonian limit. Consider a perfect fluid with energy density μ and pressure p whose flow lines are the integral curves of the Killing vector (i.e. the fluid is at rest in the static frame). The energy–momentum tensor is given by (3.8). Putting this in (3.13) and (3.11), one finds

$$f_{;ab}(g^{ab} + V^a V^b) = f(\tfrac{1}{2}\kappa(\mu + 3p) - \Lambda). \tag{3.14}$$

In the Newtonian limit the pressure p is normally very small compared to the energy density μ. (We are using units in which the speed of light is unity. In units in which the speed of light is c, the expression $\mu + 3p$ should be replaced by $\mu + 3p/c^2$.) One would therefore obtain approximate agreement with Newtonian theory if $\kappa = 8\pi G$ and if $|\Lambda|$ is very small. We shall use units of mass in which $G = 1$. In these units, a mass of 10^{28} gm corresponds to a length of 1 cm. Sandage's (1961, 1968) observations of distant galaxies place limits on $|\Lambda|$ of the order of $10^{-56}\,\text{cm}^{-2}$; we shall normally take Λ to be zero, but shall bear in mind the possibility of other values.

One may then integrate (3.14) over a compact region $\mathscr{F}$ of the three-surface $\{t = \text{constant}\}$ and transform the left-hand side into an integral of the gradient of f over the bounding two-surface $\partial\mathscr{F}$:

$$\int_{\mathscr{F}} f(4\pi(\mu + 3p))\,\mathrm{d}\sigma = \int_{\mathscr{F}} f_{;ab}(g^{ab} + V^a V^b)\,\mathrm{d}\sigma$$
$$= \int_{\partial\mathscr{F}} f_{;a}(g^{ab} + V^a V^b)\,\mathrm{d}\tau_b,$$

where $d\sigma$ is the volume element of the three-surface $\{t = \text{constant}\}$ in the induced metric, and $d\tau_b$ is the surface element of the two-surface $\partial\mathscr{F}$ in the three-surface. This gives the analogue of the Newtonian formula for the total mass contained within a two-surface. There are however two important differences from the Newtonian case:

(i) a factor f appears in the integral on the right-hand side. This means that matter placed in a region where f is considerably less than one (a large negative Newtonian potential) makes a smaller contribution to the total mass than does the same matter in a region where f is almost one (small negative Newtonian potential);

(ii) the pressure contributes to the total mass. This means that in some circumstances it can actually assist rather than prevent gravitational collapse.

The equations
$$R_{ab} = 8\pi(T_{ab} - \tfrac{1}{2}Tg_{ab}) + \Lambda g_{ab}$$
are called the *Einstein equations* and are often written in the equivalent form
$$(R_{ab} - \tfrac{1}{2}Rg_{ab}) + \Lambda g_{ab} = 8\pi T_{ab}. \tag{3.15}$$
Since both sides are symmetric, these form a set of ten coupled non-linear partial differential equations in the metric and its first and second derivatives. However the covariant divergence of each side vanishes identically, that is,
$$(R^{ab} - \tfrac{1}{2}Rg^{ab} + \Lambda g^{ab})_{;b} = 0$$
and
$$T^{ab}{}_{;b} = 0$$
hold independent of the field equations. Thus the field equations really provide only six independent differential equations for the metric. This is in fact the correct number of equations to determine the space–time, since four of the ten components of the metric can be given arbitrary values by use of the four degrees of freedom to make co-ordinate transformations. Another way of looking at this is that two metrics $\mathbf{g}_1$ and $\mathbf{g}_2$ on a manifold $\mathscr{M}$ define the same space–time if there is a diffeomorphism θ which takes $\mathbf{g}_1$ into $\mathbf{g}_2$. Therefore the field equations should define the metric only up to an equivalence class under diffeomorphisms, and there are four degrees of freedom to make diffeomorphisms.

We shall consider the Cauchy problem for the Einstein equations in chapter 7, and shall show that, together with the equations for the matter fields, they are sufficient to determine the evolution of space–time given suitable initial conditions, and that they satisfy the causality postulate (a).

The Einstein equations can be derived by requiring that the action

$$I = \int_{\mathscr{D}} (A(R-2\Lambda)+L)\,\mathrm{d}v \tag{3.16}$$

be stationary under variations of g_{ab}, where L is the matter Lagrangian and A a suitable constant. For

$$\Delta((R-2\Lambda)\,\mathrm{d}v) = ((R-2\Lambda)\,\tfrac{1}{2}g^{ab}\Delta g_{ab} + R_{ab}\Delta g^{ab} + g^{ab}\Delta R_{ab})\,\mathrm{d}v.$$

The last term can be written

$$\begin{aligned} g^{ab}\Delta R_{ab}\,\mathrm{d}v &= g^{ab}((\Delta\Gamma^c{}_{ab})_{;c} - (\Delta\Gamma^c{}_{ac})_{;b})\,\mathrm{d}v \\ &= (\Delta\Gamma^c{}_{ab}g^{ab} - \Delta\Gamma^d{}_{ad}\,g^{ac})_{;c}\,\mathrm{d}v. \end{aligned}$$

Thus it may be transformed into an integral over the boundary $\partial\mathscr{D}$, which vanishes as $\Delta\Gamma^a{}_{bc}$ vanishes on the boundary. Therefore

$$\left.\frac{\partial I}{\partial u}\right|_{u=0} = \int_{\mathscr{D}} \{A((\tfrac{1}{2}R-\Lambda)g^{ab} - R^{ab}) + \tfrac{1}{2}T^{ab}\}\Delta g_{ab}\,\mathrm{d}v, \tag{3.17}$$

and so if $\partial I/\partial u$ vanishes for all Δg_{ab}, one obtains the Einstein equations on setting $A = (16\pi)^{-1}$.

One might ask whether varying an action derived from some other scalar combination of the metric and curvature tensors might not give a reasonable alternative set of equations. However the curvature scalar is the only such scalar linear in second derivatives of the metric tensor; so only in this case can one transform away a surface integral and be left with an equation involving only second derivatives of the metric. If one tried any other scalar such as $R_{ab}R^{ab}$ or $R_{abcd}R^{abcd}$ one would obtain an equation involving fourth derivatives of the metric tensor. This would seem objectionable, as all other equations of physics are first or second order. If the field equations were fourth order, it would be necessary to specify not only the initial values of the metric and its first derivatives, but also the second and third derivatives, in order to determine the evolution of the metric.

We shall assume the field equations do not involve derivatives of the metric higher than the second. If these field equations are derived from a Lagrangian, then the action must have the form (3.16). One could however obtain a system of equations other than the Einstein equations, if one restricted the form of the variations Δg_{ab} for which the action was required to be stationary.

For example, one could restrict the metric to be conformal to a flat metric, i.e. assume

$$g_{ab} = \Omega^2\eta_{ab},$$

where η_{ab} is a flat metric as in Special Relativity. Then

$$\Delta g_{ab} = 2\Omega^{-1}\Delta\Omega g_{ab}$$

and the action will be stationary if

$$\{(A(\tfrac{1}{2}R-\Lambda)g^{ab}-R^{ab})+T^{ab}\}\Delta\Omega g_{ab} = 0$$

for all $\Delta\Omega$, that is if $\qquad R+A^{-1}T = 4\Lambda.$

From (2.30),

$$R = -6\Omega^{-3}\Omega_{|bc}\eta^{bc} = -6\Omega^{-1}\Omega_{;bc}g^{bc}+12\Omega^{-2}\Omega_{;c}\,\Omega_{;d}g^{cd},$$

where | denotes covariant differentiation with respect to the flat metric η_{ab}. If the metric is static, Ω will be constant along the integral curves of the Killing vector **K** (it will be independent of the time t). The magnitude of **K** will be proportional to Ω. Therefore

$$\begin{aligned} f_{;ab}(g^{ab}+V^aV^b)f^{-1} &= \Omega_{;ab}(g^{ab}+V^aV^b)\,\Omega^{-1} \\ &= -\tfrac{1}{6}R+2\Omega^{-2}\Omega_{;a}\,\Omega_{;b}g^{ab}-\Omega^{-1}\Omega_{;a}V^a{}_{;b}V^b \\ &= -\tfrac{1}{6}R+f^{-2}f_{;a}f_{;b}g^{ab}. \end{aligned}$$

Thus the Laplacian of f will be equal to $-\frac{1}{6}R$ plus a term proportional to the square of the gradient of f. This last term may be neglected in a weak field. From the field equations, $-\frac{1}{6}R$ will be equal to $\frac{1}{6}A^{-1}T-\frac{2}{3}\Lambda$. For a perfect fluid, $T = -\mu+3p$. One will therefore get agreement with Newtonian theory if Λ is small or zero and $A^{-1} = -24\pi$.

This theory in which the metric is restricted to be conformally flat is known as the Nordström theory. It can be reformulated as a theory in which the metric is the flat metric $\boldsymbol{\eta}$ and in which the gravitational interaction is represented by an additional scalar field ϕ. As mentioned before, this sort of theory would be inconsistent with the observed deflection of light by massive objects, and it would not account for the measured advance of the perihelion of Mercury.

One could in fact obtain the observed deflection of light and the advance of the perihelion of Mercury if the metric was restricted to be of the form

$$g_{ab} = \Omega^2(\eta_{ab}+W_aW_b),$$

where W_a is an arbitrary one-form field. This would give the Newtonian limit in a static metric in which W_a was parallel to the timelike Killing vector. There could however also be other static metrics where W_a was not parallel to the Killing vector and these would not give the Newtonian limit. Further this restriction on the form of the metric seems rather artificial. It appears more natural not to restrict the metric, apart from requiring that it be Lorentzian.

We therefore adopt as our third postulate,

Postulate (*c*): *Field equations*

Einstein's field equations (3.15) hold on $\mathscr{M}$.

The predictions of these field equations agree, within the experimental errors, with the observations that have been made so far on the deflection of light and the advance of the perihelion of Mercury, though the question of whether there exists a long range scalar field which ought to be included in the energy–momentum tensor remains open at the present time.